A Proof of the Riemann Hypothesis

Jorma Jormakka

jorma.o.jormakka@gmail.com

Abstract. Zeros and the pole of the Riemann zeta function $\zeta(s)$ correspond to simple poles of the logarithmic derivative $f(s) = \frac{d}{ds}\ln\zeta(s)$. In $Re\{s\} > 1$ the function $f(s)$ has an absolutely convergent sum expression

$$f(s) = -\sum_{j=1}^{\infty} \ln(p_j) p_j^{-s}(1 - p_j^{-s})^{-1}$$

and an analytic continuation to the complex plane except for a discrete set of simple poles in the area $Re\{s\} \le 1$. Close to a pole s_k the function $f(s)$ is $r_k/(s-s_k)$+finite terms. Omitting the finite terms, we can evaluate this function into a Taylor series at the x-axis point $x > 1$. The absolute values of the coefficients of the Taylor series of each pole decrease as x^{-i} for some $i > 0$ as a function of x. The absolute values of the coefficients of the Taylor series of $f(s)$ decrease as a negative exponent of x when x grows. That means that all terms $a_i x^{-i}$, $a_i \in \mathbb{R}$, are cancelled by other terms in $f(s)$ when $x \to \infty$. These other terms must contain terms $-a_i x^{-i}$. Such terms arise only from poles. It follows that in the sum of all poles of $f(s)$, at the point x, poles must cancel other poles when $x \to \infty$. The poles of $f(s)$ in $Re\{s\} \ge 1$ and $Re\{s\} \le 0$ are known. They are the only poles that give a negative coefficients of x^{-j}, $j > 0$, while the remaining poles, the non-trivial zeros of $\zeta(s)$, give positive coefficients. It is shown that the poles of $f(s)$ cancel when $x \to \infty$ if and only if every pole s_k at $0 < Re\{s_k\} < 1$ satisfies $Re\{s\} = \frac{1}{2}$, i.e., the Riemann Hypothesis is true.

Key words: Riemann zeta function, Riemann Hypothesis, Number Theory.

AMS Mathematic Subject Classification: 11M26

Let p_j, $j = 1, 2, \ldots$, denote the primes in the increasing order. The Riemann zeta function $\zeta(s)$ has the expressions

$$\zeta(s) = \prod_{j=1}^{\infty} (1 - p_j^{-s})^{-1} = \sum_{k=1}^{\infty} \frac{1}{k^s}. \tag{1}$$

The infinite product and the sum converge absolutely if $Re\{s\} > 1$. See e.g. [1] for basic properties of zeta. The function $\zeta(s)$ satisfies for $Re\{s\} > 1$ the equation

$$\zeta(s)^{-1}\zeta'(s) = \frac{d}{ds} \ln \zeta(s) = -\frac{d}{ds} \sum_{j=1}^{\infty} \ln(1 - p_j^{-s})$$

$$= -\sum_{j=1}^{\infty} \ln(p_j) p_j^{-s} (1 - p_j^{-s})^{-1}. \tag{2}$$

As $\zeta(s)$ is analytic in the complex plane with the exception of $s = 1$, the function

$$f(s) = \zeta'(s)\zeta(s)^{-1} = \frac{d}{ds} \ln \zeta(s) \tag{3}$$

is analytic in the complex plane with the exception of points where $\zeta(s) = 0$ or $s = 1$. In $Re\{s\} > 1$ the function $f(s)$ has the absolutely convergent sum expression

$$f(s) = -\sum_{j=1}^{\infty} \ln(p_j) p_j^{-s} (1 - p_j^{-s})^{-1}. \tag{4}$$

Since $\zeta(s^*) = \zeta(s)^*$, the poles of $f(s)$ that are not on the x-axis appear in pole pairs (s_k, s_k^*), where s_k^* is a complex conjugate of $s_k \in \mathbb{C}$. Expressing $\zeta(s)$ as a Taylor series centered at a zero s_k of $\zeta(s)$, derivating to get $\zeta'(s)$, and dividing gives $f(s)$ in (3). Similarly, at the pole $s = 1$ expressing $\zeta(s)$ as a power series, derivating to get $\zeta'(s)$, and dividing yields $f(s)$. This operation shows that all poles of $f(s)$ are first-order poles: close to a pole s_k, where $Im\{s_k\} > 0$, $f(s)$ has an expression of the form

$$f(s) = \frac{r_k}{s - s_k} + f_{1,k}(s) \tag{5}$$

and close to the pole s_k^*, $f(s)$ of the form

$$f(s) = \frac{r_k}{s - s_k^*} + f_{2,k}(s). \tag{6}$$

If $\zeta(s)$ has a pole at s_k then r_k is a negative integer, while if $\zeta(s)$ has a zero, then r_k is a positive integer. The functions $f_{1,k}(s)$ and $f_{2,k}(s)$ are analytic close to s_k and s_k^* respectively. If the pole is at the x-axis, there is only one pole of the type (5) with $Im\{s_k\} = 0$.

