Derivation of the Schrödinger Equation from a Fractal Vortex Particle Model

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Abstract

A particle model developed elsewhere is presented very briefly, in which particles are seen as vortex structures in a kind of fluid. The particle core is tied by vortex lines to the core of other particles in the vicinity. The model is shown to explain spin ½ as a topological effect. It also sheds some light on the process of intrication. But vortex lines also induce velocities on themselves and on other vortex lines. When the Hasimoto transformation is used, the velocity induction equation is shown to give rise to the Schrödinger equation for the particle. We also consider the multi-particle systems and we give a tentative explanation for the Pauli exclusion principle.

1 Introduction

The Schrödinger equation was not found from first principles. Schrödinger obtained it from a dubious variational method. But it gained immediate acceptance because it reproduced the energy levels of the hydrogen atom. Since then, it has furnished a host of solutions to many different problems in quantum theory and they all agreed with experimental results. Here we are going to derive it starting from a fractal vortex model of particles [1,2]. As a bonus, the model will furnish an explanation for spin ½. It also might explain the intrication problem.

2 The fractal model of particles

My fractal particle model [1,2] was initially devised to try and explain the spin ½ phenomenon. It is a synthesis of several well-known ideas:

- Mandelbrot's fractals [3].
- The Einstein-Rosen particle model [4].
- Thomson's (lord Kelvin) vortex atom model [5].
- Jehle's flux tubes [6,7,8].
- Dirac's sphere in the box [9].

Mandelbrot coined the word "fractal" in 1975 to designate objects that reproduce homothetically on different scales. He began seeing fractals everywhere and this induced me to make the working hypothesis that the Universe itself could be a fractal. The homothetical scales that I postulated were in particular the particle and the galaxy scale.

A spiral galaxy looks like a vortex structure in a kind of fluid. The fluid is composed of stars, planets, asteroids, gas and finally particles. Thus, what we will call a particle on scale (n), is a vortex structure composed of particles on scale (n-1). This is a recurrent definition of the fractal.

On the particle scale, the fluid can be seen as an aether. Despite the negative connotation attached to the aether, this concept must be accepted as real if the fractal hypothesis is correct. Indeed, on the galactic scale the aether can be seen with the naked eye by directly looking at galaxy pictures. It cannot be denied. So, if the fractal hypothesis is correct, it must exist on the particle scale too. Moreover, the usual objections against the aether are answered in [2].

Thus, we see particles as vortex structure in an ethereal fluid. They have two Killing vectors. One is time-like and accounts for the fact that the particle is stationary. The second one is space-like and implies that the structure is axially symmetric.

The particle core (the wormhole throat) is defined as the surface where the aether velocity attains the light velocity of scale (n-1). Different particles are distinguished by the topology of their cores. The electron is supposed to have the simplest topology, namely S^2.

3 The spin ½ of the electron

We supposed that the electron is the homothetical equivalent of the spiral galaxy. The core in this case is S^2. Another feature of our model is that we associated the core with an Einstein-Rosen bridge, or more likely a traversable wormhole establishing a bridge between two regions of space-time.

But how can such a structure display spin ½ ? In the representation below we have shown an open Einstein-Rosen bridge. We have kept as usual only 2 dimensions for the visual representation of the spatial section, which should count 3 dimensions (see figure 1).

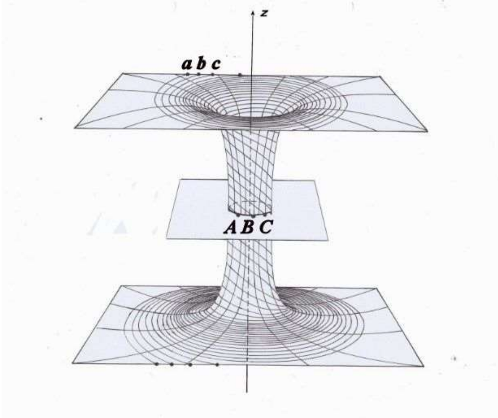

Figure 1. The particle core is shown as an Einstein-Rosen bridge, or better, a traversable wormhole, linking two regions of space-time.

The space-like section represented here can be seen as a universe membrane of finite thickness and the electron as a hole in it. The vortex lines *a, b, c…* come from infinity and wind up around the axis. These lines are extensible but cannot be cut by virtue to a theorem due to Helmoltz. Their intersections *A, B, C…* with the throat of the wormhole defines a ring which rotates with the fluid because vortex lines are frozen in the fluid and move with it. If one adds the third dimension of space this ring becomes a spherical surface to which the vortex lines are attached. Besides, one can imagine that vortex lines are attached in the same way to the cores of neighbouring particles, so that the situation becomes that represented below.

