A Proof of Riemann Hypothesis by Vector Properties of Riemann Zeta Function and Rubber Strip Model

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Abstract: The Riemann Hypothesis (RH) states that the non-trivial zeros of the Riemann Zeta Function (RZF) $\zeta(s)$ or the Dirichlet Eta Function(DEF) $\eta(s)$ for a complex variable $s = x + iy$ is of the form $s = 0.5 + iy$. In this thesis, we treat each term of the RZF(we only mention the RZF instead of 'the RZF or the DEF') as a vector. We showed some vector properties of the RZF by tracing term vectors. If there exist zeros whose real part is not 0.5, such as $\zeta(\alpha - i\beta) = \zeta(1 - \alpha + i\beta) = 0$, the trajectory of $\zeta(\alpha - iy)$ and $\zeta(1 - \alpha + iy)$ must intersect at the origin when $y = \beta$. To check if this can happen, we introduced the rubber strip model, and by using the Cauchy-Riemann differential equations, we induced a contradiction, $\zeta(s) = constant$, which proves the RH. In appendices, we provided the source programs for visualizing vector traces of the RZF. We also suggested three other possible proofs of the RH for further studies.

1. Introduction

The RZF [1][2][3][4][5] $\zeta(s)$ and the DEF [6] $\eta(s)$ are functions of a complex variable $s = x + iy$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \tag{1.1}$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s) \tag{1.2}$$

The RH [1][7][8] states that all the non-trivial zeros of $\zeta(s)$ are of the form $s = 0.5 + iy$. The line $x = 0.5$ is called the *critical line*. The RH remains one of the most important unsolved problems in mathematics.

Despite of the tremendous effort [8] to prove the RH, we were somewhat surprised to find that, there, as far as we know, are few efforts to prove the RH by considering each term of the RZF as a vector. The fact that each term of the RZF is a complex number, and a complex number is equivalent to a 2-dimensional vector, was the stimulus to visually trace the individual term as a vector.

Our effort does not focus on how to find zeros of the RZF, but focus on *how the infinite series of term vectors of the RZF approaches to the origin*. It was our intuition that, for any complex variable $s = x + iy, x \neq 0.5$, if we can geometrically prove that the infinite series of term vectors of the RZF can't approach to the origin, then the RH is true.

If there exist zeros whose real part is not 0.5, such as $\zeta(\alpha - i\beta) = \zeta(1 - \alpha + i\beta) = 0$, the trajectory of $\zeta(\alpha - iy)$ and $\zeta(1 - \alpha + iy)$ must intersect at the origin when $y = \beta$. To check if this can happen, we introduced the rubber strip model, and by using the Cauchy-Riemann differential equations, we induced a contradiction, $\zeta(s) = constant$, which proves the RH. The rubber strip model is just the trajectory of $\zeta(x + iy), \alpha \leq x \leq 1 - \alpha$, which resembles a thin flexible rubber band, as depicted in Figure 12.

2. Symmetry Properties of the Zeros of the RZF

It is well known that the following three equations are true, where $\xi(s)$ is the Riemann's Xi-function [8][10].

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{\frac{-s}{2}} \tag{2.1}$$

$$\xi(s) = \xi(1-s) \tag{2.2}$$

$$\zeta(\bar{s}) = \overline{\zeta(s)} \tag{2.3}$$

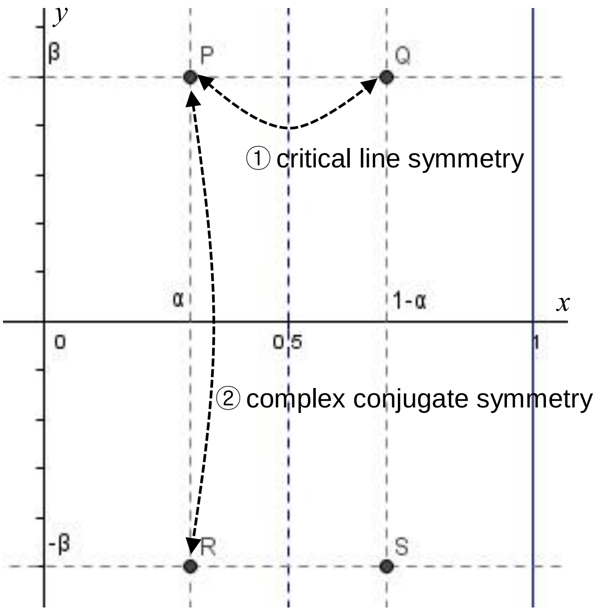
The right side of the equations (1.2) and (2.1) include $\zeta(s)$, so, the zeros of $\zeta(s)$ are also the zeros of $\eta(s)$ and $\xi(s)$.

Lemma 2.1. Equations (2.2) and (2.3) means that there exist two types of symmetries of the zeros of the RZF, as in Figure 1.

① **Critical line symmetry:** Symmetry of (2.2), which means that if $s = \alpha + i\beta$ is zero, then $1 - \alpha + i\beta$ is also a zero.

② **Complex conjugate symmetry:** Symmetry of (2.3), which means that if $s = \alpha + i\beta$ is a zero, then $s = \alpha - i\beta$ is also a zero.

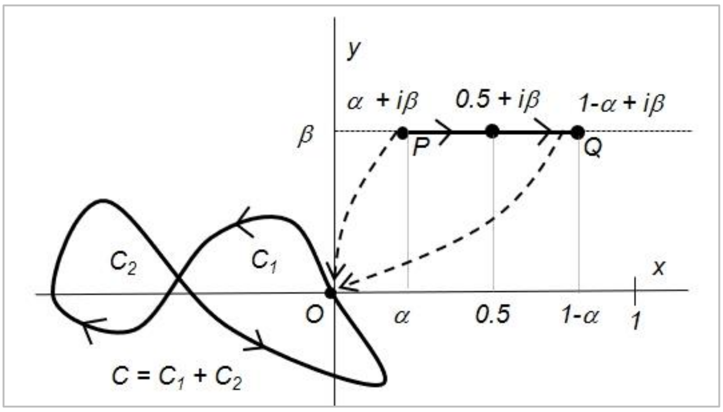
Figure 1. Zero symmetries of the RZF.



Proof. Let $s = \alpha + i\beta$. First, in (2.3), $\zeta(\alpha - i\beta) = \overline{\zeta(\alpha + i\beta)} = 0$, which is same as $\zeta(R) = \overline{\zeta(P)} = 0$, in Figure 1. So, the complex conjugate symmetry is true. Second, in (2.2), $\xi(\alpha + i\beta) = \xi\{1 - (\alpha + i\beta)\} = 0$, which is same as $\xi(P) = \xi(S) = 0$, in Figure 1. Because of the complex conjugate symmetry, $\xi(S) = \xi(Q) = 0$. So, $\xi(P) = \xi(Q) = 0$, which is the critical line symmetry. ■

Lemma 2.2. By moving x from α to $1-\alpha$, where $\zeta(\alpha+i\beta)=\zeta(1-\alpha+i\beta)=0$, a logically closed trajectory must be drawn, which starts from the origin and ends at the origin, as in Figure 2.

Figure 2. A closed curve trajectory for $\alpha \leq x \leq 1-\alpha, 0 \leq \alpha \leq 0.5$.



Proof. A closed curve trajectory *C* in Figure 2 must be drawn, as x moves from *P*(α, β) to *Q*(1 - α, β), by the following 3 steps.

① **Initial state at *P*(α, β):** At *P*(α, β), the trajectory remains at the origin.

② **Movement to (0.5, β):** The trajectory will leave the origin and will reach somewhere on the curve *C*.

③ **Movement to *Q*(1 - α, β):** At *Q*(1 - α, β), the trajectory will come back to the origin.

