Assessing the black hole solution

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Abstract

The generally accepted solution in *GR* that predicts black holes and an event horizon is shown here to be physically unsound. Recognising that Newton's inverse-square law of gravitation is based strictly on Euclidean geometry, enables one to determine exactly how time is curved in the geometrical representation of *GR*. Using Einstein's field equations for the field outside a point mass can then be used to prove that the spacetime is completely regular, i.e. there is no event horizon or black hole, and velocities do not exceed the speed of light.

1 Gravitational field due to a point mass

A solution for the gravitational field due to a point mass was first obtained using Albert Einstein's theory of general relativity (*GR*) [1] in 1916 by Karl Schwarzschild [2]. A year later Droste [3] and Weyl [4] independently obtained a further variant of the solution. Subsequently, Hilbert [5] extended Droste and Weyl's solution in such a way that it showed a discontinuity in spacetime which is now interpreted as an event horizon obscuring the central mass, which is called a black hole.

The procedure for obtaining the solution is summarised, as follows. A metric line element in a curved 4D spacetime (t, r, θ, ϕ) with spherical spatial symmetry about the origin can be written as

$$d\tilde{s}^2 = c^2 dt'^2 = A(r)\, c^2 dt^2 - B(r)\, dr^2 - C(r)\, d\Omega^2 \tag{1}$$

where $d\tilde{s}$ is a spacetime increment, c the speed of light, dt' an increment of proper time t', dt an increment of coordinate time t, dr an increment of radial coordinate distance r, and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$; A, B and C are radially dependent functions describing the curvature of the time, radial and angular metric coefficients, respectively. Lagrangian formalism is then used to obtain the geodesic equations in t, r, θ, ϕ, respectively:

$$\ddot{t} + \frac{A'}{A}\,\dot{t}\dot{r} = 0$$

$$\ddot{r} + c^2\frac{A'}{2B}\dot{t}^2 + \frac{B'}{2B}\dot{r}^2 - \frac{C'}{2B}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0 \qquad ...(ii)$$

$$\ddot{\theta} + \frac{C'}{C}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0 \qquad ...(iii)$$

$$\ddot{\phi} + \frac{C'}{C}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0 \qquad ...(iv) \qquad (2)$$

where A', B', C' are the first derivatives of A, B, C respectively, with respect to the variable r. The non-zero Christoffel symbols are

$$\Gamma^t_{tr} = \Gamma^t_{rt} = A'/2A;\ \Gamma^r_{tt} = c^2A'/2B;$$

$$\Gamma^r_{rr} = B'/2B;\ \Gamma^r_{\theta\theta} = -C'/2B;\ \Gamma^r_{\phi\phi} = -C'\sin^2\theta/2B$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = C'/2C;\ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = C'/2C;\ \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$$

The Ricci tensor components for the curvature, defined by

$$R_{ab} = \frac{\partial\Gamma^c_{ab}}{\partial x_c} - \frac{\partial\Gamma^c_{ac}}{\partial x_b} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{bd}\Gamma^d_{ac} \qquad (3)$$

are then given by:

$$R_{tt} = c^2\left(\frac{A''}{2B} - \frac{A'B'}{4B^2} - \frac{A'^2}{4AB} + \frac{A'C'}{2BC}\right) \qquad [g_{tt} = c^2A]$$

$$R_{rr} = -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'C'}{2BC} - \frac{C''}{C} + \frac{C'^2}{2C^2} \qquad [g_{rr} = -B]$$

$$R_{\theta\theta} = 1 - \frac{C''}{2B} - \frac{A'C'}{4AB} + \frac{B'C'}{4B^2} \qquad [g_{\theta\theta} = -C] \qquad (4)$$

plus a similar equation to the last one for $R_{\phi\phi}$.

The required vacuum solution is then found by setting all the Ricci tensor components equal to zero ($R_{ab} = 0$), and after some manipulation of Equations 4 one obtains the following pair of simultaneous equations relating A, B and C:

$$\frac{A'}{A} + \frac{B'}{B} = \frac{2C''}{C'} - \frac{C'}{C}\ ;\ \frac{A'}{A} - \frac{B'}{B} = -\frac{2C''}{C'} + \frac{4B}{C'} \qquad (5)$$

This provides two independent equations for the three variables, A, B, and C, which means they cannot be solved explicitly, without some additional condition or assumption.