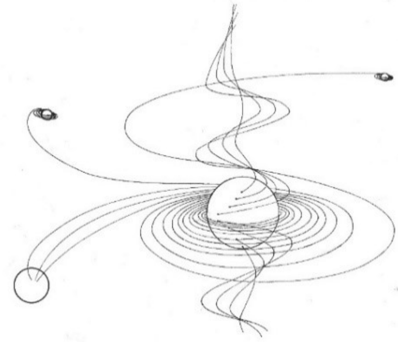

Figure 2. The particle core is tied by vortex lines to the core of other particles in the vicinity.

From the point of view of topology, the model is in all points equivalent to a central sphere (the wormhole throat) attached by elastic threads to distant particles (the box sides), as represented below.

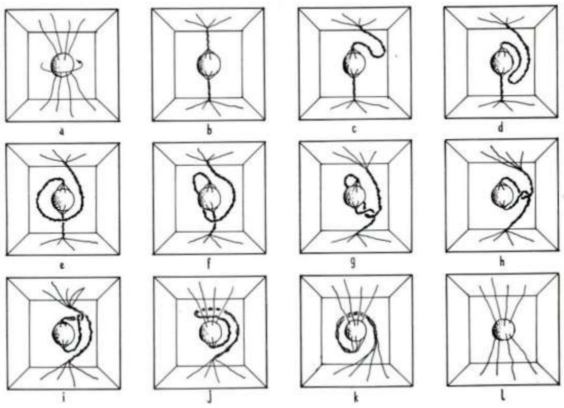

Figure 3. A sphere attached to a box by extensible strings needs two complete turns to come back to its original situation. This was presented by Dirac in its conferences as a model of spin 1/2.

The central sphere needs two full turns in the same sense to come back to its original state. This is typical of spin ½. Our model takes thus correctly into account the behaviour of a particle like the electron in its rotation about itself.

4 Intrication

Let us note in passing the implications that this model could have for the curious phenomenon of intrication. We have supposed that each particle is bound to its neighbours by vortex lines. Let us consider first two particles A and B. The number of vortex lines binding them together is large but finite. Indeed, each vortex line is in fact a quantised and very thin vortex tube. Each tube has thus a small but well determined cross section. So, the number of vortex tubes that can be attached to a particle core is large but finite. When all sites of A are occupied by vortex tubes coming from B, one can say that A is 100 percent intricated with B. No other particle can then intricate with A. If A possesses sites not connected to B, it can bind partially with other particles C, D, … One could then define the ratio of intrication between say A and B, as the number of A sites connected to B sites, divided by the total number of A sites.

5 Vorticity Induction

The main part of the velocity induction by a vortex filament has been found by Callegari and Ting [10] as

$$\frac{\partial \boldsymbol{X}}{\partial t} = \boldsymbol{u} = \frac{\Gamma}{4\pi} ln\left(\frac{L}{a}\right) \kappa \boldsymbol{b} \tag{5.1}$$

Where $\boldsymbol{X}$ is the position of a point of the filament, a is the filament core radius, Γ is its circulation, L is its length, κ its curvature and $\boldsymbol{b}$ is the binormal of the Frenet-Serret (FS) triad attached to the vortex filament axis. If a is sufficiently small compared to L, the variation of $ln(L/a)$ with L is weak. $ln(L/a)$ may be approximated by a constant c, and the above equation (5.1) becomes

$$\boldsymbol{u} = c\kappa\boldsymbol{b} \tag{5.2}$$

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Then a rescaling of time and length may be performed to put this in the form

$$\boldsymbol{X}_t = \boldsymbol{u} = \kappa \boldsymbol{b} \tag{5.3}$$

But $\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$ and $\boldsymbol{t} = \frac{\partial \boldsymbol{X}}{\partial s} = \boldsymbol{X}_s$. By the Frenet-Serret (FS) formula, we have:

$$\boldsymbol{t}_s = \kappa \boldsymbol{n} \qquad \boldsymbol{n}_s = -\kappa \boldsymbol{t} + \tau \boldsymbol{b} \qquad \boldsymbol{b}_s = -\tau \boldsymbol{n} \tag{5.4}$$

Where τ is the torsion of the curve. After some manipulations of (5.3) we find:

$$\boldsymbol{X}_t = \boldsymbol{X}_s \times \boldsymbol{X}_{ss} \tag{5.5}$$

This is the differential evolution equation of the filament.