So, the trajectory drawn while x moves from α to 1 - α, must be a logically closed trajectory *C*. Here, a logically closed trajectory means that, *C* may cross itself, resulting multiple loops $C_1, C_2, \ldots, C_i$. ■

3. Vector Properties of the RZF

3.1 Considering Each Term of the RZF as a Vector

In (1.1) and (1.2), let's denote each term of the RZF or the DEF as $f_n(s)$ and $g_n(s)$, $s = x+iy$, respectively.

$$f_n(s) = \frac{1}{n^s} = e^{-xlnn}e^{-iylnn} = r_n(x)e^{i\theta_n(y)} = u_n(x,y) + iv_n(x,y) \quad (3.1)$$

$$r_n(x) = e^{-xlnn} \quad (3.2)$$

$$\theta_n(y) = -ylnn \quad (3.3)$$

$$u_n(x,y) = r_n(x)cos\theta_n(y) \quad (3.4)$$

$$v_n(x,y) = r_n(x)sin\theta_n(y) \quad (3.5)$$

$$f_1(s) = \frac{1}{1^s} = e^{-xln1}e^{-iyln1} = 1$$

$$f_2(s) = \frac{1}{2^s} = e^{-xln2}e^{-iyln2} = e^{-xln2}\{\cos(yln2) - isin(yln2)\}$$

$$\zeta(s) = \sum_{n=1}^{\infty} f_n(s) = \sum_{n=1}^{\infty} e^{-xlnn}e^{-iylnn}$$

$$= 1 + \sum_{n=2}^{\infty} e^{-xlnn}e^{-iylnn} \tag{3.6}$$

$$g_n(s) = \frac{(-1)^{n+1}}{n^s} = (-1)^{n+1}e^{-xlnn}e^{-iylnn} = (-1)^{n+1}f_n(s) \tag{3.7}$$

$$\eta(s) = \sum_{n=1}^{\infty} g_n(s) = \sum_{n=1}^{\infty}(-1)^{n+1}e^{-xlnn}e^{-iylnn}$$

$$= 1 + \sum_{n=2}^{\infty}(-1)^{n+1}e^{-xnn}e^{-iylnn} \tag{3.8}$$

We consider $f_n(s)$ or $g_n(s)$ as a 2-dimensional vector in (x, y) plane, and by using computer programs we traced the series of vectors. By doing so, we can visually observe how the RZF or the DEF approaches to the origin or to the other point.

Lemma 3.1. Each term of the RZF can be considered as a 2-dimentional vector.

Proof. Each term of the RZF is a complex number. A complex number can be represented as a point in (x, y) plane, which can be considered as a 2-dimentional vector. So, each term of the RZF can be considered as a 2-dimentional vector. ■

Definition 3.2. Term vector: A term of the RZF that is equivalent to a 2-dimensional vector.

Lemma 3.3. The following three properties of $f_n(s), s = x + iy$, are true.

① *x* determines the magnitude of $f_n(s)$, which is e^{-xlnn}.

② *y* determines the argument of $f_n(s)$, which is $-ylnn$.

③ *x* and *y* are independent.

Proof. The magnitude and argument of $f_n(s)$ are as follows.

$$r_n(x) = |f_n(s)| = e^{-xlnn} \tag{3.9}$$

$$\theta_n(y) = arg\{f_n(s)\} = -ylnn \tag{3.10}$$

So, obviously, the above three properties are true for $f_n(s)$. ■

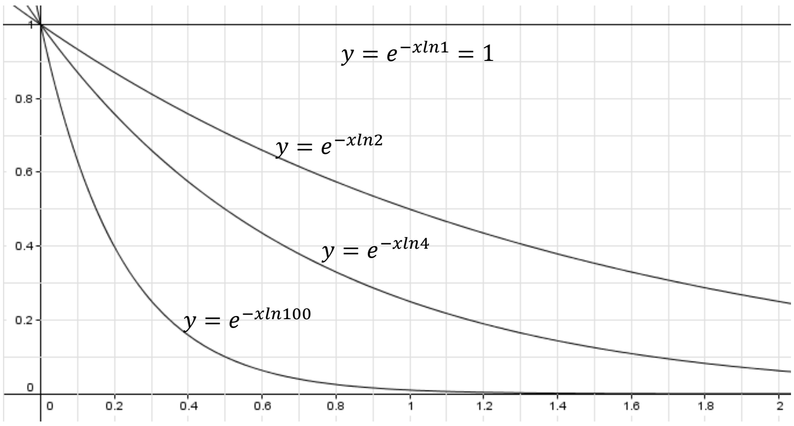
Lemma 3.3 also applies to $g_n(s)$ of (3.7). What is important is that x and y are independent. Table 1 shows some examples of x vs $r_n(x)$ relationships.

Table 1. Magnitude *$r_n(x)$* examples.

x	**0**	**1/3**	**1/2**	**2/3**	**1**
$r_n(x)$	1	$(1/n)^{1/3}$	$(1/n)^{1/2}$	$(1/n)^{2/3}$	$(1/n)^1$
$r_2(x)$	1	0.794	0.707	0.630	0.500
$r_3(x)$	1	0.693	0.577	0.481	0.333
$r_{100}(x)$	1	0.215	0.100	0.046	0.010
$r_{10000}(x)$	1	0.046	0.010	0.002	0.0001

Figure 3 shows some graphs of these relationships as functions, $y = r_n(x) = |f_n(s)| = e^{-xlnn}$, for some n. For $n = 1$, $y = r_1(x) = e^{-xln1} = e^0 = 1$, which corresponds to the constant term 1 in (3.6) and (3.8). As n increases, the graphs decreases more sharply.

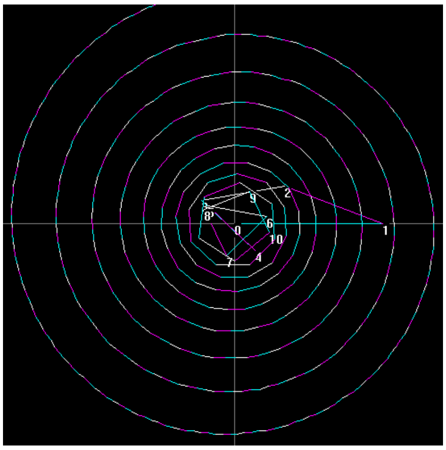
Figure 3. Graphs of $y = e^{-xlnn}$ for $n = 1, 2, 4, 100$.



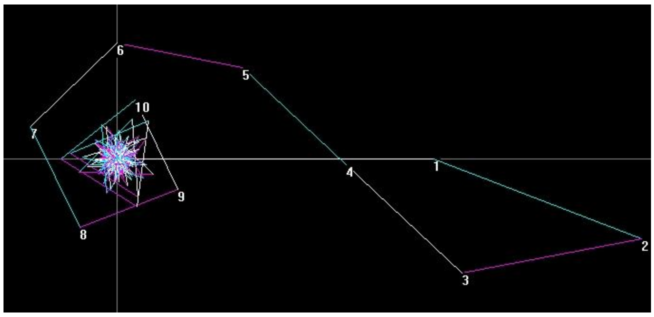
3.2 Vector Trace Graphs

We used PureBasic [11] free version to plot the trace. The source program and some videos are given in appendix A and C. Figure 4 shows some vector trace graphs of the RZF and the DEF for $s = 0.5 + 14.13i$ and $s = 0.5 + 24499.24i$, which are two zeros of the RZF.

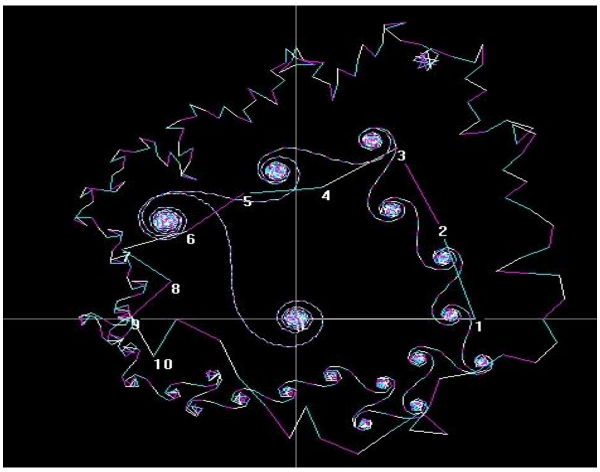
Figure 4. Sample vector trace graphs.



