A Different Way to Prove a Prime Number between 2N and 3N

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Abstract

In this paper we will use a different way to prove that there exists at least a prime number p in between $2n$ and $3n$ where n is a positive integer. The proof extends the Bertrand's postulate / Chebyshev's theorem which states that a prime number exists between n and $2n$. The method to prove this proposition is to analyze the binomial coefficient, a similar method used by Erdős in the proof of Bertrand's postulate.

Introduction

The Bertrand's postulate / Chebyshev's theorem states that for any positive integer n, there is always a prime number p such that $n < p \le 2n$. It was proved in 1850 [1]. In 1932, Paul Erdős [2] used a much simpler method to prove the theorem by carefully analyzing the central binomial coefficient $\binom{2n}{n}$. In 2006, M. El Bachraoui [3] extended the theorem by proving that for any positive integer n, there is a prime number p such that $2n < p \le 3n$. In this paper, the author will use a different method to prove the same extension by analyzing the binomial coefficient $\binom{3n}{n}$. First, we will define and clarify some terms and concepts. Then we will propose the subject of the thesis.

Definition: $\Gamma_{a\ge p>b}\{n\}$ denotes the prime number decomposition operator. It is the product of the prime numbers in the decomposition of a positive integer n or a positive integer expression. In this operator, p is a prime number, a and b are real numbers, and $n \ge a \ge p > b \ge 1$.
It has some properties: By definition, it is always true that $\Gamma_{a\ge p>b}\{n\} \ge 1$ **— (1)**

If no prime number in $\Gamma_{a\ge p>b}\{n\}$, then $\Gamma_{a\ge p>b}\{n\} = 1$, or vice versa, if $\Gamma_{a\ge p>b}\{n\} = 1$, then no prime number in $\Gamma_{a\ge p>b}\{n\}$ as in $\Gamma_{12\ge p>4}\{12\} = 11^0 \cdot 7^0 \cdot 5^0 = 1$. **— (2)**

If there is at least one prime number in $\Gamma_{a\ge p>b}\{n\}$, then $\Gamma_{a\ge p>b}\{n\} > 1$, or vice versa, if $\Gamma_{a\ge p>b}\{n\} > 1$, then there is at least one prime number in $\Gamma_{a\ge p>b}\{n\}$ as in $\Gamma_{4\ge p>2}\{12\} = 3 > 1$. **— (3)**

Similar to Paul Erdős' paper [2], we define *R(p)* by the inequalities $p^{R(p)} \le 3n < p^{R(p)+1}$, and determine the $p\text{-}adic\ valuation$ of $\binom{3n}{n}$.

$v_p\left(\binom{3n}{n}\right) = v_p((3n)!) - v_p((2n)!) - v_p(n!) = \sum_{i=1}^{R(p)} \left(\left\lfloor\frac{3n}{p^i}\right\rfloor - \left\lfloor\frac{2n}{p^i}\right\rfloor - \left\lfloor\frac{n}{p^i}\right\rfloor\right) \le R(p)$

because for any real numbers a and b, the expression of $\lfloor a + b\rfloor - \lfloor a\rfloor - \lfloor b\rfloor$ is 0 or 1.

Thus, if p divides $\binom{3n}{n}$, then $v_p\left(\binom{3n}{n}\right) \le R(p) \le \log_p(3n)$, or $p^{v_p\left(\binom{3n}{n}\right)} \le p^{R(p)} \le 3n$ **— (4)**

And if $3n \ge p > \lfloor\sqrt{3n}\rfloor$, then $0 \le v_p\left(\binom{3n}{n}\right) \le R(p) \le 1$. **— (5)**

From the prime number decomposition,

$\binom{3n}{n} = \frac{(3n)!}{n!\cdot(2n)!} = \Gamma_{3n\ge p>n}\{\frac{(3n)!}{n!\cdot(2n)!}\} \cdot \Gamma_{n\ge p>\lfloor\sqrt{3n}\rfloor}\{\frac{(3n)!}{n!\cdot(2n)!}\} \cdot \Gamma_{\lfloor\sqrt{3n}\rfloor\ge p}\{\frac{(3n)!}{n!\cdot(2n)!}\}$.

Since all prime numbers in $n!$ are not in the range of $3n \ge p > n$,

$\Gamma_{3n\ge p>n}\{\frac{(3n)!}{n!\cdot(2n)!}\} = \Gamma_{3n\ge p>n}\{\frac{(3n)!}{(2n)!}\}$.

Referring to **(5)**, $\Gamma_{n\ge p>\lfloor\sqrt{3n}\rfloor}\{\frac{(3n)!}{n!\cdot(2n)!}\} \le \prod_{n\ge p} p$.