Following Hasimoto [11], we combine the second and third FS formula (5.4) to obtain:

$$(\boldsymbol{n} + i\boldsymbol{b})_s = -\kappa \boldsymbol{t} - i\tau(\boldsymbol{n} + i\boldsymbol{b}) \tag{5.6}$$

This suggests introducing the complex vector

$$\boldsymbol{N} = (\boldsymbol{n} + i\boldsymbol{b})e^{i\phi} \tag{5.7}$$

Where the complex phase $\phi(s,t)$ is such that its derivative with respect to s should simply give $\tau(s,t)$. This will be the case if $\phi(s,t) = \int_0^s \tau(s',t)\, ds'$. We find thus from (5.7)

$$\boldsymbol{N}_s = -\kappa e^{i\phi} \boldsymbol{t} \tag{5.8}$$

Let us set

$$\psi = \kappa e^{i\phi} = \kappa(s,t) e^{i\int_0^s \tau(s',t) ds'} \tag{5.9}$$

This is a complex function replacing two scalar functions $\kappa(s,t)$ and $\tau(s,t)$. It drives the evolution of the filament. Equation (5.8) thus becomes

$$\boldsymbol{N}_s = -\psi \boldsymbol{t} \tag{5.10}$$

This is as a new complex FS equation where the variables $\boldsymbol{n}(s,t)$ *and* $\boldsymbol{b}(s,t)$ have been replaced by the complex ψ. This is called the Hasimoto transformation [11]. Still following Hasimoto, we search for an equation to replace $\boldsymbol{X}_t = \boldsymbol{X}_s \times \boldsymbol{X}_{ss}$ **.** We first notice that

$$\boldsymbol{t} \cdot \boldsymbol{t} = 1 \quad \boldsymbol{N} \cdot \boldsymbol{N}^* = 2 \quad \boldsymbol{N} \cdot \boldsymbol{N} = \boldsymbol{N}^* \cdot \boldsymbol{N}^* = 0 \tag{5.11}$$

$(\boldsymbol{t}, \boldsymbol{N}, \boldsymbol{N}^*)$ is a new basis replacing the FS basis $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$.

As we have shown above, $\boldsymbol{N}_s = -\psi \boldsymbol{t}$ replaces the FS equations 2 and 3. We must still account for the first FS equation (5.4). To separate $\boldsymbol{n}$ from $\boldsymbol{N}$ we must multiply by the phase $e^{-i\phi}$ which would come from ψ^*. So, let us compose

$$\psi^* \boldsymbol{N} + \psi\, \boldsymbol{N}^* = 2\kappa \boldsymbol{n} \tag{5.12}$$

Hence, we obtain the first FS equation in the new variables as

$$\boldsymbol{t}_s = \frac{1}{2}(\psi^* \boldsymbol{N} + \psi\, \boldsymbol{N}^*) = \kappa \boldsymbol{n} \tag{5.13}$$

Let us now derive the induction equation (5.3) with respect to s. We find

$$\frac{\partial}{\partial s}\left(\frac{\partial \boldsymbol{X}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{X}}{\partial s}\right) = \frac{\partial}{\partial t}\boldsymbol{t} = \kappa_s \boldsymbol{b} + \kappa \boldsymbol{b_s} = \kappa_s \boldsymbol{b} - \kappa\tau \boldsymbol{n} \tag{5.14}$$

Now let us express this in terms of the new variables $\psi, \boldsymbol{N}$. We form

$$\psi_s\, \boldsymbol{N}^* - \psi_s^* \boldsymbol{N} = -2i\boldsymbol{t}_t \tag{5.15}$$

For expressing $\boldsymbol{N}_t$ in the new variables, we set

$$\boldsymbol{N}_t = \alpha \boldsymbol{N} + \beta\, \boldsymbol{N}^* + \gamma \boldsymbol{t} \tag{5.16}$$

Using the orthogonality relations (5.11), we find

$$\boldsymbol{N} \cdot \boldsymbol{N}_t^* + \boldsymbol{N}_t \cdot\, \boldsymbol{N}^* = 2(\alpha + \alpha^*) = 4\, Re(\alpha) = 0 \tag{5.17}$$

Similarly, we have

$$(\boldsymbol{N} \cdot \boldsymbol{N})_t = 4\,\beta = 0 \tag{5.18}$$

And finally

$$\boldsymbol{t} \cdot \boldsymbol{N}_t = \gamma = -i\psi_s \tag{5.19}$$

Thus, it comes

$$\boldsymbol{N}_t = Im(\alpha)\boldsymbol{N} - i\psi_s \boldsymbol{t} \tag{5.20}$$