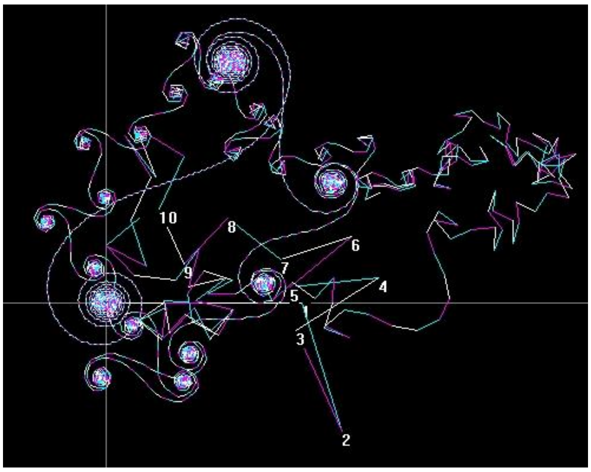
(a) the RZF: *x* = 0.5, *y* = 14.13



(b) the DEF: *x* = 0.5, *y* = 14.13



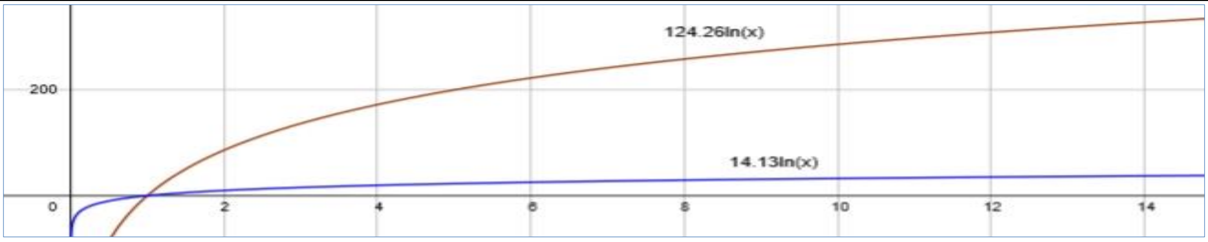
(c) the RZF: *x* = 0.5, *y* = 24499.24



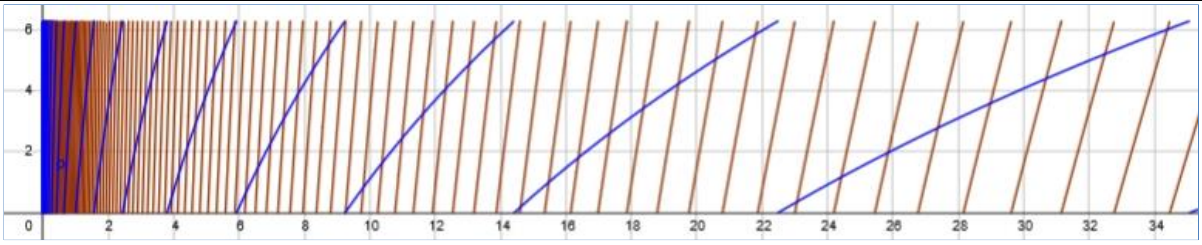
(d) the DEF: *x* = 0.5, *y* = 24499.24

To understand how Figure 4 is drawn, please watch vector trace videos in Appendix C. You can see vectors spiral in and spiral out making lumps, according to the vector argument graphs in Figure 5. More lumps appear when *y* is large.

Figure 5. Example graphs of vector argument.



(a) $y = 14.13$(blue), $y = 124.26$(brown)



(b) mod(arg, 2π), $y = 14.13$(blue), $y = 124.26$(brown)

In Figure 5 (a), the argument, $\theta_n(y) = arg\{f_n\,(s)\} = -ylnn$, is depicted by replacing n by x. The blue graph is, $\theta_x(14.13) = arg\{f_x\,(s)\} = -14.13lnx, y = 14.13$ and the brown graph is $\theta_x(124.26) = arg\{f_x\,(s)\} = -124.26lnx,, y = 124.26$. Figure 5 (b) is the modulo of the Figure 5 (a), $mod\{\theta_n(y), 2\pi\}$. The blue graph is, $mod\{\theta_x(14.13), 2\pi\}$ and the brown graph is $mod\{\theta_x(124.26), 2\pi\}$.

You can see that for larger y, the graph increases more rapidly, which means that the change of the direction of vector is more severe.

The movement of vectors can be roughly classified as follows.

① **Zigzag:** Vectors zigzag when the argument of vector changes severely.

② **Spiral in:** Vectors shrink to a point. It occurs when sequence of vectors with $arg(v_{n+1}) - arg(v_n) > 90°$ are dominant.

③ **Spiral out:** Vectors are inverted from spiral-in to spiral-out. It occurs when sequence of vectors with $arg(v_{n+1}) - arg(v_n) < 90°$ are dominant.

④ **Smooth moving:** Vectors move to other place smoothly.

3.3 Trace of *x* and *y*

We used GeoGebra [12] to trace the RZF with respect to *x* and *y*. GeoGebra provides zeta function and we set the following parameter and function, and animated *x* and *y*.

$$s = x + iy$$

$$w = zeta(s)$$

Graphs for the following cases are traced.

(a) $x = 0.5$, $14.13 \leq y \leq 32.94$.

(b) $x = 0.25$, $14.13 \leq y \leq 32.94$.

(c) $x = 0.75$, $14.13 \leq y \leq 32.94$.

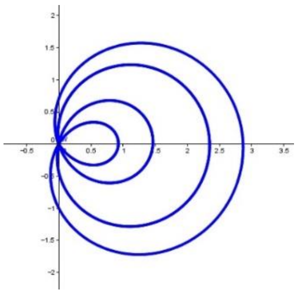
(d) $0 \leq x \leq 1$, $y = 14.13$.

(e) $0 \leq x \leq 1$, $y = 124.26$.

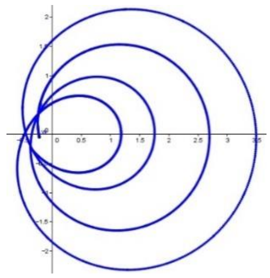
(f) $0 \leq x \leq 1$, $y = 294014.13$.

Figure 6 shows above 6 graphs.

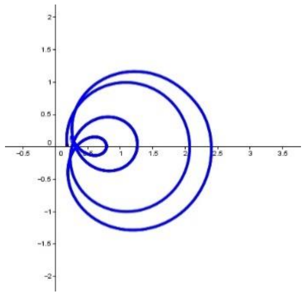
Figure 6. Trace of the RZF with respect to *x* and *y*.



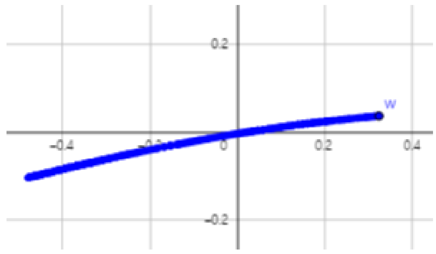
(a) $x = 0.5$, $14.13 \leq y \leq 32.94$.



(b) $x = 0.25$, $14.13 \leq y \leq 32.94$.

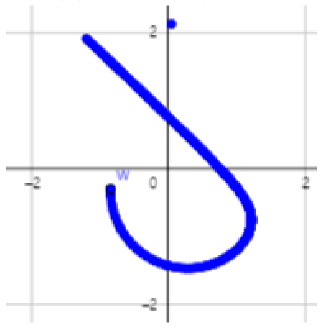


(c) $x = 0.75$, $14.13 \leq y \leq 32.94$.



(d) $0 \leq x \leq 1$, $y = 14.13$.

(e) $0 \leq x \leq 1$, $y = 124.26$



(f) $0 \leq x \leq 1$, $y = 294014.13$

From the graphs in Figure 6, we can see the followings.

(a) When $x = 0.5$, graphs pass the origin because there exist some zeros of the RZF.
(b) When $x = 0.25$, graph swells with some bias to the left, because the magnitude of each vector $e^{-x\ln n}$ increases.
(c) When $x = 0.75$, graph shrinks with some bias to the right, because the magnitude of each vector $e^{-x\ln n}$ decreases.
(d) When $y = 14.13$, as x moves from 0 to 1, an open curve which passes the origin at $x = 0.5$ is drawn, because $s = 0.5 + 14.13i$ is a zero of the RZF.

(e) When y = 124.26, as α moves from 0 to 1, an open curve which passes the origin at x = 0.5 is drawn, because $s = 0.5 + 124.26i$ is a zero of the RZF.

(f) When y = 294014.13, as x moves from 0 to 1, an open curve which does not pass the origin is drawn, because $s = 0.5 + 294014.13i$ is not a zero of the RZF.

4. Three Vector Properties of the RZF

Lemma 4.1. The RZF and the DEF can be geometrically represented as the sum of three vectors.