Let us consider a function $g(s)$ that is analytic at s_0 and at all points s where $Re\{s\} > 1$. Let us assume that $g(s)$ has first-order poles at some points $s_k \in \mathbb{C}$, $k \in K$, $0 \notin K$. K is an index set and can be taken as integers. It can be finite or infinite. We ignore the analytic part of a pole (5) and the term pole function at s_k is used as a name of the function $r_k/(s - s_k)$. Let $t = x^{-1}$. We calculate the t-function in the following way: we expand $g(s)$ into a Taylor series at the point s_0 and z_1 is the parameter of the Taylor series. We consider $|z_1|$ small and there is a radius of convergence as $g(s)$ is analytic at s_0. Then we set $z_2 = x - z_1$ and consider $|z_2|$ as small. This gives a Taylor series of $g(s)$ at x, with a negative parameter $-z_2$. As $g(s)$ is analytic at $s_0 + x = x$, the Taylor series converges in some radius. Then we set $z_1 = z_2 = 0$ and insert $t = x^{-1}$. Setting $z_1 = 0$ at the Taylor series at s_0 we get $g(s_0)$, and setting $z_2 = 0$ at the Taylor series at x we get $g(x)$. Inserting $t = x^{-1}$ we get $g(t^{-1})$, which will be called the t-function of $g(s)$. There are several reasons for defining the procedure of evaluating the t-function in this way. One is that the coefficients of the Taylor series at s_0 and x have a conversion formula, thus if we have a convergent series expression for $g(s)$ at x, then we can calculate the Taylor series at s_0. We do not need an analytic continuation of $g(s)$ to the whole complex plane. Another, and more important, reason for using the Taylor series in the definition is that in this way we get the correct sign of t-functions of the poles in the negative x-axis, as is seen in Lemma 4. The reason for changing the variable x to $t = x^{-1}$ is that for poles the t-function has positive powers of t and we can write recursion formulas in Lemma 2 with $O(t^2)$ and $t \to 0$. The t-functions of functions of the form ce^{-ax}, which appear in (4), are not power series with positive powers of t. If we formally expand e^{-ax} into a power series of x and then set $t = x^{-1}$, we get negative powers of t. When $x \to \infty$ such a power series diverges. This means that functions of the form ce^{-ax} cannot be understood as power series of t with positive powers. They can only be understood as functions of x. They appear as x-dependent coefficients of positive powers of t. In Lemma 1 we prove that the coefficients of the positive powers of t decrease to zero when $x \to \infty$. This means that all coefficients of positive powers of t need not be numbers. They can be x-dependent coefficients, functions of x. Such coefficients are used in the following way: assume that the x-dependent coefficient of t^j, $j > 0$, coming from one set of poles does not decrease to zero when $x \to \infty$. Then the sum of x-dependent or fixed coefficients of t^j from the other sets of poles must partially cancel this coefficient so that the sum decreases to zero when $x \to \infty$. Since $t = x^{-1}$ one may wonder if the positive powers of t are well defined or if x-dependent coefficients should be expanded as powers of t. Should we do so, then it is unclear what are positive powers of t and what are x-dependent coefficients. There is no such confusion in the proof. Only one x-dependent coefficient is needed in the proof Theorem 1. Other coefficients that are needed in the proof of Theorem 1 are fixed real numbers. The x-dependent coefficient that is needed is $C(x)$, which is derived in Lemma 4. We show in Lemma 4 that $C(x)$ is a valid x-dependent coefficient and cannot be understood as a power series of t. Let us calculate the t-function of a pole function. We write

$$s - s_k = (s - s_0) - (s_k - s_0) = z_1 - (s_k - s_0) = x - z_2 - (s_k - s_0). \qquad (7)$$

Let the index set K be finite. At the point $s_0 + x$ the set k of pole functions of $g(s)$ is

$$\sum_{k \in K} \frac{r_k}{s - s_k} = \sum_{k \in K} \frac{r_k}{x - z_2 - (s_k - s_0)}. \qquad (8)$$

Let $a_k = s_k - s_0$, $z_2 = 0$ and $x = l^{-1}$. Then

$$\sum_{k\in K} \frac{r_k}{s - s_k} = x \sum_{k\in K} \frac{r_k}{1 - a_k t} \tag{9}$$

is a function of t, the t-function of a set of pole functions of $g(s)$. The t-functions have values on the x-axis and the area that we are interested in is when $x \to \infty$. In this proof t-functions are always real. Expanding (9) as a power series of x gives, assuming we can change the summation order

$$t \sum_{k\in K} \frac{r_k}{1 - a_k t} = t \sum_{i=0}^{\infty} t^i \sum_{k\in K} r_k a_k^i. \tag{10}$$

The series for each pole function converges when $t = x^{-1}$ is sufficiently small and when K is finite, there is a convergence radius for the whole sum. Infinite sets K appear in this proof only in Lemma 4 and in Lemma 2 in two equations, (32) and (34). In Lemma 4 divergent sums are mentioned as an impossible case and are discarded. In (32) and (34) there is no sum over the powers of t. These equations only give a coefficient of a power of x, and this coefficient can be infinite, as is mentioned in (22) and (43). In Theorem 1 it is shown that the sums (32) and (34) converge.

The absolute value of the coefficient of the power of x^{-i-1} for a pole function for s_k in (10) is $|r_k a_k^i|$. Unless each coefficient of each positive power of $t = x^{-1}$ decreases to zero when $x \to \infty$, the function of t for one pole decreases along the x-axis as a hyperbolic term ax^{-i-1}, $a \in \mathbb{R}$, for some $i \geq 0$ when x grows. If the absolute value of each coefficient of each power of t in the sum of all poles of $g(s)$ does not go to zero as $x \to \infty$, then the sum of the t-functions of the poles decreases along the x-axis as a hyperbolic term ax^{-i-1}, $a \in \mathbb{R}$, for some $i \geq 0$ when x grows. Let us assume that we have the inequality

$$|g(s_0 + x)| < e^{-bx}|g(s_0)| \tag{11}$$

for some $b > 0$. This means that the the absolute values of the coefficients of the Taylor series at $s_0 + x$ decrease as fast or faster than e^{-bx} as a function of x. This is faster than the decrease of any function ax^{-i-1}. Let us assume that $g(s)$ has a pole with $r_k = -1$ at $s = 1$. It yields the t-function $-t(1-t)^{-1}$. Why does $g(s)$ not have positive powers of t when $x \to \infty$ and decrease as ax^{-i-1} for some $a \in \mathbb{R}$ and $i \geq 0$? Obviously some other terms of $g(s)$ cancel these positive powers of t. We formulate this phenomenon into a lemma for $g(s) = f(s)$ in (3).

Lemma 1. *The absolute value of the coefficient of each positive power of x in the t-function of the sum of the poles of $f(s)$ in (3), when evaluated at x decreases to zero when x grows to infinity.*

Proof. The limit of $\zeta(s)$ can be determined from the series expression

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s} \tag{12}$$

which converges when $Re\{s\} > 1$. Letting s be real and growing to infinity leaves only the first term of the sum. Thus, the limit is one. It follows that the limit of $\zeta'(s)$ is zero when $s \to \infty$ and s is real. By derivating the series expression of $\zeta(s)$ we notice that in $Re\{s\} > 1$ in (13), which we satisfy by requiring $s_0 > 1$ and $x \geq 0$, holds

$$|\zeta'(s_0 + x)| = |\sum_{k=2}^{\infty} - \ln(k) k^{-s_0 - x}| < e^{-\ln(2)x} |\zeta'(s_0)|. \tag{13}$$

As $\lim_{s \to \infty} \zeta(s) = 1$ and $f(s) = \zeta(s)^{-1}\zeta'(s)$, we have the inequality

$$|f(s_0 + x)| < e^{-ax} |f(s_0)| \tag{14}$$

for some a with $0 < a < \ln(2)$ and $s_0 = 0$. Thus, $f(s)$ in (3) is a function where the absolute values of the coefficients of the power series of t at $x >> 1$, decrease exponentially as a function of x. This implies that coefficients c of terms cx^j,