Let us set $\alpha = iR$ where R is real. We have:

$$\boldsymbol{N}_t = i(R\boldsymbol{N} - \psi_s \boldsymbol{t}) \tag{5.21}$$

We now combine (5.21) with $\boldsymbol{t}_s$ and $\boldsymbol{t}_t$ to compute $\boldsymbol{N}_{st}$ and $\boldsymbol{N}_{ts}$ by two different ways. This will allow us to find the last unknown R and in turn to find the equation for ψ. Since $\boldsymbol{N}_{st} = \boldsymbol{N}_{ts}$, we can equate the components in the basis $(\boldsymbol{t}, \boldsymbol{N},\, \boldsymbol{N}^*)$. It yields:

$$i\partial_t \psi = -\psi_{ss} - R\psi \tag{5.22}$$

$$\frac{i}{2}\psi\psi_s^* = iR_s - \frac{i}{2}\psi_s\psi^* \tag{5.23}$$

The second equation (5.23) gives

$$R_s = \frac{1}{2}(\psi\psi_s^* + \psi_s\psi^*) = \frac{1}{2}|\psi|_s^2 \tag{5.24}$$

From which we deduce by integration over *s* that

$$R(s,t) = \frac{1}{2}|\psi|^2 + A(t) \tag{5.25}$$

Where *A* is some arbitrary function of *t*. So finally, the equation for ψ (5.22) is

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$$i\frac{\partial}{\partial t}\psi = -\frac{\partial^2}{\partial s^2}\psi - \left(\frac{1}{2}|\psi|^2 + A(t)\right)\psi \qquad (5.26)$$

If we set $\frac{\Gamma}{2\pi} = \hbar$ and $m = \left(ln(L/a)\right)^{-1}$, and we retrace all the developments, we find that equation (5.26) becomes

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial s^2}\psi - \left(\frac{\hbar}{4m}|\psi|^2 + A(t)\right)\psi \qquad (5.27)$$

6 Integration over all space

We first want to find the value of the wave function at the position of the particle. As we will see, extension to any other position in space will be trivial.

If we suppose that the space around the particle core is nearly densely filled with vortex lines linking the core to the surroundings, we must integrate the above induction equation over all space (outside the particle core). We will choose a spherical coordinate system centred on the particle, defined by:

$$\begin{cases} x = r\ sin\theta\ cos\varphi \\ y = r\ sin\theta\ sin\varphi \\ z = r\ cos\theta \end{cases} \qquad (6.1)$$

ψ becomes a function of x, y, z, t. One must thus find the generalization of equation (5.27) to be integrated over the whole of space to represent the contribution from all vortex lines.

We must express $\frac{\partial^2}{\partial s^2} = \frac{\partial^2}{\partial r^2}$ in the *x, y, z* system. We first find:

$$\frac{\partial}{\partial r} = sin\theta\ cos\varphi\ \frac{\partial}{\partial x} + sin\theta\ sin\varphi\frac{\partial}{\partial y} + cos\theta\ \frac{\partial}{\partial z} \qquad (6.2)$$

and then:

$$\begin{aligned}\frac{\partial^2}{\partial r^2} = sin\theta\ cos\varphi \Bigg\{ & \left[\frac{\partial(sin\theta)}{\partial x}\ cos\varphi\ \frac{\partial}{\partial x} + sin\theta\ \frac{\partial(cos\varphi)}{\partial x}\frac{\partial}{\partial x} + sin\theta\ cos\varphi\ \frac{\partial^2}{\partial x^2}\right] \\ & + \left[\frac{\partial(sin\theta)}{\partial x}\ sin\varphi\ \frac{\partial}{\partial y} + sin\theta\ \frac{\partial(sin\varphi)}{\partial x}\frac{\partial}{\partial y} + sin\theta\ sin\varphi\ \frac{\partial^2}{\partial x \partial y}\right] \\ & + \left[\frac{\partial(cos\theta)}{\partial x}\ \frac{\partial}{\partial z} + cos\theta\ \frac{\partial^2}{\partial x \partial z}\right]\Bigg\} \\ + sin\theta\ sin\varphi \Bigg\{ & \left[\frac{\partial(sin\theta)}{\partial y}\ cos\varphi\ \frac{\partial}{\partial x} + sin\theta\ \frac{\partial(cos\varphi)}{\partial y}\frac{\partial}{\partial x} + sin\theta\ cos\varphi\ \frac{\partial^2}{\partial y \partial x}\right] \\ & + \left[\frac{\partial(sin\theta)}{\partial y}\ sin\varphi\ \frac{\partial}{\partial y} + sin\theta\ \frac{\partial(sin\varphi)}{\partial y}\frac{\partial}{\partial y} + sin\theta\ sin\varphi\ \frac{\partial^2}{\partial y^2}\right] \\ & + \left[\frac{\partial(cos\theta)}{\partial y}\ \frac{\partial}{\partial z} + cos\theta\ \frac{\partial^2}{\partial y \partial z}\right]\Bigg\}\end{aligned}$$