Proof. We can rewrite the RZF in (3.6) and the DEF in (3.8) as follows.

$$\zeta(s) = 1 + \sum_{n=2}^{\infty} e^{-xlnn} e^{-iylnn} \tag{3.6}$$

$$= 1 + \sum_{n=3}^{\infty} e^{-xlnn} e^{-iylnn} + e^{-xln2} e^{-iyln2}$$

$$= \overrightarrow{A_x} + \overrightarrow{B_x} + \overrightarrow{C_x} \tag{4.1}$$

$$\overrightarrow{A_x} = (1,0) \tag{4.2}$$

$$\overrightarrow{B_x} = \sum_{n=3}^{\infty} e^{-xlnn} e^{-iylnn} \tag{4.3}$$

$$\overrightarrow{C_x} = e^{-xln2} e^{-iyln2} \tag{4.4}$$

$$\eta(s) = 1 + \sum_{n=2}^{\infty} (-1)^{n+1} e^{-xlnn} e^{-iylnn} \tag{3.8}$$

$$= 1 + \sum_{n=3}^{\infty} (-1)^{n+1} e^{-xlnn} e^{-iylnn} - e^{-xln2} e^{-iyln2}$$

$$= \overrightarrow{D_x} + \overrightarrow{E_x} + \overrightarrow{F_x} \tag{4.5}$$

$$\overrightarrow{D_x} = (1,0) \tag{4.6}$$

$$\overrightarrow{E_x} = \sum_{n=3}^{\infty} (-1)^{n+1} e^{-xlnn} e^{-iylnn} \tag{4.7}$$

$$\overrightarrow{F_x} = -e^{-xln2} e^{-iyln2} \tag{4.8}$$

The sum of three vectors in (4.1) and (4.5) are the geometric representations of the RZF and the DEF, respectively. ■

The vector representations of the RZF and the DEF are similar, so, from now on, we mention only for the RZF, but the logic applied to the RZF can also be applied to the DEF.

Definition 4.2. Last vector: Vector $\overrightarrow{C_x}$ in (4.1).

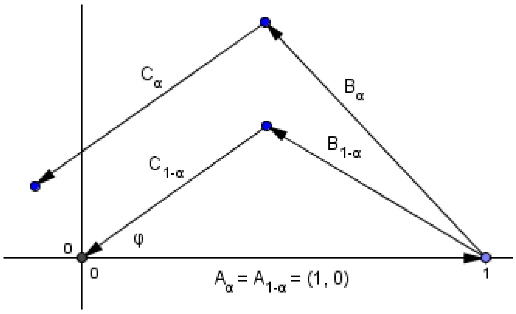
Definition 4.3. Three vector set(TVS): Set of three vectors in (4.1), $V_x = \{\overrightarrow{A_x}, \overrightarrow{B_x}, \overrightarrow{C_x}\}$.

Definition 4.4. Two last vectors: The two last vectors $\overrightarrow{C_\alpha}$ and $\overrightarrow{C_{1-\alpha}}$.

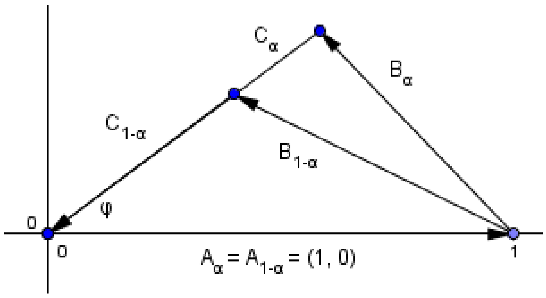
Lemma 4.5. If $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) = 0, 0 < \alpha < 0.5$, then the two last vectors $\overrightarrow{C_\alpha}$ and $\overrightarrow{C_{1-\alpha}}$ must be on the same line.

Proof. Figure 7 shows two types of TVSs, $V_\alpha = \{\overrightarrow{A_\alpha}, \overrightarrow{B_\alpha}, \overrightarrow{C_\alpha}\}$ and $V_{1-\alpha} = \{\overrightarrow{A_{1-\alpha}}, \overrightarrow{B_{1-\alpha}}, \overrightarrow{C_{1-\alpha}}\}$.

Figure 7. Last vector examples.



(a) Two last vectors can't end at the origin.



(b) Two last vectors end at the origin.

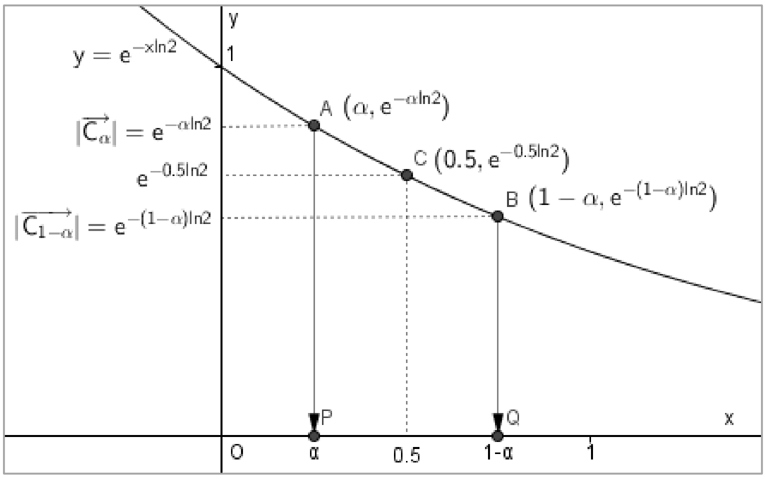
The two last vectors are $\overrightarrow{C_\alpha} = e^{-\alpha ln2}e^{-i\beta ln2}$ and $\overrightarrow{C_{1-\alpha}} = e^{-(1-\alpha)ln2}e^{-i\beta ln2}$, respectively. The arguments of the two last vectors are same, so, they are parallel to each other. If the two last vectors are not on the same line as in Figure 7 (a), it can't be $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) = 0$. So, the two last vectors should be on the same line, as in Figure 7 (b). ■

Definition 4.6. Same line restriction: The result of Lemma 4.5.

Lemma 4.7. The magnitude of the last vector $\overrightarrow{C_x}$ monotonously decreases as x moves from α to $1 - \alpha$, $0 < \alpha < 0.5$.

Proof. The magnitude of the two last vectors are $e^{-\alpha ln2}$ and $e^{-(1-\alpha)ln2}$, respectively, as in Figure 8.

Figure 8. Last vector magnitude graph.

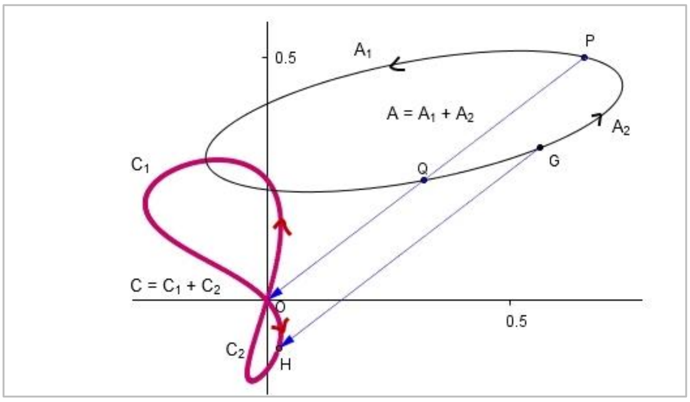


∗ The graph e^{-xln2} represents the magnitude of all vectors for *n* = 2.

The graph $y = e^{-xln2}$ represents the magnitude of the term vector for $n = 2$, which monotonously decreases as x moves from α to $1 - \alpha$. So, the magnitude of the last vector $\overrightarrow{C_\alpha}$ monotonously decreases as α moves from α to $1 - \alpha$. ■

Lemma 4.8. The trajectory $\overrightarrow{A_\alpha}, +\overrightarrow{B_x} + \overrightarrow{C_x}$, while x moves from α to $1 - \alpha$, will draw a closed curve, as an example trajectory in Figure 9, which is a vector aspect of Lemma 2.2.

Figure 9. An example trajectory of $\overrightarrow{A_\alpha}, +\overrightarrow{B_x} + \overrightarrow{C_x}$, $\alpha \le x \le 1-\alpha$.



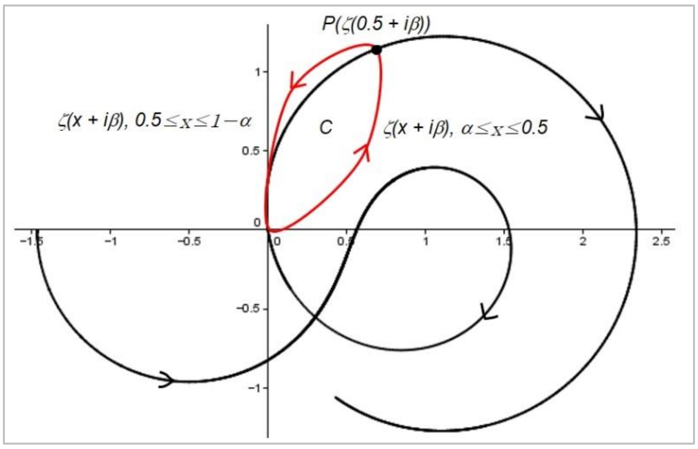
∗ While *G* moves around *A*, H draws *C*.