$j>0$, $c \in \mathbb{R}$, go to zero when $x \to \infty$. Let $Re\{s\}>1$, we define $f_{1,0}=0$ and

$$f_{1,m}(s) = -\sum_{j=1}^{m} \ln(p_j) p_j^{-s}(1-p_j^{-s})^{-1}. \tag{15}$$

For a finite m the function $f_{1,m}(s)$ is analytic everywhere. When $m \to \infty$ the sequence $f_{1,m}(s)$ converges to $f(s)$ in $Re\{s\}>1$. The difference $f_{2,m}(s) = f(s) - f_{1,m}(s)$ contains all poles of $f(s)$ for every finite m. When $m \to \infty$ and $Re\{s\}>1$ the sequence $f_{2,m}(s)$ converges to zero. There cannot remain any analytic term of $f(s)$ in this limit, because the term is zero in $Re\{s\}>1$ and thus must be zero in the whole complex plane. This means that every analytic term of $f(s)$ is contained in $f_{1,m}(s)$ for $m \geq m_1$ if m_1 is sufficiently large. What remains in the limit function $f_{2,m}(s)$ is the sum of the poles of $f(s)$. The limit function $f_{2,m}(s)$, $m \to \infty$, is not analytic for $Re\{s\} \leq 1$, and it is zero for $Re\{s\}>1$. In Lemma 2 we only get recursion formulas for the coefficients of powers of t for pole functions of $f(s)$. These recursion equations are $O(t^2)$, thus they become exact only when $t \to \infty$, i.e., $x \to \infty$. This is why we only look at the cancellation of pole functions at $x \to \infty$. The t-function $f_{s,m}(t^{-1}) = f_{2,m}(x) = f(x) - f_{1,m}(x)$ decreases exponentially as a function of x when $x \to \infty$. If any $f_{1,m}(x)$ partially cancels any terms ct^j, $j>0$, $c \in \mathbb{R}$, there is the first index m when $f_{1,m}(x)$ partially cancels such terms. Then the set of the poles of $f_{2,m-1}(x) = f(x) - f_{1,m-1}(x)$ must be different, or some pole has a different residue r_k than in the function $f_{2,m}(x) = f(x) - f_{1,m}(x)$. Yet, both $f_{s,m}(s)$ and $f_{s,m-1}(s)$ have the same poles as $f(s)$ and with the same residues. This is so because $f_{1,m}(s)$ does not have poles. Therefore only t-functions of pole functions can cancel t-functions of other pole functions when $x >> 1$. As $f(s)$ does not have any positive powers t^j when $x \to \infty$, this means that the t-functions of the sum of the poles of $f(s)$ cancel when $x \to \infty$ so well that every positive power t^j vanishes. So well is expressed by the condition in Lemma 1. □

Comment: The reason why the everywhere analytic part of $f(s)$ does not cancel terms ct^j, $c \in \mathbb{R}$, $j>0$, is because the poles of $f(s)$ are always in $f_{2,m}(s)$.

It is true that an everywhere analytic function $g(s)$ cannot have finitely many additive terms ct^j. The t-function of $g(s)$ is $g(t^{-1})$. If $x = 0$, the parameter t becomes infinite, but the value of $g(t^{-1})$ is finite for every x as $g(s)$ is analytic everywhere. Terms t^j, $j > 0$, become infinite at $x = 0$, thus the t-function of $g(s)$ cannot contain finitely many additive terms ct^j, $j > 0$, $c \in \mathbb{R}$. However, it is not true that everywhere analytic functions cannot contain a convergent infinite series or t or grow as ct^j when $x \to \infty$. For that we need additional conditions. Let us consider a function $g(s)$ that can cancel a term $(x^j)^{-1}$ and what remains is $h(x) = (x^j)^{-1} - g(s)$. We assume that $h(x)$ satistifes $x^{2j}h(x) \to 0$ when $x \to \infty$, $h(x) \to \infty$ when $x \to -\infty$, and $h(x) > 0$. The exponential terms in $f(s)$ in $Re\{s\} > 1$ in (4) have these properties. By direct calculation

$$g(s) = \frac{1}{ax^j} - h(x) = \frac{1}{ax^j + b(x)} \tag{16}$$

where $a \in \mathbb{R}$, $a \neq 0$, $j > 0$ is an integer and

$$b(x) = \frac{a^2x^{2j}h(x)}{1 - ax^jh(x)} \tag{17}$$

giving the inverse relation

$$h(x) = \frac{b(x)}{ax^j(ax^j + b(x))}. \tag{18}$$

Let $a > 0$. Let $x \to -\infty$, then $h(x) \to \infty$ and

$$b(x) = -ax^j \frac{ax^jh(x)}{ax^jh(x) - 1}. \qquad (19$$

Then $ax^jh(x) - 1 < ax^jh(x)$, thus $b(x) < -ax^j$, meaning that $ax^j + b(x) < 0$. Let $x \to \infty$, then $x^{2j}h(x) \to 0$, so $b(x) \to 0$ and $ax^j + b(x) > 0$. There must be a value x giving $ax^j + b(x) = 0$. Let $a < 0$. Let $x \to -\infty$, then $h(x) \to \infty$ and $b(x) = -ax^j \frac{ax^jh(x)}{ax^jh(x)-1}$. Then $ax^jh(x) - 1 > ax^jh(x)$, thus $b(x) > -ax^j$, meaning that $ax^j + b(x) > 0$. Let $x \to \infty$, then $x^{2j}h(x) \to 0$, so $b(x) \to 0$ and $ax^j + b(x) < 0$. There must be a value x giving $ax^j + b(x) = 0$. Thus, for any $a \in R$ the function $g(x)$ has a pole and it is not everywhere analytic, but if $h(x)$ does not satisfy the required conditions, $g(s)$ can be analytic everywhere. Consequently, it can also be possible to have a convergent infinite series of powers of t in an everywhere analytic function.

The function $f(s)$ in (3) has the following poles in $Re\{s\} > 0$:

(i) There is a pole with $r = -1$ at $s = 1$.

(ii) There is a set A of pole pairs $f(s)$ at s_k and s_k^* where s_k has a nonzero imaginary part, and the r-value r_k is positive. We know of s_k is that $0 < Re\{s_k\} < 1$, and that that there exist poles s_k with the real part $\frac{1}{2}$.

(iii) There may be a set A_1 of poles $s_{k,1}$ of $f(s)$ with $r_{k,1}$ a positive integer, the pole $s_{k,1}$ is real and $0 < s_{k,1} < 1$. No such pole is known.