(6.3)

$$+\cos\theta \left\{ \left[\frac{\partial(\sin\theta)}{\partial z}\ \cos\varphi\ \frac{\partial}{\partial x} + \sin\theta\ \frac{\partial(\cos\varphi)}{\partial z}\frac{\partial}{\partial x} + \sin\theta\ \cos\varphi\ \frac{\partial^2}{\partial z\partial x}\right] + \left[\frac{\partial(\sin\theta)}{\partial z}\ \sin\varphi\ \frac{\partial}{\partial y} + \sin\theta\ \frac{\partial(\sin\varphi)}{\partial z}\frac{\partial}{\partial y} + \sin\theta\ \sin\varphi\ \frac{\partial^2}{\partial z\partial y}\right] + \left[\frac{\partial(\cos\theta)}{\partial z}\ \frac{\partial}{\partial z} + \cos\theta\ \frac{\partial^2}{\partial z^2}\right] \right\}$$

Let us write:

$$\sin\theta = \frac{\rho}{r} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \qquad \cos\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$
$$\sin\varphi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}} \qquad \cos\varphi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}} \tag{6.4}$$

This allows us to compute some of the derivatives in $\frac{\partial^2}{\partial r^2}$:

$$\frac{\partial^2}{\partial r^2} = \sin\theta\ \cos\varphi \left\{ \left[\frac{x\,z^2}{\rho\,r^3}\ \cos\varphi\ \frac{\partial}{\partial x} + \frac{y^2}{\rho^3}\ \sin\theta\frac{\partial}{\partial x} + \sin\theta\ \cos\varphi\ \frac{\partial^2}{\partial x^2}\right] + \left[\frac{x\,z^2}{\rho\,r^3}\ \sin\varphi\ \frac{\partial}{\partial y} - \frac{x\,y}{\rho^3}\ \frac{\partial(\sin\varphi)}{\partial x}\frac{\partial}{\partial y} + \sin\theta\ \sin\varphi\ \frac{\partial^2}{\partial x\partial y}\right] + \left[-\frac{x\,z}{r^3}\ \frac{\partial}{\partial z} + \cos\theta\ \frac{\partial^2}{\partial x\partial z}\right] \right\}$$
$$+ \sin\theta\ \sin\varphi \left\{ \left[\frac{y\,z^2}{\rho\,r^3}\ \cos\varphi\ \frac{\partial}{\partial x} - \frac{x\,y}{\rho^3}\sin\theta\ \frac{\partial}{\partial x} + \sin\theta\ \cos\varphi\ \frac{\partial^2}{\partial y\partial x}\right] + \left[\frac{y\,z^2}{\rho\,r^3}\ \sin\varphi\ \frac{\partial}{\partial y} + \frac{x^2}{\rho^3}\sin\theta\ \frac{\partial}{\partial y} + \sin\theta\ \sin\varphi\ \frac{\partial^2}{\partial y^2}\right] + \left[-\frac{y\,z}{r^3}\ \frac{\partial}{\partial z} + \cos\theta\ \frac{\partial^2}{\partial y\partial z}\right] \right\} \tag{6.5}$$

$$+\cos\theta \left\{ \left[-\frac{\rho\,z}{r^3}\ \cos\varphi\ \frac{\partial}{\partial x} + 0 + \sin\theta\ \cos\varphi\ \frac{\partial^2}{\partial z\partial x}\right] + \left[-\frac{\rho\,z}{r^3}\ \sin\varphi\ \frac{\partial}{\partial y} + 0 + \sin\theta\ \sin\varphi\ \frac{\partial^2}{\partial z\partial y}\right] + \left[\frac{\rho^2}{r^3}\ \frac{\partial}{\partial z} + \cos\theta\ \frac{\partial^2}{\partial z^2}\right] \right\}$$