Proof. In Figure 9, let $\overrightarrow{C_\alpha} = \overrightarrow{PO}$, $\overrightarrow{C_{1-\alpha}} = \overrightarrow{QO}$. Let *G* be a point on the trajectory *A,* which is the trajectory of $\overrightarrow{A_\alpha} + \overrightarrow{B_x}$. We assumed *A* is a closed curve. Then $\overrightarrow{C_x} = \overrightarrow{GH}$, $|\overrightarrow{GH}| = e^{-x ln2}$. While *G* moves on the trajectory A_1, the trajectory of $\overrightarrow{A_\alpha} + \overrightarrow{B_x} + \overrightarrow{C_x}$ will draw a closed curve C_1. Likewise, while *G* moves on the trajectory A_2, the trajectory of $\overrightarrow{A_\alpha} + \overrightarrow{B_x} + \overrightarrow{C_x}$ will draw a closed curve C_2. The curves C_1 and C_2 may intersect themselves, but we can consider them as logically closed curves as in Figure 2. So, the trajectory $\overrightarrow{A_\alpha} + \overrightarrow{B_x} + \overrightarrow{C_x}$, while x moves from α to $1-\alpha$, will draw a closed curve. ■

Lemma 4.8 is just a vector aspect of Lemma 2.2. We draw Figure 9 by using GeoGebra just to see how a closed curve is generated, according to the same line restriction.

Lemma 4.9. The trajectory of Lemma 4.8, will intersect the curve $\zeta(0.5+iy)$, where $\zeta(\alpha+iy) = \zeta(1-\alpha+iy) = 0$, as in Figure 10.

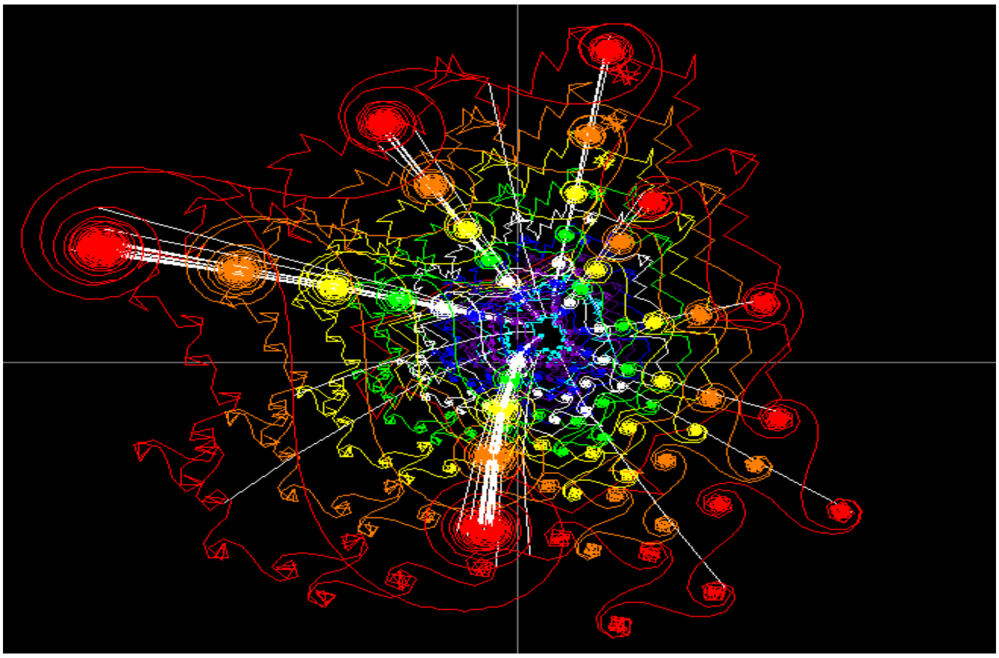
Figure 10. An example trajectory of $\zeta(x+iy), \alpha \le x \le 1-\alpha$.



Proof. In Figure 10, the black curve is the trace of $\zeta(0.5+iy), 0 \le y \le 20$. If there exist zeros such that $\zeta(\alpha+i\beta) = \zeta(1-\alpha+i\beta) = 0$, then while *x* moves between $\alpha \le x \le 1-\alpha$, a closed curve which starts at the origin and ends at the origin will be drawn and that curve will meet the curve at $P(\zeta(0.5+iy))$, when $x = 0.5$. ■

Figure 11 shows nine parallel vector trace graphs for *y* = 24499.249265478 and *x* = 0.3(outside red), 0.35, 0.4, 0.45, 0.5, 0.55, 0.6, 0.65, 0.7(inside aqua).

Figure 11. Parallel vector trace example.

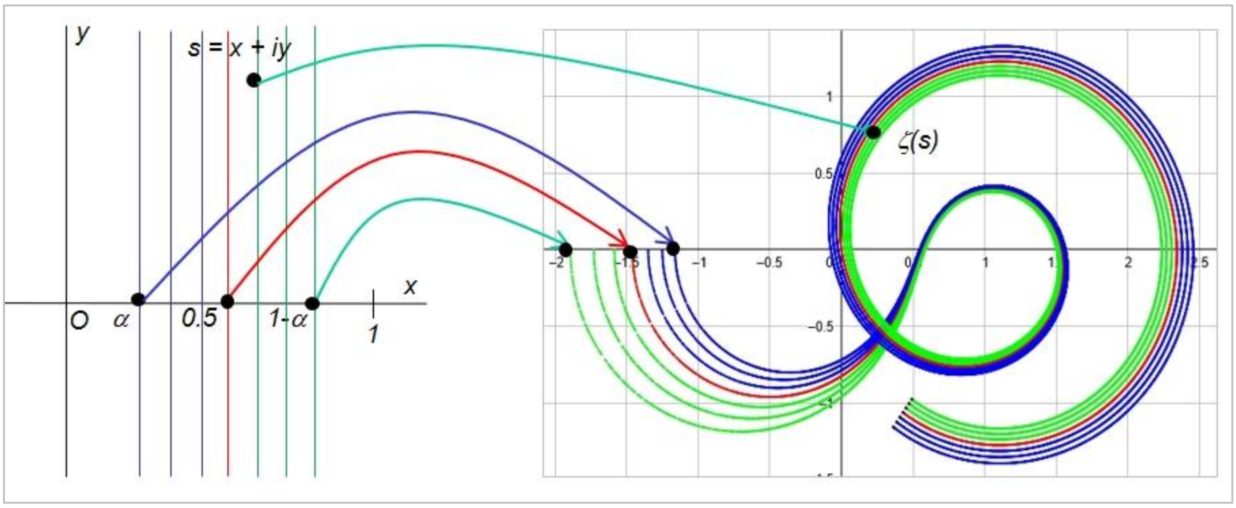


In Figure 11, the white lines from outside to inside are the lines that connect the end points for every 100'th term vectors. The parallel vector trace program source is provided in Appendix B.

5. Proof of the RH by the Rubber Strip Model

Definition 5.1. Rubber strip model: A model to explain the trajectory of $\zeta(x+iy)$, $\alpha \le x \le 1-\alpha$, $-\infty \le y \le \infty$. Figure 12 depicts an exemplary rubber strip model for $x = 0.44/0.47(blue), 0.5(red), 0.53/0.56(green), 0 \le y \le 20$. The rubber strip can be flexibly bended and the width may change along with y.

Figure 12. An exemplary rubber strip model.

