About the properties of prime numbers in the form $md^m + 1$ and $d^m + 1$

By Shazly Abdullah

ABSTRACT .In this study we used an algebraic method that uses elementary algebra and binomial theorem. To create series We used these series to study the prime numbers of the from $p = md^m + 1$ and $q = d^m + 1$,We found several characteristics . for example, we proved If , p prime number and $p = md^m + 1$where then $m^{d^m} \equiv -1(mod\ \ p)$ .We also obtained several results in finite series .

Key words: binomial theorem, series , prime numbers , finite series

1.INTRODUCTION

Numbers of the form $b_n = k2^n + 1$called broth and $C_n = n2^n + 1$in There are also other similar formulas, such as Mersenne counter and Fermat, such numbers. It is known that they provide us with large prime numbers, [see James Tattersall 146] In this paper, elementary algebra and binomial theorem, and difference of tow nth power are used to created finite series in an algebraic method, then we used series to create congruence with specific properties. Through this process, we reached the theorem. 1 We used Theorem 1 to study the prime numbers in the form $p = md^m + 1$ and $q = d^m + 1$of which we found about these numbers theorem.2 and several results in finite series. According binomial theorem and difference of tow nth power theorem if n a positive integer and x y real numbers then [see K.H 22]

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^j y^{n-j}$$

And

$$x^n - y^n = (x - y)\sum_{j=1}^{n} x^{n-j} y^{j-1}$$

2.basic series

Theorem.1 let a and d real number and n a positive integer then

$$a^{n-1}(d^n - 1) = \frac{d-1}{a-1}(a^n - 1) + (d-1)\sum_{j=1}^{n-1} \frac{(ad-1)^j}{(a-1)^{j+1}}\left(a^n - \left(\binom{n}{0} + \binom{n}{1}(a-1) \dots\dots\dots \binom{n}{j}(a-1)^j\right)\right)$$

Theorem.2 let p q primers numbers a m a positive integers $p = md^m + 1$ and $q = d^m + 1$

$$\begin{cases} m^{d^m} \equiv 1(mod\ pq) & if\ \ d\ in\ even \\ m^{d^m} \equiv -1(mod\ p) & if\ \ d\ is\ odd \end{cases}$$

This section first we will create the basic series

Basic series. let $k, g, u,$ *real numbers and m constant then*

$$L_n(k,g,u) = V_n{}^n(k,g,u) + S_n(k,g)$$

Where

$$L_n\ (k,g,u) = (u-k+g)^n - m(1+g)^n$$

And

$$V_n{}^n(k,g,u) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(u^{n-j} - m\right)(k-g)^j$$

And

$$S_n(k,g) = mk \sum_{j=0}^{n-1} (-1)^{j+1} \frac{(k-g)^j}{g^{j+1}} \left((1+g)^n - \left(\binom{n}{0} + \binom{n}{1} g + \binom{n}{2} g^2 \dots\dots\dots . \binom{n}{j} g^j \right) \right)$$

Proof. let $g\ k\ u$, real numbers then according to difference of tow nth power theorem we have that

$$(k-g)^n - (-g)^n = \mathrm{k} \sum_{j=1}^{n} (k-g)^{j-1}(-g)^{n-j}$$

Then

$$-(-g)^n = -(k-g)^n + \mathrm{k} \sum_{j=1}^{n} f^{j-1}(k,h) g^{n-j}(-h)$$

let $q \in R$, $n \in N$ where m constant then by multiplying m and adding $u^q(k-g)^n$ from both sides

$$u^q(k-g)^n - m(-g)^n = u^q(k-g)^n - m(k-g)^n + \mathrm{m\ k} \sum_{j=1}^{n} (k-g)^{j-1}(-g)^{n-j}$$

Then

$$(1) \qquad u^q(k-g)^n - m(-g)^n = (u^q - m)(k-g)^n + m\ \mathrm{k} \sum_{j=1}^{n} (k-g)^{j-1}(-g)^{n-j}$$