It has been proved [4] that $\prod_{n\ge p} p < 2^{2n-3}$ when $n \ge 3$.

Thus for $n \ge 3$, $\binom{3n}{n} < \Gamma_{3n\ge p>n}\{\frac{(3n)!}{(2n)!}\} \cdot 2^{2n-3} \cdot \Gamma_{\lfloor\sqrt{3n}\rfloor\ge p}\{\frac{(3n)!}{n!\cdot(2n)!}\}$ **— (6)**

Proposition

For every positive integer n, there exists at least a prime number p such that $2n < p \le 3n$.

Proof:

By induction on n, for $n = 3$, $\frac{3^{3n-2}}{n(2^{2n-2})} = \frac{3^6}{2^4} = 45\frac{9}{16} < \binom{3n}{n} = \binom{9}{3} = 84$.

If $\binom{3n}{n} > \frac{3^{3n-2}}{n(2^{2n-2})}$ for n stands, then for $n+1$,

$\binom{3(n+1)}{n+1} = \frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1)} \cdot \binom{3n}{n} > \frac{3(3n+2)(3n+1)}{(2n+2)(2n+1)} \cdot \frac{3^{3n-2}}{n(2^{2n-2})} > \frac{3^{3(n+1)-2}}{(n+1)(2^{2(n+1)-2})}$

because $\frac{3(3n+2)(3n+1)}{(2n+2)(2n+1)} \cdot \frac{3^{3n-2}}{n(2^{2n-2})} = 3 \cdot \frac{3n+2}{2n+1} \cdot \frac{3n+1}{2n} \cdot \frac{3^{3n-2}}{(n+1)(2^{2n-2})} > 3^3 \cdot \frac{3^{3n-2}}{(n+1)(2^{2n-2})}$

Thus for $n \ge 3$, $\binom{3n}{n} > \frac{3^{3n-2}}{n(2^{2n-2})}$ **— (7)**

Applying **(7)** into **(6)**:

For $n \ge 3$, $\frac{3^{3n-2}}{n(2^{2n-2})} < \Gamma_{3n\ge p>n}\{\frac{(3n)!}{(2n)!}\} \cdot 2^{2n-3} \cdot \Gamma_{\lfloor\sqrt{3n}\rfloor\ge p}\{\frac{(3n)!}{n!\cdot(2n)!}\}$ **— (8)**

Let $\pi(x)$ be the number of prime numbers less than or equal to x, where x is a positive real number. For the first six sequential natural numbers, there are three prime numbers 2, 3, and 5. For adding any successive set of six sequential natural numbers, there are at most two prime numbers added, $p \equiv 1$ (MOD 6) and $p \equiv 5$ (MOD 6). Thus, $\pi(x) \le \left\lfloor \frac{x}{3} \right\rfloor + 2 \le \frac{x}{3} + 2$. **— (9)**

Referring to **(4)** and **(9)**,

$$\Gamma_{\lfloor\sqrt{3n}\rfloor \ge p}\left\{\frac{(3n)!}{n!\cdot(2n)!}\right\} = \Gamma_{\lfloor\sqrt{3n}\rfloor \ge p}\left\{\binom{3n}{n}\right\} \le (3n)^{\pi(\sqrt{3n})} \le (3n)^{\frac{\sqrt{3n}}{3}+2} \qquad \textbf{— (10)}$$

Applying **(10)** into **(8)**: $\dfrac{3^{3n-2}}{n(2^{2n-2})} < \Gamma_{3n \ge p > n}\left\{\dfrac{(3n)!}{(2n)!}\right\} \cdot 2^{2n-3} \cdot (3n)^{\frac{\sqrt{3n}}{3}+2}$

Since both $2^{2n-3} > 0$ and $(3n)^{\frac{\sqrt{3n}}{3}+2} > 0$ for $n \ge 3$,

$$\Gamma_{3n \ge p > n}\left\{\frac{(3n)!}{(2n)!}\right\} > \frac{3^{3n-2}}{n(2^{2n-2})(2^{2n-3})(3n)^{\frac{\sqrt{3n}}{3}+2}} = \frac{32\cdot\left(\frac{27}{16}\right)^n}{3\cdot(3n)^{\frac{\sqrt{3n}}{3}+3}} = \frac{32}{3}\cdot\frac{\left(\frac{27}{16}\right)^n}{(3n)^{\frac{\sqrt{3n}+9}{3}}} \qquad \textbf{— (11)}$$