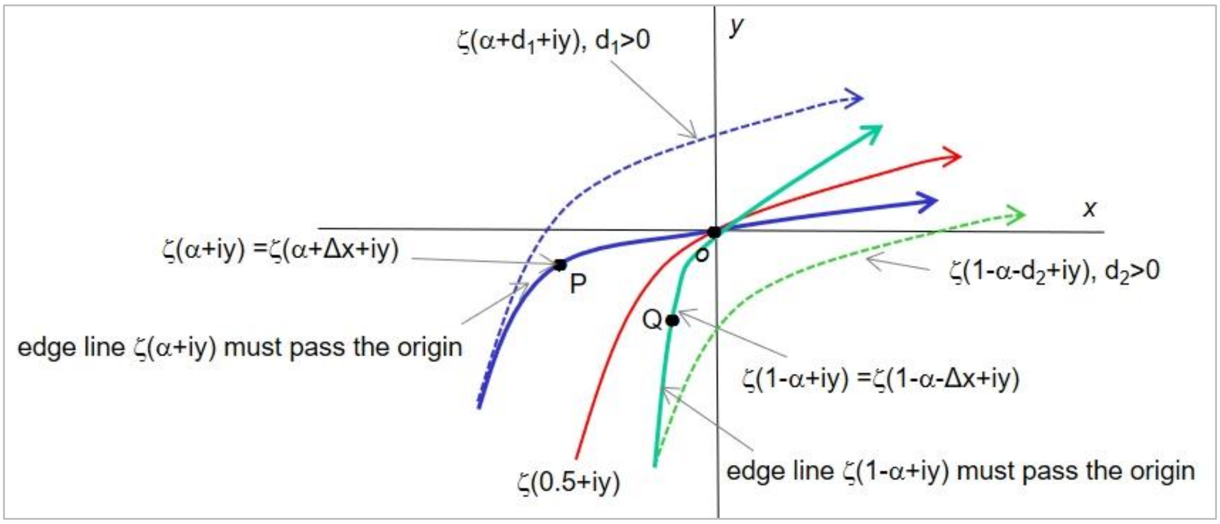


* A rubber strip model for $\alpha = 0.44/0.47(blue), 0.5(red), 0.53/0.56(green), 0 \le y \le 20$.

Definition 5.2. Edge lines: Two edge lines of the rubber strip, which are $\zeta(\alpha+iy)$ and $\zeta(1-\alpha+iy)$.

Lemma 5.3. To have two zeros such as $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) = 0$, two edge lines should intersect at the origin when $y = \beta$, as in Figure 13.

Figure 13. Two edge lines intersect at the origin.



Proof. If two edge lines intersect at $(x, y) \neq (0, 0)$, which is not the origin, $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) \neq (0,0)$, so, two edge lines must intersect at the origin. ■

Lemma 5.4. To make two edge lines intersect at the origin, two edge lines should cross the other lines within the rubber strip, passing infinitely many points like P or Q in Figure 13, which leads to a contradiction $\zeta(s) = constant$. So, the RH is true.

Proof. Suppose at $y = \beta - l_1, l_1 > 0$, the two edge lines begin to approach to the origin. Then the two edge lines must step into the rubber strip, which makes the two edge lines cross the other lines within the rubber strip and makes infinitely many points like P or Q in Figure 13. So, the following two equations must be satisfied.

$$\zeta(\alpha + iy) = \zeta(\alpha + \Delta x + iy), \text{ at } P. \tag{5.1}$$

$$\zeta(1 - \alpha + iy) = \zeta(1 - \alpha - \Delta x + iy), \text{ at } Q. \tag{5.2}$$

Let $s = x + iy$.

$$\zeta(s) = \zeta(x + iy) = u(x, y) + iv(x, y)$$

$$\Delta s = \Delta x + i\Delta y$$

$$\zeta'(s) = \lim_{\Delta s \to 0} \frac{\zeta(s+\Delta s) - \zeta(s)}{\Delta s}$$

Let's select (5.1) for our proof.

$$\zeta(\alpha + iy) = \zeta(\alpha + \Delta x + iy)$$

$$u(\alpha, y) + iv(\alpha, y) = u(\alpha + \Delta x, y) + iv(\alpha + \Delta x, y)$$

$$u(\alpha + \Delta x, y) - u(\alpha, y) + iv(\alpha + \Delta x, y) - iv(\alpha, y) = 0$$

Let $x = \alpha + \Delta x$, then $\alpha = x - \Delta x$.

$$u(x,y) - u(x-\Delta x, y) + iv(x,y) - iv(x - \Delta x, y) = 0$$

$$\lim_{\Delta x \to 0} \frac{u(x,y)-u(x-\Delta x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x,y)-v(x-\Delta x,y)}{\Delta x} = 0$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$$

$\frac{\partial u}{\partial x} = 0$ and $\frac{\partial v}{\partial x} = 0$

The Cauchy-Riemann differential equations states that the real part and the imaginary part of an analytic function $h(s) = u(x,y) + iv(x,y)$ satisfy the following equations at each point where $h(z)$ is analytic [13][14][15][16].

$\frac{\partial \mathrm{u}}{\partial \mathrm{x}} = \frac{\partial \mathrm{v}}{\partial \mathrm{y}}$ and $\frac{\partial \mathrm{u}}{\partial \mathrm{y}} = -\frac{\partial \mathrm{v}}{\partial \mathrm{x}}$ (5.3)

The RZF is analytic to all complex plane except $s = 1$ [4], so, the RZF is analytic to all points $s = x + iy, y > 0$. To satisfy (5.3),

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0.$

$$\zeta'(s) = 0$$

$$\zeta(s) = constant \tag{5.4}$$

The result (5.4) contradicts, so, the RH is proved. ■

6. Conclusion

In this thesis, we proved the RH by analyzing the vector properties of the RZF and the DEF. We treated each term of the RZF as a vector and showed some vector properties of the RZF by tracing term vectors. For a complex variable $s = x + iy$, x affects the magnitude of term vectors and y affects the argument of each term vectors. If there exist zeros whose real part is not 0.5, such as $\zeta(\alpha - i\beta) = \zeta(1 - \alpha + i\beta) = 0$, the trajectory of $\zeta(\alpha - iy)$ and $\zeta(1 - \alpha + iy)$ must intersect at the origin when $y = \beta$. To check if this can happen, we introduced the rubber strip model, and by using the Cauchy-Riemann differential equations, we induced a contradiction, $\zeta(s) = constant$, which proves the RH.