According binomial theorem

$$(u-k+g)^n = u^n - \binom{n}{1} u^{n-1}(k-g) + \binom{n}{2} u^{n-2}(k-g)^2 - \binom{n}{3} u^{n-3}(k-g)^3 \dots\dots\dots\dots\dots\dots\dots (k-g)^n$$

And

$$m(1+g)^n = m + m\binom{n}{1} g + m\binom{n}{2} g^2 + m\binom{n}{3} g^3 \dots\dots\dots\dots\dots . mg^n$$

By subtracting $m(k-g)^n\ from\ (u-k+g)^n$ then

$$(u-k+g)^n - m(1+g)^n$$
$$= u^n - m - \binom{n}{1} u^{n-1}(k-g) - m\binom{n}{1} g + \binom{n}{2} u^{n-2}(k-g)^2 - m\binom{n}{2} g^2$$
$$- \binom{n}{3} u^{n-3}(k-g)^3 - m\binom{n}{3} g^3 \dots\dots\dots\dots\dots..(k-g)^n - mg^n$$

By extracting the common factor $\binom{n}{j}$ between the terms

(2) $$(u-k+g)^n - m(1+g)^n$$
$$= u^n - m - \binom{n}{1}(u^{n-1}(k-g) + mg) + \binom{n}{2}(u^{n-2}(k-g)^2 - mg^2)$$
$$- \binom{n}{3}(u^{n-3}(k-g)^3 + mg^3) \dots\dots\dots\dots ((k-g)^n - mg^n)$$

So we note in (2) limit (1) equal $u^n - m$ and limit (2) equal $\binom{n}{1}(u^{n-1}(k-g)+mg)$ and limit 2 equal $\binom{n}{2}(u^{n-2}(k-g)^2 - mg^2)$ and 3 equal $\binom{n}{3}(u^{n-3}(k-g)^3 + mg^3)$ and last limit $(k-g)^n - mg^n$ then

According equation (1)

$$u^q(k-g)^n - m(-g)^n = (u^q - m)(k-g)^n + mk\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

let

(3) $$w^q{}_n(k,g,u) = u^q(k-g)^n - m(-g)^n$$

(4) $$z^q{}_n(k,g,u) = (u^q - m)(k-g)^n$$

(5) $$c_n(k,g) = m\mathrm{k}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

So

(3) $$W_n{}^q(k,g,u) = Z_n{}^q(\mathrm{k},g,\mathrm{u}) + C_n(k,g)$$

From (3) and term (1) in equation (2)

$$\binom{n}{0}(u^n - m) = \binom{n}{0} W_0{}^{n-0}(k,g,u)$$

From equation (3) and term (2) in equation (2)

$$\binom{n}{1}(u^{n-1}(k-g) + mg) = \binom{n}{1} W_1{}^{n-1}(k,g,u)$$

term (3)

$$\binom{n}{2}(u^{n-2}(k-g)^2 - mg^2) = \binom{n}{2} W_2{}^{n-2}(k,g,u)$$

term (4) in equation (2)

$$\binom{n}{3}(u^{n-3}(k-g)^3 + mg^3) = \binom{n}{3} W_3{}^{n-3}(k,g,u)$$

Last term

$$\binom{n}{n}((k-g)^n - mg^n) = \binom{n}{n} W_n{}^{n-n}(k,g,u)$$

So from (1) and (3) equations

$$(u-k+g)^n - m(1+g)^n = \sum_{j=0}^{n}(-1)^j\binom{n}{j} w^{n-j}{}_j(k,g,u)$$

Let

$$L_n(k,g,u) = (u-k+g)^n - m(1+g)^n$$

Then

$$L_n(k,g,u) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} w^{n-j}{}_j(k,g,u)$$