2 The solution set

To proceed, it is customary to define a new radial coordinate $\tilde{r}$ in terms of C, where

$$\tilde{r} = \sqrt{C(r)} \tag{6}$$

and then the metric in Equation 1 becomes

$$d\tilde{s}^2 = c^2 A(\tilde{r})\, dt^2 - B(\tilde{r})\, d\tilde{r}^2 - \tilde{r}^2 d\Omega^2 \tag{7}$$

With this coordinate transformation, Equations 5 can then be solved in terms of $(t, \tilde{r}, \theta, \phi)$, often called Schwarzschild coordinates, resulting in

$$d\tilde{s}^2 = c^2 \left(1 - \frac{\alpha}{\tilde{r}}\right) dt^2 - \left(1 - \frac{\alpha}{\tilde{r}}\right)^{-1} d\tilde{r}^2 - \tilde{r}^2 d\Omega^2 \tag{8}$$

i.e.

$$A(\tilde{r}) = \frac{1}{B(\tilde{r})} = 1 - \frac{\alpha}{\tilde{r}} \qquad ; \qquad C = \tilde{r}^2 \tag{9}$$

α is a positive, real constant of integration that is found by comparing the solution with known physics (to be discussed later). In addition, since $A(\tilde{r})B(\tilde{r}) = 1$, we have

$$A(r)B(r) = d^2\tilde{r}/dr^2 \tag{10}$$

The exact relationship between $\tilde{r}$ and r is not specified or deducible from the spherical geometry alone, so the above procedure merely expresses A and B in terms of C (or the coordinate $\tilde{r}$). Many teachers of relativity ignore any difference between $\tilde{r}$ and r, and regard $\tilde{r}$ as the "true" radial coordinate distance r from the mass M.[1] This opens the door to the idea of a black hole, since the function $(1 - \alpha/r)$ changes sign when r goes from above to below α.

In his original solution, Schwarzschild [2] realised that $(1 - \alpha/\tilde{r})$ becomes negative if $\tilde{r} < \alpha$. However, he regarded this as non-physical, and to prevent it happening defined the auxiliary radial coordinate $\tilde{r}$ by the function

$$\tilde{r} = (r^3 + \alpha^3)^{1/3}$$

[1] By "true" is meant here the distance according to an observer in a flat frame of reference a very long way from the mass causing gravitation, or in a flat frame that would have existed if the effect of the gravitational mass M were somehow removed.

so that as $r \to 0$, $\tilde{r} \to \alpha$, and the discontinuity is forced to occur at the origin. Thus, Schwarzschild himself rejected the idea of A or B changing sign or becoming infinite in free space, and he did not predict black holes, even though current popular science suggests he did.

Soon afterwards, both Droste and Weyl published a different variant, essentially in which $\tilde{r} = r$, but they avoided the discontinuity by limiting the range of $\tilde{r}$ to $\alpha < \tilde{r} < \infty$. Subsequently, Hilbert extended Droste and Weyl's solution to the range $\tilde{r} < \alpha$, i.e. into the region where the functions A and B become negative, and then spacetime is discontinuous. In modern language this is called an event horizon, behind which a black hole is obscured. Hilbert has a good argument for extending the solution to $\tilde{r} \to 0$, even if $\tilde{r}$ is not equal to r, on the grounds that GR is intended to be a generally covariant theory, meaning *inter alia* that a change of coordinates should not alter the physics of the situation.

Although this extension by Hilbert is currently the generally accepted solution, several mathematicians have questioned it. According to Abrams [6] and Crothers [7], the complete set of possible solutions for $\tilde{r}$ as a function of r is given by

$$\tilde{r} = (|r - r_0|^n + \alpha^n)^{1/n} \tag{11}$$

where n is a positive integer, r_0 an arbitrary constant, and α a real positive constant. Schwarzschild's original solution is represented here by $n = 3$ and $r_0 = 0$. Droste and Weyl's solution has $n = 1, r_0 = \alpha, r > r_0$, and in a paper by Brillouin [8], there is a solution with $n = 1, r_0 = 0$. Crothers [7] points out that Hilbert's extension of Droste and Weyl's solution does not belong to this set, since the modulus of $(r - r_0) \geq 0$, and therefore $\tilde{r} \geq \alpha$. The function $(1 - \alpha/\tilde{r})$ then cannot become negative, and so black holes are not actually predicted mathematically. However, this objection has not been accepted by the scientific community to date, and so Hilbert's extension to $\tilde{r} < \alpha$ is still regarded as correct.