References

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List of Tables

List of Figures

Appendix A : Source Progeam for RZF or DEF Trace

;Code for zeta or eta vector trace visualization using PureBasic evaluation version.

```
;[1] graph window.
#Window1    = 0
#Image1     = 0
#ImgGadget = 0
#width          = 1370
#height         = 735

;[2] variables.
Define.d Dim x(10000000)      ;Re(z)
Define.d Dim y(10000000)      ;Im(z)
Define.d Dim t(10000000)      ;Arg(z), radian
Define.d Dim deg(10000000)   ;Arg(z), degree
Define.d Dim r(10000000)      ;r = |z|
Define.d a, b, r, t, delta, x0, y0, x1, y1, xsum, ysum, x2, y2, x3, y3, lnn, rr
Define.q i, m, n, thresh

;[3] sample zero values.
a = 1/2
;a = 0.501
;b = 14.134725141734693790
;b = 236.5242296658162058
;b = 5565.566217327
b = 24499.249265478
;b = 74908.108191005

;[4] font.
LoadFont (0, "Courier", 15) ;load Courier Font, Size 15.
LoadFont (1, "Arial", 24)     ;load Arial Font, Size 24.
OpenConsole()
If b>0
  header$ = "Riemann Zeta : s=" + a + "+" + b + "i"
  ;header$ = "Dirichlet Eta : s=" + a + "+" + b + "i"
Else
  header$ = "Riemann Zeta : s=" + a + "" + b + "i"
  ;header$ = "Dirichlet Eta : s=" + a + "" + b + "i"
EndIf
delta = 0.007   ;image zoom factor: small value for zoom in.
m = 7000 ;#vectors to plot.
pi.d = 3.1415926535

;[5] calculate vectors.
For n=1 To m Step 1
  lnn = Log(n)
  r(n) = Exp(-a*lnn)
  t(n) = -b*lnn
  deg(n) = Mod(Round(Degree(t(n)), #PB_Round_Down), 360)   ;#PB_Round_Up, #PB_Round_Nearest
  If deg(n)<0
    deg(n) = deg(n) + 360
  EndIf
  dg = deg(n) - deg(n-1)
```

```
  ;PrintN("n=" + n + " r(n)=" + r(n) + " θ=" + deg(n) + "° dθ=" + dg + "°" + " t(n)=" + t(n))    ;print values.
;for eta function, remove following 5 comments in If...Else...EndIf block.
;    If Mod(n, 2)=1
       x(n) = r(n)*Cos(t(n))
       y(n) = r(n)*Sin(t(n))
;    Else
;        x(n) = -r(n)*Cos(t(n))
;        y(n) = -r(n)*Sin(t(n))
;    EndIf
Next

;[6] graph origin.
x0 = #width/2
y0 = #height/2
xsum = 0
ysum = 0
If OpenWindow(#Window1, 0, 0, #width, #height, header$, #PB_Window_SystemMenu ) ;If 1
   If CreateImage(#Image1, #width, #height)                                          ;If 2
      ImageGadget(#ImgGadget, 0, 0, #width, #height, ImageID(#Image1))
      StartDrawing(ImageOutput(#Image1))
      Delay(2000)
      DrawingFont(FontID(1))    ;use the 'Courier' font
      c$ = "Riemann Zeta Function Vector Trace : s = " + a + " + " + b + "i"
      ;c$ = "Dirichlet Eta Function Vector Trace : s = " + a + " + " + b + "i"
      DrawText(200,200, c$, RGB(255, 255, 255))
      StopDrawing()
      ImageGadget(#ImgGadget, 0, 0, #width, #height, ImageID(#Image1))
      StartDrawing(ImageOutput(#Image1))
      Delay(1000)
      DrawingFont(FontID(1))
      c$ = "
"
      DrawText(150, 200, c$, RGB(0, 0, 0))    ;erase previous text.
      StopDrawing()
      ImageGadget(#ImgGadget, 0, 0, #width, #height, ImageID(#Image1))
      StartDrawing(ImageOutput(#Image1))
      ;axis.
      LineXY(0, y0, #width, y0, RGB(128,128,128))
      LineXY(x0, 0, x0, #height, RGB(128,128,128))
      x1 = Int(xsum/delta) + x0
      y1 = -Int(ysum/delta) + y0
      StopDrawing()

      ;[7]plot vectors.
      For i = 1 To m
         If Not(i>=startVector And i<=endVector)
            Gosub plotVector
         EndIf
      Next
      SetGadgetState(#ImgGadget, ImageID(#Image1))
      StartDrawing(ImageOutput(#Image1))
     EndIf    ;If 2
   Repeat
```

```
      Event = WaitWindowEvent()
   Until Event = #PB_Event_CloseWindow
EndIf ;If 1

;======================= plotVector Subroutine ===============================
plotVector:
          xsum = xsum + x(i)
          ysum = ysum + y(i)
          xx = Int(xsum/delta)
          yy = Int(ysum/delta)
          x2 = xx + x0
          y2 = -yy + y0
          SetGadgetState(#ImgGadget, ImageID(#Image1))
          StartDrawing(ImageOutput(#Image1))
          ;vector colors.
          If Mod(i, 3) = 1
             color = RGB(255, 255, 255) ;white.
          ElseIf Mod(i, 3) = 2
             color = RGB(0, 255, 255)
          Else
             color = RGB(255, 0, 255)
          EndIf
          LineXY(x1, y1, x2, y2, color)
          x3 = Int(xsum*100)/100
          y3 = Int(ysum*100)/100
          rr = Sqr(xsum*xsum + ysum*ysum)
          c$ = "n = " + Str(i+jump) + " : (x, y) = (" + xsum + ", " + ysum + "), r = " + rr + ", θ = " + deg(i) + "°, dθ = "
+ Str(deg(i)-deg(i-1)) + "°                    "
          DrawText(20, 20, c$)
          PrintN(c$)
          If i=1 ;mark 0 and 1
             DrawText(x1, y1, "0")
             DrawText(x2, y2, "1")
          EndIf
          If i>=2 And i<=10    ;mark first 10 points
             DrawText(x2, y2, Str(i))
          EndIf
          Delay(1)    ;plot speed.
          StopDrawing()
          x1 = x2
          y1 = y2
Return
```

Appendix B: Source Program for Parallel RZF or DEF Trace

;Code for parallel zeta or eta vector trace visualization using PureBasic evaluation version.

```
#Window1     = 0
#Image1      = 0
#ImgGadget = 0
#width             = 1370
#height            = 735

Define.d Dim x(100000, 9)
Define.d Dim y(100000, 9)
Define.d Dim t(100000, 9)
Define.d Dim deg(100000, 9)
Define.d Dim r(100000, 9)
Define.d Dim a2(9)
Define.s Dim header$(9)

Define.d a, b, r, t, delta
Define.d Dim x0(9), Dim y0(9), Dim x1(9), Dim y1(9), Dim xsum(9), Dim ysum(9), Dim x2(9), Dim y2(9), Dim
x3(9), Dim y3(9)
Define.d lnn, rr
Define.q i, m, n, Dim color(9)

b = 14.134725141734693790
;b = 21.022039638771554993
;b = 69.546401711
;b = 124.256818554
;b = 236.5242296658162058
;b = 570.051114782
;b = 572.419984132
;b = 1201.810334857
;b = 2210.850941099
;b = 3156.300357947
;b = 5565.566217327
;b = 7776.955377123
;b = 9457.289938949
;b = 10000.065345417
;b = 10000.651847322
;b = 10000.918178956
;b = 12571.195309379
;b = 15536.816303095
;b = 24499.249265478
;b = 33945.406726423
;b = 48596.896626512
;b = 53243.675588739
;b = 74908.108191005

maxj = 9

a2(1) = 0.3
a2(2) = 0.35
a2(3) = 0.4
```

```
a2(4) = 0.45
a2(5) = 0.5
a2(6) = 0.55
a2(7) = 0.6
a2(8) = 0.65
a2(9) = 0.7

color(1) = RGB(255, 0, 0)
color(2) = RGB(255, 127, 0)
color(3) = RGB(255, 255, 0)
color(4) = RGB(0, 255, 0)
color(5) = RGB(255, 255, 255)
color(6) = RGB(0, 0, 255)
color(7) = RGB(75, 0, 130)
color(8) = RGB(148, 0, 211)
color(9) = RGB(0, 255, 255)

delta = 0.015 ;image size zoom factor
d = 0 ;delay
markN = 1 ;1=mark, 0=do not mark n on the image

m = 5500 ;# terms

LoadFont (0, "Courier", 15)                    ; Load Courier Font, Size 15
LoadFont (1, "Arial", 24)                      ; Load Arial Font, Size 24

OpenConsole()

For j=1 To maxj
   header$(j) = "Riemann Zeta : s" + j + "=" + a2(j) + "+" + b + "i"
Next

pi.d = 3.1415926535

For n=1 To m Step 1
   For j=1 To maxj

      lnn = Log(n)
      r(n, j) = Exp(-a2(j)*lnn)
      t(n, j) = -b*lnn
      deg(n, j) = Mod(Round(Degree(t(n, j)), #PB_Round_Down), 360) ;#PB_Round_Up, #PB_Round_Nearest
      If deg(n, j)<0
         deg(n, j) = deg(n, j) + 360
      EndIf
      dg = deg(n, j) - deg(n-1, j)
      ;PrintN("n=" + n + " r(n, j)=" + r(n, j) + " θ" + j + "=" + deg(n, j) + "° dθ" + j + "=" + dg + "°" + " t(n, j)=" + t(n,
j))
   ;   If Mod(n, 2)=1 ;For eta function, remove comments of If...Else...EndIf block.
         x(n, j) = r(n, j)*Cos(t(n, j))
         y(n, j) = r(n, j)*Sin(t(n, j))
   ;   Else
   ;      x(n, j) = -r(n, j)*Cos(t(n, j))
   ;      y(n, j) = -r(n, j)*Sin(t(n, j))
```

```
   ;    EndIf

   Next
Next

;origin
For j=1 To maxj
   x0(j) = #width/2
   y0(j) = #height/2
   xsum(j) = 0
   ysum(j) = 0
Next

j = 1

If OpenWindow(#Window1, 0, 0, #width, #height, header$(1) + "/" + header$(2), #PB_Window_SystemMenu ) ;If
1
   If CreateImage(#Image1, #width, #height)                                                  ;If 2

      ImageGadget(#ImgGadget, 0, 0, #width, #height, ImageID(#Image1))
      StartDrawing(ImageOutput(#Image1))

      Delay(1000)

      DrawingFont(FontID(1)) ; Use the 'Courier' font

      c$ = "Riemann Zeta Function Vector Trace : s = " + a2(1) + " + " + b + "i"
      DrawText(200,200, c$, RGB(255, 255, 255))
      c$ = "Riemann Zeta Function Vector Trace : s = " + a2(maxj) + " + " + b + "i"
      DrawText(200,300, c$, RGB(255, 255, 255))

      StopDrawing()

      ImageGadget(#ImgGadget, 0, 0, #width, #height, ImageID(#Image1))
      StartDrawing(ImageOutput(#Image1))

      Delay(2000)

      DrawingFont(FontID(1))
      c$                                          =                                          "
"
      DrawText(150, 200, c$, RGB(0, 0, 0))
      DrawText(150, 300, c$, RGB(0, 0, 0))

      StopDrawing()

      ImageGadget(#ImgGadget, 0, 0, #width, #height, ImageID(#Image1))
      StartDrawing(ImageOutput(#Image1))


       LineXY(0, y0(j), #width, y0(j), RGB(128,128,128))
       LineXY(x0(j), 0, x0(j), #height, RGB(128,128,128))
```

```
        For j=1 To maxj
          x1(j) = Int(xsum(j)/delta) + x0(j)
          y1(j) = -Int(ysum(j)/delta) + y0(j)
        Next

        xo = 150
        yo = 150
        deg(0, 1) = 0
        deg(0, 2) = 0

        StopDrawing()

        lastMile = 0

        For i = 1 To m    ;For loop
          For j=1 To maxj
            Gosub plotVector
          Next


          If Mod(i, 100)=0
            For j=2 To maxj+0
              Delay(1)
              SetGadgetState(#ImgGadget, ImageID(#Image1))
              StartDrawing(ImageOutput(#Image1))
              LineXY(x1(j-1), y1(j-1), x1(j), y1(j), color(5))
              StopDrawing()
            Next
          EndIf

        Next ;For loop

      EndIf ;If 2

    Repeat
      Event = WaitWindowEvent()
    Until Event = #PB_Event_CloseWindow

EndIf ;If 1

;======================= plotVector Subroutine ==============================

plotVector:

  Delay(d)

          xsum(j) = xsum(j) + x(i, j)
          ysum(j) = ysum(j) + y(i, j)
          xx = Int(xsum(j)/delta)
          yy = Int(ysum(j)/delta)
          x2(j) = xx + x0(j)
          y2(j) = -yy + y0(j)
```

```
        SetGadgetState(#ImgGadget, ImageID(#Image1))
        StartDrawing(ImageOutput(#Image1))

        LineXY(x1(j), y1(j), x2(j), y2(j), color(j))

        x3(j) = Int(xsum(j)*100)/100
        y3(j) = Int(ysum(j)*100)/100

        rr = Sqr(xsum(j)*xsum(j) + ysum(j)*ysum(j))

        c$ = "n = " + Str(i+jump) + " : (x, y) = (" + xsum(j) + ", " + ysum(j) + "), r = " + rr + ", θ = " + deg(i, j) + "°,
dθ = " + Str(deg(i, j)-deg(i-1, j)) + "°                    "
        DrawText(20, 20, c$)
        ;PrintN(c$)

        If i=1 ;mark 0 and 1
          DrawText(x1(j), y1(j), "0")
          DrawText(x2(j), y2(j), "1")
        EndIf

        StopDrawing()

        x1(j) = x2(j)
        y1(j) = y2(j)
Return
```