From equation(4) $w^q{}_n(k,g,u) = z_n{}^q(k,g,u) + c_n(k,g)$ then we have that

(4) $$L_n(k,g,u) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} z_j{}^{n-j}(k,g,u) + \sum_{j=0}^{n} (-1)^j \binom{n}{j} c_j(k,g)$$

We note from the equation (4)

$$Z_n{}^q(\mathrm{k},g,\mathrm{u}) \;\; = (u^q - m)(k-g)^n$$

And

$$C_n(k,g) = \; m\, k \sum_{j=1}^{n} (k-g)^{j-1}(-g)^{n-j}$$

Then

$$L_n(k,g,u) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(u^{n-j} - m\right)(k-g)^j + \mathrm{mk} \sum_{j=1}^{n} \sum_{r=1}^{j} (-1)^j \binom{n}{j} (k-g)^{r-1}(-g)^{j-r}$$

Let

$$V^n{}_n(k,g,u) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(u^{n-j} - m\right)(k-g)^j$$

And

$$S_n(k,g) = \mathrm{mk} \sum_{j=1}^{n} \sum_{r=1}^{j} (-1)^j \binom{n}{j} (k-g)^{r-1}(-g)^{j-r}$$

Then we have that

(5) $$L_n(k,g,u) = V_n{}^n(k,g,u) + S_n(k,g)$$

we find in $S_n(k,g)$ tow signs $(-1)^j(-1)^{j-r} = (-1)^r$ if r j even or odd so they can by combined in $(-1)^r$ then we have

$$S_n(k,g) = \mathrm{mk} \sum_{j=1}^{n} \sum_{r=1}^{j} (-1)^r \binom{n}{j} (k-g)^{r-1} g^{j-r}$$

Where

$$s_n(k,g) = \mathrm{mk}\left(\sum_{r=1}^{1} (-1)^r \binom{n}{1} (k-g)^{r-1} g^{1-r} + \sum_{r=1}^{2} (-1)^r \binom{n}{2} (k-g)^{r-1} g^{2-r} \right.$$
$$\left. + \sum_{r=1}^{3} (-1)^r \binom{n}{3} (k-g)^{r-1} g^{3-r} \dots\dots\dots\dots. \sum_{r=1}^{n} (-1)^r \binom{n}{n} (k-g)^{r-1} g^{n-r} \right)$$

In $S_n(k,g)$ a all compound terms have been dismantled note if we add for every first term in the complex term we find that $-\left(\binom{n}{1}+\binom{n}{2}g\dots\dots\dots\binom{n}{n}g^{n-1}\right)$ then we adding the terms to include that $(k-g)$ finding that $(k-g)$ $\left(\binom{n}{2}+\binom{n}{3}g\dots\dots\binom{n}{n}g^{n-2}\right)$ then the term that include $(k-g)^2$ we find that $(k-g)^2\left(-\left(\binom{n}{3}+\binom{n}{4}g\dots\dots\binom{n}{n}g^{n-j-1}\right)\right)$ if the method is equal all the terms can be added $1\le j\le n-1$ until we reach the last terms $(k-g)^{n-1}$ then

(6) $$s_n(k,g)$$
$$= \mathrm{mk}\Bigg(-\left(\binom{n}{1}+\binom{n}{2}g+\binom{n}{3}g^2\dots\dots\dots\binom{n}{n}g^{n-1}\right)$$
$$+(k-\mathrm{g})\left(\left(\binom{n}{2}+\binom{n}{3}g+\binom{n}{4}g^2+\binom{n}{5}g^3\dots\dots\dots\binom{n}{n}g^{n-2}\right)\right)$$
$$-(k-g)^2\left(\binom{n}{3}+\binom{n}{5}g+\binom{n}{6}g^2+\binom{n}{7}g^3\dots\dots\dots\binom{n}{n}g^{n-3}\right)\dots\dots\dots\binom{n}{n}g^{n-n}\Bigg)$$