Inserting $s = s_0 + l$, $s_0 = 0$, $t = x^{-1}$ to the expression of a pole (5) on the t-axis gives the function (we omit the analytic function part in (5))

$$\frac{r_k}{s - s_k} = \frac{r_k}{s_0 + l - s_k} = \frac{r_k}{l - s_k} = x^{-1}\frac{r_k}{1 - s_k x^{-1}} = \frac{tr_k}{1 - s_k t} \qquad (20)$$

where s_k is a real number. For a pole pair in the positive and negative y-axis the equation (18) takes the form

$$\frac{r_k}{s - s_k} = \frac{tr_k}{1 - (1 + i\alpha_k)a_k t} \qquad (21)$$

$$\frac{r}{s - s_k^*} = \frac{tr_k}{1 - (1 - i\alpha_k)a_k t}.$$

Here $a_k = Re\{s_k\}$ and $\alpha_k = a_k^{-1} Im\{s_k\}$ is chosen positive by numbering the pole pairs.

Lemma 2. *The sum of the functions of t of the forms (20)-(21) for the poles of $f(s)$ in $Re\{s\} > 0$ gives for $x \to \infty$ a function of t of the form*

$$b_1 t + b_2 t(1-t)^{-1} \tag{22}$$

if and only if the Riemann Hypothesis is true, b_1 and b_2 are finite x-dependent coefficients of the powers of t with values in reals, or both are infinite.

Proof. Let s_k, s_k^* be a pole pair in the set A. The two functions for a pole pair in (21) have a real sum:

$$\frac{tr_k}{1-a_k(1+i\alpha_k)t} + \frac{tr_k}{1-a_k(1-i\alpha_k)t} = tr_k \frac{2(1-a_k t)}{1-2a_k t+(1+\alpha_k^2)(a_k t)^2}. \tag{23}$$

We expand the sum S:

$$S = \frac{2(1-a_k t)}{1-2a_k t+\alpha_k^2(a_k t)^2} = \frac{2-2a_k t}{1+\alpha_k^2(a_k t)^2}\frac{1}{1-2a_k t\gamma_k^{-1}} \tag{24}$$

where we have written $\gamma_k = 1+\alpha_k^2(a_k t)^2$. Thus

$$S = \frac{2-2a_k t}{\gamma_k}\sum_{i=0}^{\infty}(2a_k t\gamma_k^{-1})^i. \tag{25}$$

Defining $\beta_{k,i} = (2a_k)^i\gamma_k^{-i-1}$ we get

$$S = 2\sum_{i=0}^{\infty}\beta_{k,i}t^i - 2a_k\sum_{i=0}^{\infty}\beta_{k,i}t^{i+1} = \sum_{i=0}^{\infty}2\beta_{k,i}t^i - 2a_k\sum_{i=1}^{\infty}\beta_{k,i-1}t^i$$

$$= 2\beta_{k,0} + \sum_{i=1}^{\infty}(2\beta_{k,i} - 2a_k\beta_{k,i-1})t^i. \tag{26}$$

We will derive a recursion equation. It comes directly from the definitions of γ_k and $\beta_{k,i}$. For $i > 0$

$$2\beta_{k,i} - 2a_k\beta_{k,i-1} = 2\frac{(2a_k)^{i-1}}{\gamma_k^i}(2a_k\gamma_k^{-1} - a_k) \tag{27}$$

$$= \frac{(2a_k)^i}{\gamma_k^{i+1}}(2-\gamma_k) = \beta_{k,i}(2-\gamma_k).$$

This gives an equation for every $i > 0$

$$2\beta_{k,i} - 2a_k\beta_{k,i-1} = 2\beta_{k,i} - \gamma_k\beta_{k,i}. \tag{28}$$

Inserting $\gamma_k = 1 + (\alpha_k a_k x)^2$ yields for $i > 0$

$$2a_k\beta_{k,i-1} = \gamma_k\beta_{k,i} = \beta_{k,i} + t^2(\alpha_k a_k)^2\beta_{k,i}. \tag{29}$$

For every k when $l >> 1$ and thus for $0 < t = x^{-1} << 1$ and $i > 0$ holds

$$2a_k\beta_{k,i-1} = \gamma_k\beta_{k,i} = \beta_{k,i} + O(t^2). \tag{30}$$

The coefficient of the the power t^i, $i > 0$, is

$$2\beta_{k,i} - 2a_k\beta_{k,i-1} = \beta_{k,i} + O(t^2). \tag{31}$$

The result (31) can be inserted to (26) to show that the coefficient of t^{i+1}, $i > 0$, in the power series of the sum of poles (ii) is

$$\sum_{k\in K} r_k(2\beta_{k,i} - 2a_k\beta_{k,i-1}) = \sum_{k\in K} r_k\beta_{k,i} + O(t^2). \tag{32}$$

For each k, when $x \to 0$ and $i > 0$, holds by (31)

$$\beta_{k,i} = 2a_k\beta_{k,i-1}. \tag{33}$$

Assume that the Riemann Hypothesis is true. Then every $a_k = Re\{s_k\} = \frac{1}{2}$ for $(s_k, s_k^*) \in A$ and A_1 is empty. As every $a_k = \frac{1}{2}$, the recursion equation (33) yields $\beta_{k,i+1} = \beta_{k,i}$ for every k and $i > 0$ when $x \to \infty$. The power series of t for $i > 1$ is of the form $t\beta_{k,1}(t + t^2 + t^3 + \cdots) = \beta_{k,1}t(1-t)^{-1}$. The coefficient

$\beta_{k,1}$ is real and positive. The pole at $s=1$ yields the function $-t(1-t)^{-1}$. The recursion equation for $\beta_{k,j}$ is $\beta_{k,i}=(2a_k/\gamma_k)\beta_{k,i-1}$. As $2a_k=1$ and since $\gamma_k\geq 1$, this implies that $\beta_{k,i-1}\geq\beta_{k,i}$ for all $i>0$. Recursion (29) for $a_k=\frac{1}{2}$ shows that for every $i>0$ the value $\beta_{k,i}$ is the same when $t\to 0$. Let

$$\beta_i=\sum_{k\in K} r_k\beta_{i,k}\ ,\ i\geq 0. \tag{34}$$

The sum β_i, $i\geq 0$, of l is positive and real and it depends on x. The sum can be finite or infinite. Since $\gamma_k\to 1$ when $t\to 0$, β_i is the same for every $i\geq 0$. Thus, the function of x given by poles of the type (ii) is

$$\beta_0 t+\beta_0 t(1-t)^{-1}. \tag{35}$$

Adding the pole at $s=1$ with the t-function $-t(1-t)^{-1}$ we get the function (22) as in the claim.