Let us rearrange the different terms:

$$\frac{\partial^2}{\partial r^2} = \left\{ \left[\sin^2\theta\ \cos^2\varphi\ \frac{\partial^2}{\partial x^2}\ + \sin^2\theta\ \sin^2\varphi\ \frac{\partial^2}{\partial y^2} + \cos^2\theta\ \frac{\partial^2}{\partial z^2}\right] \right\}$$

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$$+\left\{\left[\frac{x^2\,z^2}{\rho\,r^4}+\frac{y^2\,z^2}{\rho\;r^4}-\frac{\rho^2\,z^2}{\rho\;r^4}\right]cos\varphi\,\frac{\partial}{\partial x}+\left[\frac{x^2\,z^2}{\rho\,r^4}+\frac{y^2\,z^2}{\rho\;r^4}-\frac{\rho^2\,z^2}{\rho\;r^4}\right]sin\varphi\,\frac{\partial}{\partial y}\right.$$

(6.6)

$$\left.+\left[\frac{x\,y^2}{r\,\rho^3}-\frac{x\,y^2}{r\,\rho^3}\right]sin\theta\,\frac{\partial}{\partial x}+\left[-\frac{y\,x^2}{r\,\rho^3}+\frac{y\,x^2}{r\,\rho^3}\right]sin\theta\,\frac{\partial}{\partial y}+\left[-\frac{z\,x^2}{r^4}-\frac{z\,y^2}{r^4}+\frac{z\,\rho^2}{r^4}\right]\frac{\partial}{\partial z}\right\}$$

$$+\left\{\frac{1}{2}\left[sin^2\theta\,sin2\varphi\,\frac{\partial^2}{\partial x\partial y}+sin2\theta\,sin\varphi\,\frac{\partial^2}{\partial z\partial y}+sin2\theta\,cos\varphi\,\frac{\partial^2}{\partial x\partial z}\right]\right\}$$

Obviously, the terms in the second curly brackets cancel each other. We must then apply this operator to the wave function and integrate the result over all space. Let us consider the free particle solution. With the benefits of insight, we know what it should look like. In rectangular coordinates it will be $\psi\sim e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$, while in spherical coordinates it will be $\psi\sim j_l(kr)Y_{lm}(\theta,\varphi)$, where $j_l(kr)$ is a spherical Bessel function and $Y_{lm}(\theta,\varphi)$ is a spherical harmonic. The two are related by: $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{m=+l}i^l\,j_l(kr)Y_{lm}^*(\hat{\boldsymbol{k}})Y_{lm}(\hat{\boldsymbol{r}})$. We will choose the first form of ψ, for it will considerably simplify the calculation of derivatives.

So, we have to integrate:

$$\int d^3r\,\frac{\partial^2}{\partial r^2}\,\psi(\boldsymbol{r},t)=\int_d^L dr\,r^2\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\,sin\theta\,\frac{\partial^2}{\partial r^2}\,\psi(r,\theta,\varphi,t)$$

(6.7)

Where d is the particle core radius. The r integration essentially runs over a sphere of radius L, centred on the particle, because $L\gg d$ and d is extremely small.

The first term of the third curly bracket in (6.6) gives:

$$-\frac{1}{2}k_xk_y\int_0^{2\pi}d\varphi\,sin2\varphi\int_0^{\pi}d\theta\,sin^3\theta\int_d^L dr\,r^2\,e^{ikr}$$

(6.8)

and it cancels due to the angular integration on φ. The next two terms vanish for similar reasons.

We then concentrate on the first curly bracket. The first term becomes:

$$-(k_x)^2\int_0^{2\pi}d\varphi\,cos^2\varphi\int_0^{\pi}d\theta\,sin^3\theta\int_d^L dr\,r^2\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$=-(k_x)^2\left(\frac{4\pi}{3}\right)\int_d^L dr\,r^2\,e^{ikr}$$

(6.9)

$$=\left(\frac{4\pi}{3}\right)\int_d^L dr\,r^2\frac{\partial^2}{\partial x^2}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\,sin\theta\int_d^L dr\,r^2\frac{\partial^2}{\partial x^2}\psi$$

After angular integrations, the second and third terms also give both a factor $\frac{4\pi}{3}$. The first curly bracket in (6.6) then leads to:

$$\frac{4\pi}{3}\int_a^L dr\, r^2 \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\psi = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, sin\theta \int_a^L dr\, r^2\, \Delta_{\boldsymbol{r}}\, \psi \tag{6.10}$$