3 Relating GR to the physical world

The quantity α relates the geometrical scale of GR to gravitational forces in the real world. To obtain this connection, one considers the equation of motion of a test object free-falling from rest at infinity along a radial path towards the central mass at the coordinate origin. The required equation is found from the radial geodesic (Equation 2) by inserting $\dot{\theta} = \dot{\phi} = 0$, viz.

$$\ddot{r} + \frac{A'}{2B} c^2 \dot{t}^2 + \frac{B'}{2B} \dot{r}^2 = 0 \tag{12}$$

Using the metric to eliminate $\dot{t}$, this becomes

$$\ddot{r} + \frac{A'}{2AB} c^2 + \left(\frac{A'}{2A} + \frac{B'}{2B} \right) \dot{r}^2 = 0 \tag{13}$$

This expression in terms of the curved geometry of spacetime is then compared with Newton's classical law of gravitation:

$$a + \frac{GM}{r^2} = 0 \tag{14}$$

where a is the acceleration, G Newton's gravitational constant and M the mass causing gravity. Superficially there is no obvious agreement, but if one temporarily ignores any difference between $\tilde{r}$ and r, it follows from Equation 10 that $A = 1/B$, and then Equation 13 would become

$$\ddot{r} + \frac{1}{2} c^2 A' = 0 \tag{15}$$

Equating the proper acceleration $\ddot{r} = d^2r/dt'^2$ to the Newtonian acceleration $a = v\, dv/dr$, would then give

$$\frac{1}{2} c^2 A' = \frac{GM}{r^2} \tag{16}$$

and by integrating this equation one obtains

$$A = 1 - \frac{2GM/c^2}{r} \tag{17}$$

and from Equation 9 we find

$$\alpha = \frac{2GM}{c^2} \tag{18}$$

However, the above step that leads to the black-hole prediction in Equation 17 is fundamentally incorrect. It is necessary to realise that the radial equation of motion (Equation 13) describes gravity as a combination of time curvature (A, A') and radial space curvature (B, B'), whereas the geometry of Newton's inverse square law of gravitation is strictly Euclidean or spatially flat, i.e. in Newton's law there is by definition no space curvature.

Thus, in order to compare Equation 13 rigorously with Newton's law, we must set $B = 1$, not $B = 1/A$. Instead of Equation 15 the following equation is then obtained:

$$\ddot{r} + \frac{1}{2}\left(c^2 + \dot{r}^2\right)\frac{A'}{A} = 0 \qquad [Newton; B = 1] \tag{19}$$

which differs slightly, but crucially, from Equation 15. Inserting Newtonian expressions for acceleration and velocity, where $\ddot{r} <=> a = -GM/r^2$ and $\dot{r}^2 <=> v^2 = 2GM/r$, then gives

$$-\frac{GM}{r^2} + \frac{1}{2}\left(c^2 + \frac{2GM}{r}\right)\frac{A'}{A} = 0 \tag{20}$$

After some rearrangement we obtain

$$\frac{A'}{A} = \frac{\alpha/r}{(\alpha + r)} \tag{21}$$

with $\alpha = 2GM/c^2$. Integrating this, finally delivers the correct expression for $A(r)$, viz,

$$A = \left(1 + \frac{\alpha}{r}\right)^{-1} = \frac{r}{r + \alpha} \tag{22}$$

Newton's inverse-square law of gravity can thus be thought of as describing that aspect of gravity caused exclusively by the curvature of the time coordinate - which is dominant for most cases we consider, such as planetary motion. However, space curvature becomes significant when speeds approach the speed of light, and distances to the central mass become small, and this will modify gravity from being purely Newtonian.