Assume that the Riemann Hypothesis is false. Then either A_1 is not empty or at least for one a pole pair (s_k,s_k^*) in A the number a_k is not $\frac{1}{2}$. Assume that A_1 is not empty. At a zero $\zeta(s)$ has a Taylor series with some convergence radius, therefore zeros do not have a concentration point in $0<Re\{s\}<1$ at the x-axis. It follows that there can only be finitely many poles of type (iii). The function of x from the sum of the finite set K of indices k in A_1 is a finite sum of functions of form (20):

$$\sum_{k\in K}\frac{tr_k}{1-s_kt}=t\sum_{j=0}^{\infty}\left(\sum_{k\in K} r_k(s_k)^j\right)t^j. \tag{36}$$

This sum cannot be of the type $b_1+bt(1-t)^{-1}$ because the poles s_k of the type (iii) have $0<s_k<1$ and r_k positive integer. We would need

$$\sum_{k\in K} r_k(s_k)^2=\sum_{k\in K} r_k(s_k)^j \tag{37}$$

for all $j > 2$, but this is impossible if any r_k is nonzero. Poles of the type (ii) have positive values of r_k and the sum of functions of the form (21) for a pole pair is a positive function of t. These poles cannot cancel the function coming from (iii) if A_1 is nonempty. Thus, if A_1 is nonempty, the sum of the functions of t of the form (20)-(21) for the poles of $f(s)$ in $Re\{s\} > 0$ is not of the form (22).

Assume that A_1 is empty, but at least for one a pole pair (s_k, s_k^*) in A the number a_k is not $\frac{1}{2}$. The functional equation, proven by Riemann,

$$\zeta(s) = 2^s \pi^{s-1} \sin(2^{-1}\pi s)\Gamma(1-s)\zeta(1-s) \tag{38}$$

shows that if there exists a zero $s_0 = x_0 + iy_0$ of $\zeta(s)$ with $0 < x_0 < \frac{1}{2}$ then there exists a zero of $\zeta(s)$ at a symmetric point in $\frac{1}{2} < x < 1$. This implies that we can find $s_{k'}$ such that $2a_{k'} > 1$. If (22) holds, and because the pole at $s = 1$ yields the function $-t(1-t)^{-1} = -t - t^2 - t^3 - \ldots$, the function coming from the pole pairs in A must be of the type $c_1 t + c_2(t^2 + t^3 + \ldots)$. where c_1, c_2 are non-negative and real. c_2 must be finite since only t-functions from pole pairs in A can cancel the t-function $-t(1-t)^{-1}$ from the pole at $s = 1$ and by Lemma 1 t-functions of poles cancel when $x \to \infty$. Indeed, $c_2 = 1$ is the only possibility. Let K be the set of indices of pole pairs in A. As c_2 is finite, the sum over K in the left side of (32) is finite if (22) holds. Summing the coefficients of the powers t^i from $i = 2$ to $i = i_1 + 1$ gives an equation where the coefficients of t^j, $j > 1$, of the pole pairs (ii) must equal the coefficients of $c_2(t^2 + t^3 + \ldots)$ to the degree of $O(t^2)$:

$$c_2 i_1 = \sum_{i=2}^{i_1+1} c_2 = \sum_{i=2}^{i_1+1} \sum_{k \in K} r_k \beta_{k,i} + O(t^2). \tag{39}$$

We have the recursion

$$\beta_{k,i} = 2a_k \beta_{k,i-1} + O(t^2). \tag{40}$$

For $a_{k'}$ this recursion gives

$$\beta_{k',i} = \beta_{k',1}(2a_{k'})^i + O(t^2). \tag{41}$$

Notice that $\beta_{k,i}$ is positive by its definition, and that $r_k = 1$ and $0 < a_k < 1$ for poles of the type (ii). Inserting this equation to (39) yields

$$c_2 i_1 = \sum_{i=2}^{i_1+1} \sum_{k \in K} r_k \beta_{k,i} + O(t^2) \geq \beta_{k',1}(2a_{k'})^{i_1} + O(t^2). \tag{42}$$

The right side in (42) grows as $\beta_{k',1}(2a_{k'})^{i_1}$ as a function of i_1 while the left side is linear in i_1. This is a contradiction since $2a_{k'} > 1$. Thus, (22) does not hold if the Riemann Hypothesis is false. □

Lemma 3. *The Riemann Hypothesis is true if and only if the sum of the functions of t of the form (20)-(21) for the poles of $f(s)$ in $Re\{s\} \leq 0$ gives a function of t of the form*

$$-b_1 t - b_2 t(1-t)^{-1} \tag{43}$$

in the limit $x \to \infty$. Here b_1 and b_2 are finite x-dependent coefficients of the powers of t with values in reals, or both are infinite.

Proof. By Lemma 1 poles of $f(s)$ must cancel when $x \to \infty$. Lemma 2 shows that if the Riemann Hypothesis holds, the poles of $f(s)$ can cancel only if the sum of the functions of t of the form (20)-(21) for the poles of $f(s)$ in $Re\{s\} \leq 0$ gives a function of t of the form (43). If the Riemann Hypothesis is false, the poles still must cancel by Lemma 1. Then Lemma 2 shows that the sum of the functions of t of the form (20)-(21) for the poles of $f(s)$ in $Re\{s\} \leq 0$ does not give a function of t of the form (43). □

It remains to see what the poles of $f(s)$ in $Re\{s\} \leq 0$ give as a function of t. The zeros of $\zeta(s)$ in the area $Re\{s\} \leq 0$ are the so called trivial zeros at even negative integers. They come from the formula

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \tag{44}$$

where $B_m = 0$ if $m > 1$ is odd. Zeta does not have a zero at $s = 0$. These zeros of $\zeta(s)$ are the poles of $f(s)$ in $Re\{s\} \leq 0$. From the functional equation (38) we can deduce that the trivial zeros are zeros of $\sin(2^{-1}\pi s)$ and therefore first-order zeros. Thus, at a point $s_k = -2k$, $k > 0$ integer, the function $f(s)$ has a first-order pole with the r-value $r_k = 1$.