Let $f(x) = \dfrac{u}{w}$ where x, u, w are real numbers and $x \ge 84$, $u = \dfrac{32}{3}\cdot\left(\dfrac{27}{16}\right)^x$, $w = (3x)^{\frac{\sqrt{3x}+9}{3}}$

$$\frac{du}{dx} = \left(\frac{32}{3}\cdot\left(\frac{27}{16}\right)^x\right)' = \frac{32}{3}\cdot\left(\frac{27}{16}\right)^x \cdot ln\left(\frac{27}{16}\right) = u \cdot ln\left(\frac{27}{16}\right)$$

$$\frac{dw}{dx} = \left((3x)^{\frac{\sqrt{3x}+9}{3}}\right)' = \left((3x)^{\frac{\sqrt{3x}+9}{3}}\right)\left(\frac{ln(3x)}{2\sqrt{3x}} + \frac{\sqrt{3x}+9}{3x}\right) = w\left(\frac{ln(3x)+2}{2\sqrt{3x}} + \frac{3}{x}\right)$$

$$f'(x) = \left(\frac{u}{w}\right)' = \frac{w(u)' - u(w)'}{w^2} = \frac{u}{w}\left(ln\left(\frac{27}{16}\right) - \frac{ln(3x)+2}{2\sqrt{3x}} - \frac{3}{x}\right)$$

Let $f_1(x) = ln\left(\dfrac{27}{16}\right) - \dfrac{ln(3x)+2}{2\sqrt{3x}} - \dfrac{3}{x}$

Since $f_1{}'(x) = \dfrac{ln(3x)}{4x\sqrt{3x}} + \dfrac{3}{x^2} > 0$, when $x > 1$, $f_1(x)$ is a strictly increasing function.

When $x = 84$, $f_1(x) = ln\left(\dfrac{27}{16}\right) - \dfrac{ln(3x)+2}{2\sqrt{3x}} - \dfrac{3}{x} \approx 0.523 - 0.237 - 0.012 = 0.274 > 0$.

Thus, when $x \ge 84$, $f_1(x) > 0$.

Since when $x \ge 84$, u, w, and $f_1(x)$ are greater than zero, $f'(x) = \dfrac{u}{w}\cdot f_1(x) > 0$.

Thus $f(x)$ is a strictly increasing function for $x \ge 84$. Then when $x \ge 84$, $f(x+1) > f(x)$.

Let $n = \lfloor x \rfloor \ge 84$, then $f(n+1) > f(n) = \dfrac{32}{3}\cdot\dfrac{\left(\frac{27}{16}\right)^n}{(3n)^{\frac{\sqrt{3n}+9}{3}}}$

Since for $n = 84$, $f(n) = \dfrac{32}{3}\cdot\dfrac{\left(\frac{27}{16}\right)^n}{(3n)^{\frac{\sqrt{3n}+9}{3}}} = \dfrac{32}{3}\cdot\dfrac{\left(\frac{27}{16}\right)^{84}}{(252)^{\frac{\sqrt{252}+9}{3}}} \approx \dfrac{1.307E+20}{8.151E+19} > 1$, and since

$f(n+1) > f(n)$, by induction on n, when $n \geq 84$, $f(n) = \frac{32}{3} \cdot \frac{\left(\frac{27}{16}\right)^n}{(3n)^{\frac{\sqrt{3n}+9}{3}}} > 1$. **— (12)**

Applying **(12)** to **(11)**: When $n \geq 84$, $\Gamma_{3n \geq p > n}\{\frac{(3n)!}{(2n)!}\} > \frac{32}{3} \cdot \frac{\left(\frac{27}{16}\right)^n}{(3n)^{\frac{\sqrt{3n}+9}{3}}} > 1$.

Thus when $n \geq 84$,

$\Gamma_{3n \geq p > n}\{\frac{(3n)!}{(2n)!}\} = \Gamma_{3n \geq p > 2n}\{\frac{(3n)!}{(2n)!}\} \cdot \Gamma_{2n \geq p > \frac{3n}{2}}\{\frac{(3n)!}{(2n)!}\} \cdot \Gamma_{\frac{3n}{2} \geq p > n}\{\frac{(3n)!}{(2n)!}\} > 1$ **— (13)**

When $2n \geq p > \frac{3n}{2}$ in $\frac{(3n)!}{(2n)!}$, if $v_p((3n)!)$ has one factor of p then $v_p((2n)!)$ also has one factor of p. Thus, $v_p\left(\frac{(3n)!}{(2n)!}\right) = v_p((3n)!) - v_p((2n)!) = 1 - 1 = 0$.