Using the binomial theorem it is possible to abbreviate all the terms that include, $(k-g)$ *and* $(k-g)^2$ and $(k-g)^3$until we reach the last term $(k-g)^{n-1}$, we notice that

$$-\left(\binom{n}{1}+\binom{n}{2}g+\binom{n}{3}g^2\dots\dots\dots\binom{n}{n}g^{n-1}\right)=-\frac{(1+g)^n-\binom{n}{0}}{g}$$

$$(k-g)\left(\binom{n}{2}+\binom{n}{3}\mathrm{g}\dots\dots\dots\binom{n}{n}g^{n-2}\right)=(k-g)\left(\frac{(1+g)^n-\binom{n}{0}-\binom{n}{1}g}{g^2}\right)$$

$$-(k-g)^2\left(\binom{n}{3}+\binom{n}{4}g\ \dots\dots\dots\binom{n}{n}g^{n-3}\right)=-(k-g)^2\left(\frac{(1+g)^n-\binom{n}{0}-\binom{n}{1}g-\binom{n}{2}g^2}{g^3}\right)$$

From equation (7) we have that

$$S_n(k,g)=\mathrm{m}k\sum_{j=0}^{n-1}(-1)^{j+1}\frac{(k-g)^j}{g^{j+1}}\left((1+g)^n-\left(\binom{n}{0}+\binom{n}{1}g+\binom{n}{2}g^2\dots\dots\dots\binom{n}{j}g^j\right)\right)$$

we have that

(7) $$L_n(k,g,u)=(u-\mathrm{k}+\mathrm{g})^n-m(1+g)^n$$

(8) $$V^n{}_n(k,g,u)=\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(u^{n-j}-m\right)(k-g)^j$$

(9) $$S_n(k,g)$$
$$=\mathrm{mk}\sum_{j=0}^{n-1}(-1)^{j+1}\frac{(k-g)^j}{g^{j+1}}\left((1+g)^n-\left(\binom{n}{0}+\binom{n}{1}g+\binom{n}{2}g^2\dots\dots\dots\dots\binom{n}{j}g^j\right)\right)$$

3. proof theorem.1

In this section we will use the basic series $L_n(k,g,u)=V^n{}_n(k,g,u)+S_n(k,g)$ in prove the theorem.1 then according basic infinite series

Proof.theorem.1 if $u=1$ in $V_n{}^n(k,g,u)$ we have that

$$V_n{}^n(k,g,1)=\sum_{j=1}^{n}(-1)^j\binom{n}{j}\left((1)^{n-j}-1\right)(k-g)^j=0$$

Then according equations basic series

$$L_n(k,g,1)=V_n{}^n(k,g,1)+S_n(k,g)$$

Then

$$L_n(k,g,1)=S_n(k,g)$$

Then according to the equations, (7, 8 ,.9) we find that

$$L_n(k,g,1)=S_n(k,g)$$

$$(1-k+g)^n-(1+g)^n=\mathrm{k}\sum_{j=0}^{n-1}(-1)^{j+1}\frac{(k-g)^j}{g^{j+1}}\left((1+g)^n-\left(\binom{n}{0}+\binom{n}{1}\mathrm{g}\ldots\ldots\binom{n}{j}g^j\right)\right)$$

Let a d a positive integers where

$$g=a-1$$
$$k=-a\mathrm{d}+a$$

Then

$$(1+ad-a+a-1)^n-(1+a-1)^n$$
$$=(-ad+a)\sum_{j=0}^{n-1}(-1)^{j+1}\frac{(-ad+a-a+1)^j}{(a-1)^{j+1}}\left((1+a-1)^n-\left(\binom{n}{0}+\binom{n}{1}(a-1)\ldots\ldots\binom{h}{j}(a-1)^j\right)\right)$$