Lemma 4. *The poles of $f(s)$ in $Re\{s\} \leq 0$ give a function of $-tC(x)$, $C(x) > 0$, when $x \to \infty$. It is of the form (43).*

Proof. The pole function at $s_k = -2k$, $k > 0$, is

$$\frac{r_k}{s - s_k} = \frac{1}{s + 2k}. \tag{45}$$

At x the pole function is

$$\frac{r_k}{x - s_k} = \frac{1}{x + 2k} = x^{-1} \frac{1}{i + 2kx^{-1}}. \tag{46}$$

The t-function at x is thus

$$t \frac{1}{1 + 2kt} = t \sum_{i=0}^{\infty} (2kt)^i. \tag{47}$$

When $|2kt| < 1$ the sum converges, but there is a problem: (47) gives a positive t-function for each pole $s_k = -2k$. If so, then the only negative t-function is $-t(1-t)^{-1}$ from $s = 1$. Then t-functions of poles cannot cancel because Lemma 2 shows that $\beta_0 > 0$ in (35) and cannot be cancelled by $-t(1-t)^{-1}$. Zeta has zeros with $a_k = 1/2$, thus $K \neq \emptyset$ in (34). If the poles $s_k = -2k$ give a positive x-function, it is $tC(x)$ as the proof later shows, thus $-t(1-t)^{-1}$ must be cancelled by pole pairs of A. The term ax remains. It follows that (47) is not the correct way: the contribution from a pole $s_k = -2k$ must have negative terms. In order to get the correct form we calculate the t-function as described before by first expanding the pole function to a Taylor series at $s_0 = 0$ where $s = s_0 + z_1$, then setting $z_1 = 0$ and seeing what the function is, and then inserting $z_1 = x - z_2$ to a Taylor series at $s_0 + x = x$, setting $z_2 = 0$, and finally inserting $t = x^{-1}$. So, let us do this: we set $s = s_0 + z_1$ and assign $s_0 = z_1 = 0$ in (46). Then

$$\frac{1}{s+2k} = \frac{1}{s_0 + z_1 + 2k} = \frac{-1}{s_0 + z_1 - 2k} = \frac{-1}{s-2k} \tag{48}$$

corresponding to a pole at $2k$ evaluated at $s_0 = 0$ with $r = -1$. A pole with $r = 1$ at the negative x-axis at the place $s_k = -2k$ gives the same t-function as a pole with $r = -1$ at the positive x-axis at the place $s_k = 2k$. The pole functions of the poles of $f(s)$ at $s_k = -2k$ is negative. When a pole at $s_k = -2k$ is evaluated at x we first evaluate it at $s_0 = 0$

$$\frac{r_k}{s - s_k} = \frac{1}{s_0 + z_1 + 2k} = \frac{-1}{-s_0 - z_1 - 2k} = \frac{-1}{s_0 + z_1 - 2k}. \tag{48}$$

Then we set $z_1 = x - z_2$, and then consider $|z_2| << 1$ in the function $-1/(s_0 + x - z_2 - 2k)$. When $s_0 = z_2 = 0$, the t-function is $-1/(x - 2k) = -t/(1 - 2kt)$ is similar to the t-function from a pole with $r = -1$ placed at $2k$. The contribution is negative. Thus, the way to evaluate the t-function of $g(s)$ is not simply changing $x = t^{-1}$ in $g(x)$. The sign may be taken both ways for poles on the negative x-axis. Only one of these ways is correct. Any finite sum of poles $s_k = -2k$ can be evaluated at $s_0 + x$ and the sum is a negative contribution, but the sum of all poles $s_k = -2k$

$$\sum_{k=1}^{\infty} \frac{-1}{x - 2k} \tag{50}$$

diverges at every point x. We cannot evaluate all these poles at x by directly evaluating a pole at $s_0 + x$ and then summing over k. Let us still look deeper at the problem. The sum of all poles $s = -2k$ at $s_0 = 0$ is obtained by setting

$z_1 = 0$. We get the negative of the sum

$$\sum_{k=1}^{\infty} \frac{1}{2k}. \tag{51}$$

This sum can be calculated. Using the facts that $\zeta(s)$ has a simple pole at $s = 1$

$$\zeta(s) = \frac{a}{s-1} + g(s) \tag{52}$$

where $g(s)$ is analytic at $s = 1$ and that $\lim_{s\to 1}(s-1)\zeta(s) = 1$, so $a = 1$, we can write

$$\zeta(1) = \lim_{s\to 1} \frac{1 + (s-1)g(1)}{s-1} = \lim_{s\to 1} \frac{1}{s-1} = \lim_{s\to 0} \frac{1}{s} \tag{53}$$

This result gives

$$\sum_{k=1}^{\infty} \frac{1}{2k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^1} = \frac{1}{2}\zeta(1) = \lim_{s\to 0} \frac{1}{2}\frac{1}{s}. \tag{54}$$

Thus, the sum of the poles at $s_k = -2k$ appears as a simple pole when evaluated at $s_0 = 0$. The pole has a negative r-value with $r = -1$ at $s_0 = 0$. However, it is not a simple pole. A simple pole with $r = -1/2$ is at $s_0 = 0$

$$\frac{-1}{2}\frac{1}{s - s_0} = \frac{-1}{2}\frac{1}{z_1} \tag{55}$$

where $s_0 = z_1 = 0$. It is evaluated at $s_0 + x$ by writing $z_1 = x - z_2$ and the t-function is obtained by setting $z_2 = 0$

$$\frac{-1}{2}\frac{1}{x - z_2} = \frac{-1}{2}x^{-1} = -\frac{1}{2}t. \tag{56}$$

This pole is finite for every $x > 0$, but the sum of the poles $s_k = -2k$ is infinite at every finite x. Indeed, the sum of poles in $Re\{s\} \leq 0$ can be presented as a simple pole in any point $s_0 + x$

$$\lim_{s\to 1} \frac{1}{s-1} = \lim_{s\to x} \frac{1}{s-x}. \tag{57}$$

The infinity $\lim_{s\to s_0} -0.5/(s-s_0)$ is not caused by the pole being physically at s_0, the infinity comes because the sum of the numbers $1/(x+2k)$ is infinite.

Clearly, something is wrong in taking the limit in (57) and summing to infinity at (50). By Lemma 1 the sum of the t-functions of the poles of $f(s)$ goes to zero if $x \to \infty$. If the t-functions from the poles $-2k$ is infinite everywhere, the sum cannot go to zero. It is either infinite or undefined when $x \to \infty$. That is, adding the t-functions of other poles one by one keeps the sum at negative infinity, while adding the t-functions of other poles to positive infinity gives infinity minus infinity. As the sum is not infinite, we cannot calculate it by summing any partial sum to infinity. The problem is in taking a limit or an infinite sum that gives a function that is infinite everywhere. We can compare the situation to taking a limit $m \to \infty$ of the functions