To conclude, when integrated over (nearly) all space, equation (5.27) becomes:

$$i\hbar \int d^3r \frac{\partial}{\partial t}\psi \;= -\frac{\hbar^2}{2m}\int d^3r\, \Delta_{\boldsymbol{r}}\, \psi - \int\left(\frac{\hbar}{4m}|\psi|^2 + A(t)\right)\psi\, d^3r \tag{6.11}$$

We next study the last term of (6.11). This time the volume of integration will be a cubical box of size L. We find:

$$\int\left(\frac{\hbar}{4m}|\psi|^2 + A(t)\right)\psi\, d^3r = \left(\frac{\hbar}{4m} + A(t)\right)\int_{-L}^{+L} dx\, e^{ik_x x}\int_{-L}^{+L} dx\, e^{ik_y y}\int_{-L}^{+L} dx\, e^{ik_z z} \tag{6.12}$$

$$= \left(\frac{\hbar}{4m} + A(t)\right)\left(\frac{8}{L^3}\right) sin^3(kL)$$

When L tends to infinity, this term will decrease towards zero. This suggest the following: if future studies imply that a new non-linear term should be included in a more evolved version of the Schrödinger equation, this term will probably be very weak.

Finally, we consider a particle immerged in a force field *V(r)*. We note that the term $-(\hbar^2/2m)\Delta\psi$ represents the particle kinetic energy. So, in order to generalize to the particle total energy, we must include a term $V\psi$. In general, the integration will not vanish because the potential symmetry centre (if any) differs from the particle centre. What should remain in the integrand of (6.11) is thus the time-dependent Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = -\frac{\hbar^2}{2m}\Delta\psi(\boldsymbol{r},t) + V(\boldsymbol{r},t)\,\psi(\boldsymbol{r},t) \tag{6.13}$$

If we want to evaluate the wave equation at another position $\boldsymbol{r_b}$, with $r_{\boldsymbol{b}} = b$, we can restart the whole treatment as above, just changing the integration on r which now runs from b to L. The final result will be the same.

7 Two-particle Equation

Consider a two-particle state. Each particle has its own wave function, so that we can write the amplitude of probability of finding particle **1** in $\boldsymbol{r_1}$ and particle **2** in $\boldsymbol{r_2}$, at time t, as:

$$\psi(\boldsymbol{r_1},\boldsymbol{r_2},t) = \psi_1(\boldsymbol{r_1},t)\;\psi_2(\boldsymbol{r_2},t) \tag{7.1}$$

According to our former calculation we have:

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$$i\hbar \frac{\partial}{\partial t}\,\psi_1(\boldsymbol{r_1},t) + \left(\frac{\hbar}{4m}|\,\psi_1(\boldsymbol{r_1},t)|^2 + A_1(t)\right)\psi_1(\boldsymbol{r_1},t) = -\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial {s_1}^2}\,\psi_1(\boldsymbol{r_1},t) + V(\boldsymbol{r_1},t)\;\psi_1(\boldsymbol{r_1},t) \qquad (7.2)$$

and the same for $\psi_2(\boldsymbol{r_2},t)$. So, we find:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r_1},\boldsymbol{r_2},t) &= \left(i\hbar\frac{\partial}{\partial t}\,\psi_1(\boldsymbol{r_1},t)\right)\psi_2(\boldsymbol{r_2},t) + \psi_1(\boldsymbol{r_1},t)\left(i\hbar\frac{\partial}{\partial t}\,\psi_2(\boldsymbol{r_2},t)\right) \\ &= \left[\left(-\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial {s_1}^2} + V(\boldsymbol{r_1},t)\right)\psi_1(\boldsymbol{r_1},t)\right]\psi_2(\boldsymbol{r_2},t) \\ &\quad -\left[\left(\frac{\hbar}{4m_1}|\,\psi_1(\boldsymbol{r_1},t)|^2 + A_1(t)\right)\psi_1(\boldsymbol{r_1},t)\right]\psi_2(\boldsymbol{r_2},t) \\ &\quad +\left[\left(-\frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial {s_2}^2} + V(\boldsymbol{r_2},t)\right)\psi_2(\boldsymbol{r_2},t)\right]\psi_1(\boldsymbol{r_1},t) \\ &\quad -\left[\left(\frac{\hbar}{4m_2}|\,\psi_2(\boldsymbol{r_2},t)|^2 + A_2(t)\right)\psi_2(\boldsymbol{r_2},t)\right]\psi_1(\boldsymbol{r_1},t) \end{aligned} \qquad (7.3)$$