We have that

$$(ad)^n-a^n$$
$$=(-ad+a)\sum_{j=0}^{n-1}(-1)^{j+1+j}\frac{(ad-a)^j}{(a-1)^{j+1}}\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(a-1)\ldots\ldots\binom{n}{j}(a-1)^j\right)\right)$$

Then

$$a^{n-1}(d^n-1)=\frac{(-1)^{2j+1}-d+1}{a-1}(a^n-1)$$
$$-(\mathrm{d}-1)\sum_{j=1}^{n-1}\frac{(-1)^{2j+1}(ad-1)^j}{(a-1)^{j+1}}\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(a-1)\ldots\ldots\binom{n}{j}(a-1)^j\right)\right)$$

Now we have that

$$a^{n-1}(d^n-1)=\frac{d-1}{a-1}(a^n-1)$$
$$+(d-1)\sum_{j=1}^{n-1}\frac{(ad-1)^j}{(a-1)^{j+1}}\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(a-1)\ldots\ldots\binom{n}{j}(a-1)^j\right)\right)$$

Note the negative sign in the equation

$$a^{n-1}(d^n-1)=\frac{d-1}{a-1}(a^n-1)$$
$$+(d-1)\sum_{j=1}^{n-1}\frac{(ad-1)^j}{(a-1)^{j+1}}\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(a-1)\ldots\ldots\ldots\binom{n}{j}(a-1)^j\right)\right)$$

Then

$$a^{n-1}(d^n-1)=\frac{d-1}{a-1}(a^n-1)$$
$$+(ad-1)\left((d-1)\sum_{j=1}^{n-1}\frac{(ad-1)^{j-1}}{(a-1)^{j+1}}\left(a^n-\binom{n}{0}+\binom{n}{1}(a-1)\ldots\ldots\binom{n}{j}(a-1)^j\right)\right)$$

Proof theorem.2

In this section we will prove theorem . 2 using theorem . 1 but before that we mention according to Euler's theorem $a^{\varphi(n)}\equiv 1(mod\ n)\ \ where\ (a,n)=1$ and $\varphi(n)$ is Euler function see proof Euler theorem in [ ]

Proof. Theorem.2 If $d=-k$ in theorem.1 we have

$$a^{n-1}((-k)^n-1)=-\frac{k+1}{a-1}(a^n-1) \tag{10}$$
$$-(ka+1)\left((-k-1)\sum_{j=1}^{n-1}\frac{(-1)^{j-1}(ak+1)^{j-1}}{(a-1)^{j+1}}\left(a^n-\binom{n}{0}-\binom{n}{1}(a-1)\ldots\ldots\binom{n}{j}(a-1)^j\right)\right)$$

Let in equation (11) $k=m\ \ and\ a=n=d^m$ then

$$\mathrm{d}^{m(d^m-1)}\left((-m)^{d^m}-1\right)=-\frac{m+1}{d^m-1}\left(\mathrm{d}^{md^m}-1\right)$$
$$+(md^m+1)\left((m+1)\sum_{j=1}^{n-1}\frac{(-1)^{j-1}(md^m+1)^{j-1}}{(d^m-1)^{j+1}}\left((d^m)^n-\binom{n}{0}-\binom{n}{1}(d^m-1)..\binom{n}{j}(d^m-1)^j\right)\right)$$

Let p q prime numbers where $p=md^m+1$ and $q=d^m+1$ then we have that

$$\mathrm{d}^{m(d^m-1)}((-m)^{\mathrm{q}-1}-1)=-\frac{m+1}{d^m-1}(\mathrm{d}^{p-1}-1) \tag{11}$$
$$+\mathrm{p}\left((m+1)\sum_{j=1}^{n-1}\frac{(-1)^{j-1}\,\mathrm{p}^{j-1}}{(d^m-1)^{j+1}}\left((d^m)^n-\binom{n}{0}-\binom{n}{1}(d^m-1)..\binom{n}{j}(d^m-1)^j\right)\right)$$