$$f_{1,m}(s) = -\sum_{j=1}^{m} \ln(p_j) p_j^{-s}(1-p_j^{-s})^{-1} \tag{58}$$

when s is real and $0 < s < 1$. The functions are finite sums and analytic everywhere. The limit function is infinite at each point s real $0 < s < 1$ and it does not converge to $f(s)$, though in $Re\{s\} > 1$ the limit converges to $f(s)$. From Lemma 1 follows that in the sum of poles of $f(s)$ at $s = x$ the absolute value of the coefficient of every power of t decreases to zero when $x \to \infty$. Thus, the function of t at $s_0 + x$ for the poles $s_k = -2k$ must be a function of x that decreases to zero when $x \to \infty$. The limit (57) and the infinite sum (46) do not give $f(s)$. The sum of the poles $s_k = -2k$ must be evaluated differently. All poles $s_k = -2k$ cannot be evaluated at x at the same time because the sum diverges. We can only evaluate at each given $x > 0$ such a subset of poles that the sum gives a finite number when evaluated at x. All poles have to be evaluated at x at some point as the sum of all poles of $f(s)$ should be zero at $x \to \infty$. Thus, we must include more poles when x grows until all poles are included when $x \to \infty$. The choice of which subsums of poles are included for each x cannot influence the result. We will make a convenient choice for these sums: let us choose a suitable growing function $N(x)$ and include the subsum of poles $s_k = -2k$ satisfying $k \leq N(x)$. A finite sum up to $N(x)$ can be evaluated at x, and when $N(x)$ increases with x, all poles $-2k$ are included in the finite sum when $k \leq N(x)$. The function $N(x)$ is a piecewise constant function. We start from the fact that a pole $s_k = -2k$, $r = 1$, appears in $s_0 = 0$ as a pole at $s_k = 2k$ with $r_k = -1$. Thus,

$$\sum_{k=1}^{N(x)} \frac{r}{s - s_k} = \sum_{k=1}^{N(x)} \frac{-1}{s_0 + z_1 - 2k}. \tag{59}$$

Inserting $s_0 = 0$ and $z_1 = x - z_2$ gives

$$= \sum_{k=1}^{N(x)} \frac{-1}{x - z_2 - 2k}. \tag{60}$$

Evaluating at x we consider $|z_2| << 1$. Setting $z_2 = 0$ we have the sum at x as

$$\sum_{k=1}^{N(x)} \frac{-1}{x - 2k} = -x^{-1} \sum_{k=1}^{N(x)} \frac{1}{1 - 2kx^{-1}} \tag{61}$$

and writing $t = x^{-1}$ we get

$$= -t \sum_{k=1}^{N(x)} \frac{1}{1 - 2kx^{-1}} = -tC(x). \tag{62}$$

In case this calculation looks strange, let us explain that it is not stating that the positive entity in the left side of (59) is the negative entity in (62). In (59) the set of poles is evaluated at 0 and in (62) at x. Compare the calculation with a calculation of a pole $r/(s-1)$, $r > 0$. At $s = s_0 + z_1$, $s_0 = 0$, $z_1 = 0$, the pole is

$$\frac{r}{z_1 - 1} = \frac{r}{-1} = -r \tag{63}$$

but when evaluated at x we first insert $z_1 = x - z_2$ and then set $z_2 = 0$:

$$\frac{r}{x - z_1 - 1} = \frac{r}{x - 1} = x^{-1}\frac{r}{1 - x^{-1}} = tr/(1 - t). \tag{64}$$

Negative $-r$ at 0 changes to positive $tr/(1-t)$ at x. The function $C(x)$ is an x-dependent coefficient of the power t. By Lemma 1 the coefficients of the power series of t decrease to zero when x grows to infinity. Thus, the coefficients are functions of x. As $t = x^{-1}$ we may ask if a coefficient, like $C(x)$, should be expressed as a power series of t. If this can be done, then we cannot say that $C(x)$ is a coefficient of the power one of t. It turns out that we cannot do so. Though formally we can express

$$C(x) = C(x(t)) = \sum_{j=0}^{\infty} t^j \sum_{k=1}^{N(x)} (2k)^j, \tag{65}$$

this is not a valid expression of the power series of t. The reason is that for any selected index k the coefficient of t^j grows to infinity when j grows. By Lemma 1 the coefficients of all positive powers of t for $f(s)$ must go to zero as $x \to \infty$. Therefore, if for poles in $Re\{s\} \leq 0$ coefficients of powers of t^j go to infinity when j grows, the same must happen for some poles in $Re\{s\} > 0$, as poles must cancel when $x \to \infty$. From (20) we see that for poles of type (iii) this cannot happen: s_k is between zero and one and there are only finitely many poles of type (iii). For the pole at $s = 1$ this also cannot happen: every coefficient of a power t^j is -1 in $-t(1-t)^{-1}$. Pole pairs of type (ii) can give coefficients for t^j that grow with j, and they also give a series of x where the coefficents decrease with j. The coefficients that grow with j are obtained for any pole pair of the type (ii) that has $a_k > \frac{1}{2}$, see (38). For any finite index set the coefficient of t^j is smaller than 2^j times the size of the index set. There is in a symmetric place another pole that has $a_k < \frac{1}{2}$. The coefficient for t^j in for these poles in the finite index set is smaller than the size of the index set. For a finite $k_{\max}$ the coefficient of t^j in the subset $k \leq k_{\max}$ in the series for $C(x)$ is smaller than $-(2k_{\max})^j$. The coefficient of t^j in the sum of these three sets goes to minus infinity when j grows to infinity for any fixed finite index set K_i and any fixed finite limit $k_{\max_i}$. Therefore it is not possible to select a sequence of finite index sets $(K_i)_i \to K$ and a sequence of finite limits $(k_{\max_i})_i \to \infty$ and take them together to an infinite index set K and infinite limit $\lim k_{\max_i} = \infty$ so that all powers of t^j cancel. Thus, pole pairs do not cancel the powers of t from the power series of $C(x)$. Therefore, the poles in $Re\{s\} \leq 0$ cannot give a power series of t where the coefficients of t^j grow to infinity when $j \to \infty$. $C(x)$ is not a power series of x with several different powers of t. It is a coefficient of the first power t, and it depends on x. It is a piecewise continuous function of x as it jumps when a new pole is added to the sum. The function $-tC(x)$ is of the type (43). Therefore Lemma 3 implies that the other poles give a contribution as in (22). The variable t is real, every r_k is real, and every s_k on the x-axis is real. The sum of the poles of a pole pair is real. Thus, every t-function is real. Therefore $C(x)$ is real. As a finite sum of t-functions for poles $-2k$ the t-function $-tC(x)$ is negative, thus $C(x) > 0$. □

Theorem 1. *All zeros of $\zeta(s)$ in $0 < Re\{s\} < 1$ have the real part $\frac{1}{2}$.*

