THE TOWER FUNCTION AND APPLICATIONS

THEOPHILUS AGAMA

Abstract. In this paper we study an extension of the Euler totient function to the rationals and explore some applications. In particular, we show that

$$\#\{\frac{m}{n}\leq \frac{a}{b} \mid m\leq a,\ n\leq b,\ \gcd(m,a)=\gcd(n,b)=1,\ \gcd(n,a)>1$$
$$\vee\ \gcd(m,b)>1\ \vee\ \gcd(m,n)>1\}=\sum_{\substack{\frac{m}{n}\leq \frac{a}{b}\\ mn\leq ab\\ m>a,n\leq b\ \vee\ m\leq a,n>b\ \vee\ \gcd(m,n)>1\\ \gcd(mn,ab)=1}}1$$

provided $\gcd(a,b)=1$.

1. Introduction

The Euler totient function φ is one of the most useful function in number theory and all of mathematics, first introduced and studied by Leonard Euler [2]. It is defined in a standard manner as the formal sum

$$\varphi(n):=\sum_{\substack{k\leq n\\ \gcd(k,n)=1}}1$$

which basically counts the number of positive integers no more than a fixed $n\in\mathbb{N}$ and co-prime to n. Euler totient function has several subtle properties (see for instance [1]) making it rife and an indispensable function in number theory.
In this note we study an extension of the Euler totient function to the rational numbers. As a consequence, we prove some identities concerning the size of certain sets. We show that the cardinalites of these two seemingly disparate sets are the same under some co-primality condition. In particular we obtain the identity

Theorem 1.1. *The identity holds*

$$\#\{\frac{m}{n}\leq \frac{a}{b} \mid m\leq a,\ n\leq b,\ \gcd(m,a)=\gcd(n,b)=1,\ \gcd(n,a)>1$$
$$\vee\ \gcd(m,b)>1\ \vee\ \gcd(m,n)>1\}=\sum_{\substack{\frac{m}{n}\leq \frac{a}{b}\\ mn\leq ab\\ m>a,n\leq b\ \vee\ m\leq a,n>b\ \vee\ \gcd(m,n)>1\\ \gcd(mn,ab)=1}}1$$

provided $\gcd(a,b)=1$.

Date: November 30, 2021.
2000 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.
Key words and phrases. Euler; totient; rationals.

2. **The tower function**

In this section we introduce and study the **tower** function. We examine some elementary properties of this function and explore some applications.

Definition 2.1. By the tower function $\nu : \mathbb{Q} \longrightarrow \mathbb{N}$, we mean the counting function

$$\nu(\frac{a}{b}) := \#\{\frac{m}{n} \leq \frac{a}{b} \mid m \leq a,\ n \leq b,\ \gcd(m,a) = \gcd(n,b) = \gcd(n,a) = \gcd(m,b) = \gcd(m,n) = 1\}.$$

2.1. **Properties of the tower function.** In this section we examine some elementary properties of the tower function $\nu : \mathbb{Q} \longrightarrow \mathbb{N}$.

Proposition 2.2. (i) If $b = 1$ then $\nu(a) = \varphi(a)$ where φ is the Euler totient function.

(ii)

$$\nu(\frac{a}{b}) = \sum_{\substack{\frac{m}{n} \leq \frac{a}{b} \\ m \leq a,\ n \leq b \\ \gcd(m,a)=\gcd(n,b)=1}} 1 - \#\{\frac{m}{n} \leq \frac{a}{b} \mid m \leq a,\ n \leq b,\ \gcd(m,a) = \gcd(n,b) = 1, \gcd(n,a) > 1\ \vee\ \gcd(m,b) > 1\ \vee\ \gcd(m,n) > 1\}$$

(iii)

$$\nu(\frac{a}{b}) = \sum_{\substack{\frac{m}{n} \leq \frac{a}{b} \\ mn \leq ab \\ \gcd(mn,ab)=1}} 1 - \sum_{\substack{\frac{m}{n} \leq \frac{a}{b} \\ mn \leq ab \\ m>a, n\leq b\ \vee\ m\leq a, n>b\ \vee\ \gcd(m,n)>1 \\ \gcd(mn,ab)=1}} 1.$$

Proof. (i) Let us take $b = 1$ then we have the collapsing of the quantity

$$\begin{aligned}\nu(a) &= \nu(\frac{a}{b}) \\ &= \#\{\frac{m}{n} \leq \frac{a}{b} \mid m \leq a,\ n \leq b,\ \gcd(m,a) = \gcd(n,b) = \gcd(n,a) \\ &= \gcd(m,b) = 1\} \\ &= \#\{m \leq a \mid \gcd(m,a) = 1\} = \varphi(a).\end{aligned}$$

(ii) Property (ii) is an easy consequence of the Definition 2.1.