Let in equation (19) V equal

$$V=(m+1)\sum_{j=1}^{n-1}\frac{(-1)^{j-1}\,\mathrm{p}^{j-1}}{(d^m-1)^{j+1}}\left((d^m)^n-\binom{n}{0}-\binom{n}{1}(d^m-1)..\binom{n}{j}(d^m-1)^j\right) \tag{12}$$

Then from equation (12) and (11) we have that

$$\mathrm{d}^{m(d^m-1)}((-m)^{\mathrm{q}-1}-1)=-\frac{m+1}{d^m-1}(\mathrm{d}^{p-1}-1)+pV \tag{13}$$

So according equation (11) $q=d^m+1$ if d is odd we have $(-m)^{q-1}=m^{d^m}$ then

$$\mathrm{d}^{m(d^m-1)}(\mathrm{m}^{\mathrm{q}-1}-1)=-\frac{m+1}{d^m-1}(\mathrm{d}^{p-1}-1)+pV$$

According Euler theorem

$$m^{p-1}\equiv 1(mod\ p)$$

And

$$m^{q-1}\equiv 1(mod\ q)$$

So from equation (13) and Euler theorem we have

$$m^{q-1}\equiv 1(mod\ \ pq)$$

But $q=d^m+1$ where d is even

$$m^{d^m}\equiv 1(mod\ \ pq)$$

And if d is even in equation (13) we have that $(-m)^{q-1}=m^{q-1}$ because $q=d^m+1$

$$-\mathrm{d}^{m(d^m-1)}(\mathrm{m}^{\mathrm{q}-1}+1)=-\frac{m+1}{d^m-1}(\mathrm{d}^{p-1}-1)+pV \tag{14}$$

From Euler theorem

$$d^{p-1}\equiv 1(mod\ \ p)$$

From equation (14) and Euler theorem

$$m^{q-1}\equiv -1(mod\ \ p)$$

Then

$$m^{d^m}\equiv -1(mod\ \ p)$$

End of proof

Lemma.1 let $n\in\mathbb{N}$where d is real number then

$$(d^{n-1}-1)(d^n-1)=\sum_{j=1}^{n-1}(d+1)^j\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(d-1)\ldots\ldots\ldots\binom{n}{j}(d-1)^j\right)\right)$$

Proof. From theorem.1 we have

$$a^{n-1}(d^n-1)=\frac{d-1}{a-1}(a^n-1)$$
$$+(d-1)\sum_{j=1}^{n-1}\frac{(ad-1)^j}{(a-1)^{j+1}}\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(a-1)\ldots\ldots\ldots\binom{n}{j}(a-1)^j\right)\right)$$

Let $a=d$ in theorem.1

$$d^{n-1}(d^n-1)=\frac{d-1}{d-1}(d^n-1)$$
$$+(d-1)\sum_{j=1}^{n-1}\frac{(d^2-1)^j}{(d-1)^{j+1}}\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(d-1)\ldots\ldots\ldots\binom{n}{j}(d-1)^j\right)\right)$$

Then

$$d^{n-1}(d^n-1)-(d^n-1)$$
$$=\frac{d-1}{d-1}\sum_{j=1}^{n-1}\left(\frac{d^2-1}{d-1}\right)^j\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(d-1)\ldots\ldots\ldots\binom{n}{j}(d-1)^j\right)\right)$$

We have

$$(d^{n-1}-1)(d^n-1)=\sum_{j=1}^{n-1}(d+1)^j\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(d-1)\ldots\ldots\ldots\binom{n}{j}(d-1)^j\right)\right)$$

End of proof

References

.

1. K. M. Rose, Elementary Number Theory" 4th Edition Addison- Wesley (2000)
2. James J. Tattersall Elementary Number Theory in Nine Chapters Cambridge University Press 2005

Student: Shazly Abdullah Fdl Faculty of mathematics sciences & statistics Aleenlain University Sudan E-mail address: Shazlyabdullah3@gmail.com