Appendix C: Vector Trace Videos

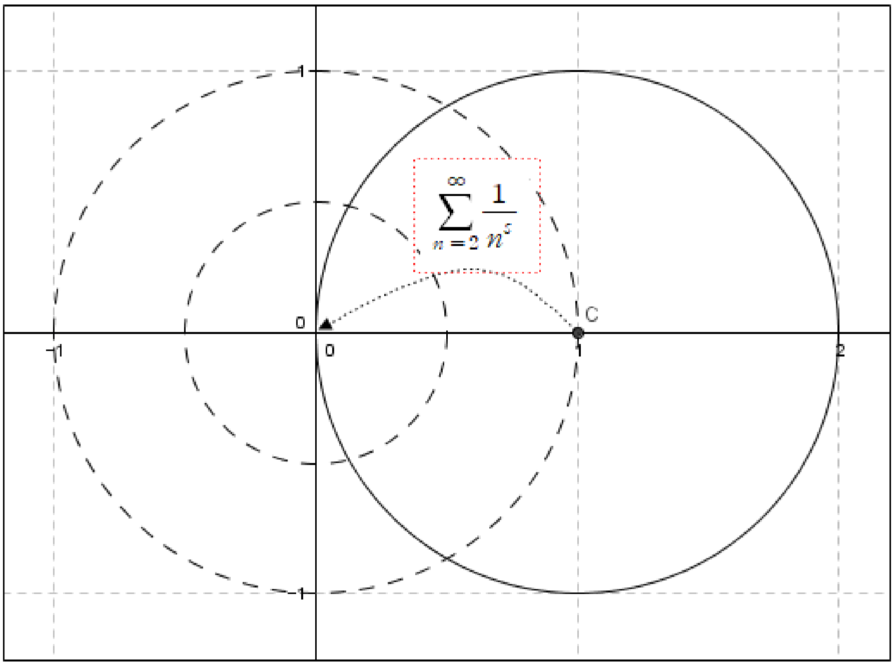
seq	type	s	link
1	the DEF	0.4 + 10000.065i	https://www.youtube.com/watch?v=cIZlImNScll
2		0.6 + 10000.065i	https://www.youtube.com/watch?v=CHjCcqthuTc
3		0.5 + 74908.108i	https://www.youtube.com/watch?v=dE2fnWLzqxw
4		0.5 + 10000.065i	https://www.youtube.com/watch?v=3c2riWMV78I
5		0.5 + 14.135i	https://www.youtube.com/watch?v=5XPmdAfBphw
6	RZF	0.4 + 10000.065i	https://www.youtube.com/watch?v=54FGRm4mb_c
7		0.6 + 10000.065i	https://www.youtube.com/watch?v=pOeANPrMIRI
8		0.5 + 74908.108i	https://www.youtube.com/watch?v=W09mzoCTHEI

Appendix D: Other Possible Proofs

D.1. Possible Proof 1: By Lattice Hitting

the RZF can be rewritten as $\zeta(s) = 1 + \sum_{n=2}^{\infty} 1/n^s$ and we can consider zero of ζ(s) as where $\sum_{n=2}^{\infty} 1/n^s$ hit the origin starting from (1, 0), as in the following figure.

Relativistic View of Zero of ζ(s)



Considering that the origin is also a lattice point, only some circle with radius $\sqrt{n}$ or $1/\sqrt{n}$ can hit the origin. To keep the radius to be of $1/\sqrt{n}$ pattern, α should be 1/2.

D.2. Possible Proof 2: By x-Axis Property

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$= A(\alpha)e^{iB(\beta)}$$

$$= A(\alpha)\{cosB(\beta) + isinB(\beta)\}$$

$$= u + iv$$

Eventually, the RZF falls into just the two sine and cosine functions, but with a variable amplitude $A(\alpha)$ and a variable argument $B(\beta)$. So, the zeros of the RZF must be on the x-axis, because the zeros must satisfy $cosB(\beta) = 0, sinB(\beta) = 0$, simultaneously.

The x-axis on the complex plane is just the critical line, where zeros are found. That is to say, zeros of sinusoidal functions are found only on *x*-axis, so, the crtical line should be the x-axis.

D.3. Possible Proof 3: By Cauchy Integral Theorem

Suppose that the trajectory *C* of Lemma 4.8 and Lemma 4.9 does not contain any zeros of $\zeta(s)$ and $\zeta'(s)$ and let's represent *C* as two parameterized closed curve $C = \zeta(t, \beta_0), \alpha \leq t \leq 1-\alpha, \zeta(\alpha + i\beta_0) = \zeta(1-\alpha+i\beta_0) = 0$. Then $\frac{d\zeta(s)}{dt}$ can't be zero in C. So, the reciprocal function of $\frac{d\zeta(s)}{dt}$, $\frac{dt}{d\zeta(s)}$ can't have any poles in C. That is to say, $\frac{dt}{d\zeta(s)}$ is analytic in C.

Let's apply the Cauchy Integral Theorem to $C = \zeta(t, \beta)$, where $\frac{dt}{d\zeta(s)}$ has no poles.

$$\oint_C h(\zeta(t,\beta_0))d\zeta(t,\beta_0) = 0$$

$$d\zeta(t,\beta_0) = \frac{d\zeta(t,\beta_0)}{dt}dt, \alpha \leq t \leq 1-\alpha$$

Let $h(\zeta(t,\beta_0)) = \frac{dt}{d\zeta(t,\beta_0)}$, which is analytic in C.

$$\oint_C h(\zeta(t,\beta_0))d\zeta(t,\beta_0) =$$

$$\int_\alpha^{1-\alpha} \frac{dt}{d\zeta(t,\beta_0)}\frac{d\zeta(t,\beta_0)}{dt}dt =$$

$$\int_\alpha^{1-\alpha} dt =$$

$$[t]\ _\alpha^{1-\alpha} =$$

$$1 - 2\alpha = 0$$

$$\alpha = 0.5.$$