Since p^0=1, referring to **(2)**, $\Gamma_{2n \geq p > \frac{3n}{2}}\{\frac{(3n)!}{(2n)!}\} = 1$

Thus, when $n \geq 84$, $\Gamma_{3n \geq p > n}\{\frac{(3n)!}{(2n)!}\} = \Gamma_{3n \geq p > 2n}\{\frac{(3n)!}{(2n)!}\} \cdot \Gamma_{\frac{3n}{2} \geq p > n}\{\frac{(3n)!}{(2n)!}\} > 1$ **— (14)**

Referring to **(1)**, $\Gamma_{3n \geq p > 2n}\{\frac{(3n)!}{(2n)!}\} \geq 1$ and $\Gamma_{\frac{3n}{2} \geq p > n}\{\frac{(3n)!}{(2n)!}\} \geq 1$, from **(14)**, at least one of these two factors is greater than one when $n \geq 84$.

If $n \geq 84$ and $\Gamma_{3n \geq p > 2n}\{\frac{(3n)!}{(2n)!}\} > 1$, since $\frac{(3n)!}{(2n)!}$ is a positive integer expression, then referring to **(3)**, there exists at least a prime number p such that $2n < p \leq 3n$. **— (15)**

$\Gamma_{\frac{3n}{2} \geq p > n}\{\frac{(3n)!}{(2n)!}\} = \Gamma_{3 \cdot (\frac{n}{2}) \geq p > 2 \cdot (\frac{n}{2})}\{\frac{(3n)!}{(2n)!}\}$.

If $\frac{n}{2} \geq 42$ and $\Gamma_{3 \cdot (\frac{n}{2}) \geq p > 2 \cdot (\frac{n}{2})}\{\frac{(3n)!}{(2n)!}\} = 1$, then from **(14)**, the factor $\Gamma_{3n \geq p > 2n}\{\frac{(3n)!}{(2n)!}\} > 1$.

Referring to **(3)**, there exists at least a prime number p such that $2n < p \leq 3n$. **— (16)**

If $\frac{n}{2} \geq 42$ and $\Gamma_{3 \cdot (\frac{n}{2}) \geq p > 2 \cdot (\frac{n}{2})}\{\frac{(3n)!}{(2n)!}\} > 1$, let $m = \frac{n}{2}$, then when $m \geq 42$, there exists at least a prime number p such that $2m < p \leq 3m$. Since $n \geq 84 \geq m \geq 42$, the statement is also valid for n.

Thus, when $n \geq 84$, there exists at least a prime number p such that $2n < p \leq 3n$. **— (17)**

From **(16)** and **(17)**, no matter $\Gamma_{\frac{3n}{2} \geq p > n}\{\frac{(3n)!}{(2n)!}\}$ is equal to 1 or greater than 1, there exists at least a prime number p such that $2n < p \leq 3n$ when $n \geq 84$. **— (18)**

Table 1 shows that when $1 \leq n \leq 84$, there is a prime number p such that $2n < p \leq 3n$. **— (19)**

Thus, the proposition is proven by combining **(15)**, **(18)**, and **(19)**: For every positive integer n, there exists at least a prime number p such that $2n < p \leq 3n$.

Table 1: For $1 \leq n \leq 84$, there is a prime number p such that $2n < p \leq 3n$.

$2n$	2	4	6	8	10	12	14	16	18	20	22	24	26	28
p	3	5	7	11	13	17	17	19	23	29	29	31	31	37
$3n$	3	6	9	12	15	18	21	24	27	30	33	36	39	42
$2n$	30	32	34	36	38	40	42	44	46	48	50	52	54	56
p	37	41	41	43	43	47	47	53	53	59	59	61	61	67
$3n$	45	48	51	54	57	60	63	66	69	72	75	78	81	84
$2n$	58	60	62	64	66	68	70	72	74	76	78	80	82	84
p	67	71	71	73	73	79	79	83	83	89	89	97	97	101
$3n$	87	90	93	96	99	102	105	108	111	114	117	120	123	126
$2n$	86	88	90	92	94	96	98	100	102	104	106	108	110	112
p	101	103	103	107	107	109	109	113	113	127	127	131	131	137
$3n$	129	132	135	138	141	144	147	150	153	156	159	162	165	168
$2n$	114	116	118	120	122	124	126	128	130	132	134	136	138	140
p	137	139	139	149	149	151	151	157	157	163	163	167	167	173
$3n$	171	174	177	180	183	186	189	192	195	198	201	204	207	210
$2n$	142	144	146	148	150	152	154	156	158	160	162	164	166	168
p	173	179	179	181	181	191	191	193	193	197	197	199	199	211
$3n$	213	216	219	222	225	228	231	234	237	240	243	246	249	252

References

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