(iii) We note that we can collapse the quantity

$$\begin{aligned}
\nu(\frac{a}{b}) :&= \#\{\frac{m}{n} \leq \frac{a}{b} \mid m \leq a,\ n \leq b,\ \gcd(m,a) = \gcd(n,b) = \gcd(n,a) \\
&= \gcd(m,b) = \gcd(m,n) = 1\} \\
&= \#\{\frac{m}{n} \leq \frac{a}{b} \mid mn \leq ab,\ m \leq a, n \leq b,\ \gcd(mn,ab) = 1\} \\
&= \sum_{\substack{\frac{m}{n} \leq \frac{a}{b} \\ mn \leq ab \\ m \leq a, n \leq b \\ \gcd(mn,ab)=1 \\ \gcd(m,n)=1}} 1 \\
&= \sum_{\substack{\frac{m}{n} \leq \frac{a}{b} \\ mn \leq ab \\ \gcd(mn,ab)=1}} 1 - \sum_{\substack{\frac{m}{n} \leq \frac{a}{b} \\ mn \leq ab \\ m>a, n \leq b\ \vee\ m \leq a, n>b\ \vee\ \gcd(m,n)>1 \\ \gcd(mn,ab)=1}} 1 \\
&= \sum_{\substack{\frac{m}{n} \leq \frac{a}{b} \\ mn \leq ab \\ \gcd(mn,ab)=1}} 1 - \sum_{\substack{\frac{m}{n} \leq \frac{a}{b} \\ mn \leq ab \\ m>a, n \leq b\ \vee\ m \leq a, n>b\ \vee\ \gcd(m,n)>1 \\ \gcd(mn,ab)=1}} 1
\end{aligned}$$

thereby ending the proof.

□

Remark 2.3. The tower function can be viewed as an extension of the euler totient function to the rationals. This assertion is obviously exemplified in Proposition 2.2.

2.2. **Applications of the tower function.** In this section we launch an application of the tower function. We use this function to prove some identity involving the cardinality of a certain set.

Theorem 2.4. *If* $\gcd(a,b) = 1$ *then we have*

$$\begin{aligned}
&\#\{\frac{m}{n} \leq \frac{a}{b} \mid m \leq a,\ n \leq b,\ \gcd(m,a) = \gcd(n,b) = 1,\ \gcd(n,a) > 1 \\
&\vee\ \gcd(m,b) > 1\ \vee\ \gcd(m,n) > 1\} = \sum_{\substack{\frac{m}{n} \leq \frac{a}{b} \\ mn \leq ab \\ m>a, n \leq b\ \vee\ m \leq a, n>b\ \vee\ \gcd(m,n)>1 \\ \gcd(mn,ab)=1}} 1
\end{aligned}$$

Proof. Appealing to Proposition 2.2, we obtain the upper bound

$$\nu(\frac{a}{b}) = \sum_{\substack{\frac{m}{n}\leq \frac{a}{b}\\ m\leq a,\ n\leq b\\ \gcd(m,a)=\gcd(n,b)=1}} 1 - \#\{\frac{m}{n}\leq \frac{a}{b} \mid m\leq a,\ n\leq b,\ \gcd(m,a)=\gcd(n,b)=1,$$
$$\gcd(n,a)>1 \ \vee\ \gcd(m,b)>1 \ \vee\ \gcd(m,n)>1\}$$
$$\leq \sum_{\substack{(m,n)\in\mathbb{N}^2\\ m\leq a,\ n\leq b\\ \gcd(m,a)=\gcd(n,b)=1}} 1 - \#\{\frac{m}{n}\leq \frac{a}{b} \mid m\leq a,\ n\leq b,\ \gcd(m,a)=\gcd(n,b)=1,$$
$$\gcd(n,a)>1 \ \vee\ \gcd(m,b)>1 \ \vee\ \gcd(m,n)>1\}$$
$$= \varphi(a)\varphi(b) - \#\{\frac{m}{n}\leq \frac{a}{b} \mid m\leq a,\ n\leq b,\ \gcd(m,a)=\gcd(n,b)=1,$$
$$\gcd(n,a)>1 \ \vee\ \gcd(m,b)>1 \ \vee\ \gcd(m,n)>1\} \tag{2.1}$$

Similarly by appealing to Proposition 2.2, we obtain the upper bound

$$\nu(\frac{a}{b}) = \sum_{\substack{\frac{m}{n}\leq \frac{a}{b}\\ mn\leq ab\\ \gcd(mn,ab)=1}} 1 - \sum_{\substack{\frac{m}{n}\leq \frac{a}{b}\\ mn\leq ab\\ m>a,n\leq b\ \vee\ m\leq a,n>b\ \vee\ \gcd(m,n)>1\\ \gcd(mn,ab)=1}} 1$$
$$\leq \sum_{\substack{mn\leq ab\\ \gcd(mn,ab)=1}} 1 - \sum_{\substack{\frac{m}{n}\leq \frac{a}{b}\\ mn\leq ab\\ m>a,n\leq b\ \vee\ m\leq a,n>b\ \vee\ \gcd(m,n)>1\\ \gcd(mn,ab)=1}} 1$$
$$\leq \varphi(ab) - \sum_{\substack{\frac{m}{n}\leq \frac{a}{b}\\ mn\leq ab\\ m>a,n\leq b\ \vee\ m\leq a,n>b\ \vee\ \gcd(m,n)>1\\ \gcd(mn,ab)=1}} 1 \tag{2.2}$$

By subtracting (2.1) from (2.2) and leveraging the condition $\gcd(a,b)=1$ the claimed equality follows immediately. □

[1]

References

1. Hardy, Godfrey Harold and Wright, Edward Maitland and others, *An introduction to the theory of numbers*, Oxford university press, 1979.
2. Euler, Leonhard and Diener, Artur and Aycock, Alexander, *Theoremata arithmetica nova methodo demonstrata*, arXiv preprint arXiv:1203.1993, 2012.

Department of Mathematics, African Institute for Mathematical Sciences, Ghana
E-mail address: theophilus@aims.edu.gh/emperordagama@yahoo.com

1