But $\partial^2/\partial {s_1}^2$ acts only on the coordinates of particle 1 (and $\partial^2/\partial {s_2}^2$ on particle 2), and so we obtain:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r_1},\boldsymbol{r_2},t) &= \left(-\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial {s_1}^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial {s_2}^2} + V(\boldsymbol{r_1},t) + V(\boldsymbol{r_2},t)\right)\psi_1(\boldsymbol{r_1},t)\;\psi_2(\boldsymbol{r_2},t) \\ &\quad -\left[\left(\frac{\hbar}{4m_1}|\,\psi_1(\boldsymbol{r_1},t)|^2 + A_1(t)\right) + \left(\frac{\hbar}{4m_2}|\,\psi_2(\boldsymbol{r_2},t)|^2 + A_2(t)\right)\right]\psi_1(\boldsymbol{r_1},t)\;\psi_2(\boldsymbol{r_2},t) \end{aligned} \qquad (7.4)$$

Now suppose that particle 1 is in position $\boldsymbol{r_a}$, particle 2 is in position $\boldsymbol{r_b}$ and we want to calculate the wave function in $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$. We place one coordinate system in $\boldsymbol{r_a}$ and another in $\boldsymbol{r_b}$. Then we integrate for particle 1 from $r = |\boldsymbol{r_1} - \boldsymbol{r_a}|$ to $r = L$ and for particle 2 from $r = |\boldsymbol{r_2} - \boldsymbol{r_b}|$ to $r = L$. Applying twice the reduction obtained above in paragraph 6, we get:

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r_1},\boldsymbol{r_2},t) = \left(-\frac{\hbar^2}{2m_1}\Delta_{r_1} - \frac{\hbar^2}{2m_2}\Delta_{r_2} + V(\boldsymbol{r_1},t) + V(\boldsymbol{r_2},t)\right)\psi(\boldsymbol{r_1},\boldsymbol{r_2},t) \qquad (7.5)$$

With $m_i = \left(ln(L/|\boldsymbol{r_i} - \boldsymbol{r_a}|)\right)^{-1}$ $i = 1,2$. Since $L \gg |\boldsymbol{r_1} - \boldsymbol{r_a}|$ *and* $|\boldsymbol{r_2} - \boldsymbol{r_b}|$ and because of the logarithm, m_1 and m_2 are close to $m = \left(ln(L/a)\right)^{-1}$. We could also include a posteriori an interparticle potential $V_{int}(\boldsymbol{r_1} - \boldsymbol{r_2}, t)$. Finally, we could also easily extend this derivation to the multi-particle systems.

8 The Pauli Exclusion Principle

If the particles in (7.5) are indistinguishable fermions, then a principle due to Pauli imposes that the wave function should be anti-symmetrized as:

$$\psi(\boldsymbol{r_1},\boldsymbol{r_2},t)=\frac{1}{\sqrt{2}}\big(\psi(\boldsymbol{r_1},\boldsymbol{r_2},t)-\psi(\boldsymbol{r_2},\boldsymbol{r_1},t)\big) \tag{8.1}$$

But this is postulated. We should search a logical explanation for it.

In our model, particles are vortex structures. Consequently, particles necessarily experience a spin-spin interaction. This form of interaction has been postulated several times [12,13,14], and many experiences have been devised to observe it, but all have failed so far. In [2], we show that the experiment geometries were badly conceived and we propose a new one that should work.

Several forms have been proposed for a spin-spin interaction potential. For example [15]:

$$\begin{gathered} V_a = g_a\,(\sigma_1\cdot\sigma_2)e^{-r/\lambda} \\ V_b = g_b\,(\sigma_1\times\sigma_2)\,r\left(1+\frac{r}{\lambda}\right)\frac{1}{r^2}e^{-r/\lambda} \end{gathered} \tag{8.2}$$

where the g's are coupling constants, λ is the interaction range and the $\sigma's$ are the spins. If we choose the second (anti-symmetrical) form, its only presence in the potential energy of the Schrödinger equation is sufficient to kill the symmetrical part of the wave function. We suggest that this could explain the anti-symmetrisation principle.

9 Conclusion

Our particle model [2] was built initially to explain the spin ½ behaviour. It is quite satisfying to find that it also leads naturally to the Schrödinger equation and maybe to a new physical interpretation of quantum intrication. This means that all the older particle models cited in section 2 probably contains some parts of truth. It also suggests that our synthetic model is on the right track.

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