There is no reason to believe that the time curvature (determined by Newton's law) deviates from Equation 22 even in strong fields, and so in order to obtain the space curvature (not described by Newton's law at all) we may now make use of the GR solution, giving

$$A = \frac{1}{B} = 1 - \frac{\alpha}{\tilde{r}} = \left(1 + \frac{\alpha}{r}\right)^{-1} \tag{23}$$

which delivers the following relationship between $\tilde{r}$ and r:

$$\tilde{r} = r + \alpha \tag{24}$$

and $C = \tilde{r}^2 = (r + \alpha)^2$. We see that it does turn out that $A = 1/B$, since $\tilde{r}$ and r are linearly related (see Equation 10), and so the proper velocity of free-fall may be written:

$$\frac{\dot{r}^2}{c^2} = 1 - A = \frac{\alpha}{r + \alpha} \tag{25}$$

whereas the (incorrect) conventional analysis gives $\dot{r}^2/c^2 = 1 - A = \alpha/r$. Thus, instead of a falling test object reaching infinite speed as $r \to 0$, we have a much more intuitive result, in which the velocity approaches c for $r \to 0$.

This analysis shows that there is only a small difference between $\tilde{r}$ and r, but that it becomes crucially significant the smaller r becomes. The coordinate $\tilde{r}$ never becomes smaller than α as $r \to 0$, meaning that A and B do not change sign. Put another way, the spacetime manifold does not exist for $\tilde{r} < \alpha$, so Hilbert's extension is invalid, and therefore an event horizon and black hole do not occur.

Finally, the free-fall acceleration is given by

$$\ddot{r} = -\frac{1}{2}c^2 \frac{\alpha}{(r+\alpha)^2} = -\frac{GM}{(r + 2GM/c^2)^2} \tag{26}$$

which shows modified inverse-square behaviour for r of the order of α, and classical Newtonian behaviour for $r \gg \alpha$.

4 Conclusion

Newton's law of gravitation relates solely to the contribution to gravity made by time curvature, space being Euclidean, and this fact must be used to link GR with classical physics. This means that the currently accepted black-hole solution is flawed in an almost trivial way. Replacing it with a solution first mentioned in a paper by Brillouin, which I have expounded here, satisfies Einstein's vacuum field equations and agrees with all the usual predictions of GR, such as the bending of starlight and the perihelion rotation of Mercury - but not the false prediction relating to black holes. The solution shows there is no horizon in spacetime, because the Schwarzschild radial coordinate $\tilde{r}$ is offset from the radial coordinate r by a distance $\alpha = 2GM/c^2$. It also predicts that the velocity of a free-falling test object cannot exceed the speed of light, which is a result that would also be intuitively expected from the kinematics of special relativity.

The deductions outlined in this short paper deviate markedly from the current paradigm. It would be satisfying if existing observations and calculations of trajectories that have previously led to the alleged presence of black holes could be re-examined on the basis of the model presented here, where there is no event horizon or black hole, and where gravity is modified in a specific way from Newtonian behaviour.

References

[1] A.Einstein, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin: 778, (11 November 1915)

[2] K.Schwarzschild, *On the gravitational field of a point mass according to Einstein's theory*, Sitzung der Koeniglich Preussischen Akademie der Wissenschaft zu Berlin, Phys.-Math.189-196, (Jan.1916)

[3] J.Droste, *The field of a single centre in Einstein's theory of gravitation*, Proc.K.Ned.Akad.Wet.19,197(1917)

[4] H.Weyl, *Zur Gravitationstheorie,* Ann.Phys(Leipzig) 54,117(1917)

[5] D.Hilbert, Math.Ann.92,1(1924), (also arXiv: physics/0310104)

[6] L.S.Abrams, *Black holes: the legacy of Hilbert's error*, Can.J.Phys, 67,919(1989) (also arXiv:gr-qc/0201044v1 14 Jan 2002)

[7] S.J.Crothers, *On the ramifications of the Schwarzschild spacetime metric*, Progress in Physics, 1,74(2005)

[8] M.Brillouin, *The singular points of Einstein's universe*, Journ Phys Radium, 23,43,(1923)
(see www.crothers.plasmaresources.com/brillouin.pdf)