Proof. The t-function from the pole at $s = 1$ is $-t(1-t)^{-1}$. Other poles of $f(s)$ must cancel this contribution because $f(s)$ does not have positive powers of t when $x \to \infty$. As this contribution is negative, it can only be cancelled by the poles of A_1 and the pole pairs of A. The poles at $s_k = -2k$ yield negative t-functions and they cannot cancel negative coefficients of positive powers of t. From Lemma 2 we notice that this contribution can be cancelled if A_1 is empty and $\beta_0 = 1$ in (35) when $x \to \infty$. In this case every $a_k = 1/2$ for pole pairs in A and the Riemann Hypothesis is true. If $\beta_0 =$ when $x \to \infty$, (35) shows that there remains the term $\beta_0 t = t$. This term must vanish in the sum of poles when $x \to \infty$. The only poles that are left to cancel this term are the poles at $s_k = -2k$. Therefore, the t-function coming from these poles must be $-t$. In Lemma 4 we derived two expressions for the t-function of the poles $s_k = -2k$. One expression is infinite at every point x and this expression is not correct since $f(s)$ is not infinite at every point x. The other expression from Lemma 4 is $-tC(x)$, $C(x) > 0$. The t-function $-tC(x)$ can give $-t$. Thus, if the Riemann Hypothesis is true $C(x) \to 1$ when $x \to \infty$. All poles cancel when $x \to \infty$. Notice that the only x-dependent coefficient is $C(x)$.

Consider the possibility that the Riemann Hypothesis is false. Also in this case we need the pole pairs that cancel $-t(1-t)^{-1}$. They are the same pole pairs as in the case the Riemann Hypothesis is true. They cancel $-t(1-t)^{-1}$, but leave the t-function t. Thus, the poles $s_k = -2k$ must cancel this function, as these poles are the only poles left yielding a negative t-function. We did not prove in Lemma 4 that there are no more than two possible expressions for the t-function coming from the poles $s_k = -2k$, but the expression $-tC(x)$ contains all poles $s_k = -2k$ when $N(x)$ grows. If there were positive contributions from any pole in A_1 or any pole pair in A that does not have $a_k = \frac{1}{2}$, then at some finite x the set $k \leq N(x)$ must include indices of poles $s_k = -2k$ that partially cancel these positive t-functions, i.e., partially cancel coefficients of powers $j > 1$ of t. This is not the case: for all values of x the sum of $s_k = -2k$ poles up to $k \leq N(x)$ gives $-tC(x)$. Thus, there are no other positive contributions than t, which $-tC(x)$ must cancel. This means that A_1 is empty and a_k for all pole pairs in A is $\frac{1}{2}$. That is, Lemma 3 shows that if the Riemann Hypothesis is false, the t-function for the poles $s_k = -2k$ cannot be as in Lemma 4. Thus, the Riemann Hypothesis is true. Notice that also here $C(x)$ is the only x-dependent coefficient that needs to be considered.

The rest of the proof of this theorem is checking if there is any obvious contradiction. In this solution $\beta_0 = 1$, $C(x) \to 1$ when $x \to \infty$, and that the absolute values of the coefficients of all powers of t go to zero when $x \to \infty$. Let us see if there is an obvious contradiction in these requirements.

Can

$$\beta_i = \sum_{k \in K} r_k \beta_{k,i} \tag{66}$$

have the value 1 in the limit $x \to \infty$ for all $i \geq 0$? Because $t \to 0$, the values of $\alpha_k = a_k^{-1} Im\{s_k\}$ must grow to infinity with k. The set A is necessarily infinite. We renumber the poles of (ii) so that (α_k) is a growing sequence and the sum $k \in K$ is the sum from $k = 1$ to infinity. Since $a_k = \frac{1}{2}$ for every k in K we can evaluate

$$\beta_{k,i} = (2a_k)^i \gamma_k^{-i-1} = \left(\frac{1}{1 + 0.25(\alpha_k x)^2} \right)^{i+1}. \tag{67}$$

Let $x >> 1$ be fixed. If $\alpha_k >> l = t^{-1}$, then $\beta_{k,i}$ is close to zero. This means that large values of α_k contribute very little to the power series of t at x. The sum in (66) can be finite and in the limit when $x \to \infty$ the number β_i can be 1. There is no obvious contradiction.

Can $C(x)$ in (62)

$$C(x) = \sum_{k=1}^{N(x)} \frac{1}{1 - 2kx^{-1}} = \sum_{k=1}^{N(x)} \frac{x}{x - 2k} \tag{68}$$

converge to 1 when $x \to \infty$? It may appear that this must be impossible as each term in (68) approaches 1 when x grows. It can. $C(x)$ s a growing function as it grows at each addition of a pole at some point x_k and also between the additions. From (54) we see that if $x = 0$ the function $C(0)$ for the sum of all poles is 1/2. Let us select an integer n_M so that for the sum up to n_M, $C(0) = (1 - M^{-1})/2$. The tail, sum from $n_m + 1$ to infinity, is $M^{-1}/2$. From (57) we notice that the tail has the same value at each point $x_k > 0$ as at $x = 0$. At the point x_k we add one pole to $C(x)$. Thus, the addition at x_k to $C(x)$ is less than $M^{-1}/2$. Let us add $[M]$ poles at points $x_1, \ldots, x_{[M]}$, $x_i > x_{i+1}$. Then $C(x_{[M]}) < (1 - M^{-1})/2 + [M]M^{-1}/2 \leq 1 - M^{-1}/2$. If we add $a[M]$ points, the upper limit is $(1 + a)/2$. We cannot say what the limit is, but as $0 < C(x)$ and a growing function, it converges to a limit or to infinity. This limit can be 1.

Notice how the problem that each term is at least one in (68) is solved. In the limit $n_M \to \infty$ all poles are added at $x = 0$. They do not give the same t-function when evaluated at x as a simple pole at $s = 0$, but a similar one. They give $-tC(x)$ instead of $-t/2$. There is no obvious contradiction with $C(x) \to 1$ as $x \to \infty$.

The convergence of the coefficients of the powers of t to zero in the sum of all poles when x grows is $O(t^2)$ for the coefficient of each power $i > 1$ of t^i separately. For the power one of t we get the result that convergence as at least $O(t)$ is possible. The term β_0 converges to β_1 as $O(t^2)$ since every $\beta_{k,i} = \beta_{k,i-1} + O(t^2)$. Each β_i converges to 1 as $O(t^2)$. The contribution from the poles at $-2k$ is $-tC(x)$. The sum $-tC(x) - t + 2\beta_0 x$ can go to zero at least as fast as $O(t)$. There is no obvious contradiction. □

Theorem 1 is the Riemann Hypothesis. For basic facts of the Riemann zeta function see standard works, like [1]. The history and background of the Riemann Hypothesis are well described in the book [2].

References

1. E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis*, Cambridge, University Press, 1952.
2. K. Sabbagh, *The Riemann Hypothesis, the greatest unsolved problem in mathematics*, Farrar, Strauss and Giroux, New York, 2002.