A New Method for the Cubic Polynomial Equation

Jose Risomar Sousa

November 6, 2021

Abstract

I present a method to solve the general cubic polynomial equation based on six years of research that started back in 1985 when, in the fifth grade, I first learned of Bhaskara's formula for the quadratic equation. I was fascinated by Bhaskara's formula and naively thought I could replicate his method for the third degree equation, but only succeeded in 1990, after countless failed attempts. The solution involves a simple transformation to form a cube and which, by chance, happens to reduce the degree of the equation from three to two (which seems to be the case of all polynomial equations that admit solutions by means of radicals). I also talk about my experiences trying to communicate these results to mathematicians, both at home and abroad.

1 Introduction

In Brazil, students learn how to solve linear equations and system of equations (on $\mathbb{Q}$) in or around the fifth grade, before being introduced to real numbers and moving on to the more challenging subject of non-linear equations in the sixth grade.

As a student with a certain gift for mathematics (even though it wasn't always like that, I struggled in the very beginning), I used to read math textbooks from more advanced grades and that's how I first learned about Bhaskara's formula for the quadratic equation, when I was in the fifth grade in 1985. With my very curious mind, I would then become so obsessed with that formula that I spent years and years trying to find a similar formula for the cubic polynomial equation. I would scratch pages and more pages of books and copybooks with vain or naive attempts to solve the cubic equation, trying to imitate the process developed by Bhaskara.

But as the saying goes, search and you will find, and after trying lots and lots of different approaches, some creative but insufficient, and some plain wrong, I finally succeeded in 1990, when I was in my second year of high school in Brazil. I was extremely happy and filled with excitement for my discovery. The winning method I discovered, along with some history background, is what this paper will talk about.

As a teenager, like most people of that age, I was a somewhat naive dreamer, I imagined that my discovery could perhaps help me to make some money, as I had been working in a kind of low paid job at a bank. So back in 1990, when I wrote a letter to professor Luis Barco, from the School of Communications and Arts of the University of São Paulo, (ECA-USP), whom I knew from his column in a Brazilian pop-science magazine called Super Interessante (*"Super Interesting"*), my boss at the bank got a phone call from professor Barco, and we both mistakenly imagined it would be about a job opportunity for me. We then scheduled a meeting at the University to chat, and it was my first time going into the very beautiful, green and pleasant campus of the University of São Paulo. I was overwhelmed with the academic atmosphere and spent some time looking at stuff around the ECA premises. Little did I know that two years later I would be attending that University myself, whose selection process is notoriously competitive and hard, once that is the best university in the country.

In 1992, the year that I got into the University of São Paulo, I tried to reach out to a few professors at the Institute of Mathematics and Statistics (IME-USP), but some of them I approached somewhat crudely and didn't get much attention. One of them was professor Odilon Otavio Luciano, who specializes in Algebra, Geometry and Topology and Logic and Foundations of Mathematics, with whom I shared my formula, and who was very interested in hearing about it. Odilon was talented and highly regarded by his peers, and had actually been recommended to me by professor Barco.

Another story worth mentioning is the blind letters I sent abroad to the department of mathematics of a few American universities. At that time there was no internet as we know it (it was mostly restricted to academic institutions), so snail mail was the preferred method of communication. I received only one response, in 1992, from professor George B. Seligman of Yale University, and some of the wording in his letter I remember to this very day:

"Dear J.R. Sousa,

I wish our undergraduate students wrote mathematics as neatly as you do, however, formulas for the cubic and quartic polynomial equations have been known for close to five centuries now, and a rigorous mathematical proof that no such expression exists for the roots of a general polynomyal of degree 5 or higher was given by Abel and Galois in 1824."

This is only the first paragraph of the letter and very close to the wording he actually used. Though I recall some of the rest, it's not very relevant for this paper. Note I don't remember exactly which year he referenced in the letter, so I'm assuming it was 1824, though that may not be the right year he used.

Shortly after solving the cubic equation, I also found a way to solve the quartic equation, with a system of equations that reduces to an equation of the third degree (that's why professor Seligman mentions it in his letter). It was much easier to figure than the cubic equation, but this paper will not cover it.

It's possible that the solution I discovered was lost in time, as I never really published it, and possibly neither did the people I showed it to. But it's by no means a certainty, if the people who've seen it shared it with others or even published it themselves somewhere, though

I really hope it wasn't the case and it's likely not the case.

2 The Solution

In the 80's, it wasn't as easy to have access to bibliographical references and run searches on a given topic, so I had no knowledge of the existing formulas due to Scipione del Ferro and Cardano (actually Tartaglia) from the 16th century[1]. If one wanted to research the literature for prior results, one had to go to a library and run some very manual searches. And then again these searches were only as good as the library's collection. But it was actually good that I didn't have access to or didn't think about looking for books on this theme, as that might have demotivated me to try and find my own solution.

Now let's see what the solution was. Here I use the very same notations that I used back in the day, so this paper recreates the solution in its very original form. I used Greek letters for the variables I introduced into the equation, except for one.

Starting from the cubic equation,

$$a\,x^3 + b\,x^2 + c\,x + d = 0 \tag{1}$$

one can transform the equation by letting $x = \pi + m$, thus obtaining:

$$a\,\pi^3 + \pi^2(3\,a\,m + b) + \pi(3\,a\,m^2 + 2\,b\,m + c) + a\,m^3 + b\,m^2 + c\,m + d = 0 \tag{2}$$

The idea I had was to form a cube like Bhaskara did in his method, hence we have the following system of equations:

$$\begin{cases} \Delta^3 = a\,m^3 + b\,m^2 + c\,m + d \\ 3\,\Delta^2\,\delta = 3\,a\,m^2 + 2\,b\,m + c \\ 3\,\Delta\,\delta^2 = 3\,a\,m + b \end{cases}$$

If one tries to solve this system, one obtains an equation of the second degree in m, which is a surprising coincidence and also greatly convenient. That is, the system reduces to:

$$m^2(b^2 - 3\,a\,c) + m(b\,c - 9\,a\,d) + c^2 - 3\,b\,d = 0, \tag{3}$$

whose solution, if $b^2 - 3\,a\,c \neq 0$, is:

$$m = \frac{-(b\,c - 9\,a\,d) \pm \sqrt{(b\,c - 9\,a\,d)^2 - 4(b^2 - 3\,a\,c)(c^2 - 3\,b\,d)}}{2(b^2 - 3\,a\,c)} \tag{4}$$

Since the biggest hurdle is now out of the way, solving equation (2) becomes really simple:

$$a\,\pi^3 + 3\,\Delta\,(\delta\pi)^2 + 3\,\Delta^2\,\delta\pi + \Delta^3 = 0, \tag{5}$$

which one can solve by completing the cube:

$$a\,\pi^3 - (\delta\pi)^3 + (\delta\pi)^3 + 3\,\Delta\,(\delta\pi)^2 + 3\,\Delta^2\,\delta\pi + \Delta^3 = 0, \tag{6}$$

and from there the equation can be further simplified:

$$(\delta\pi + \Delta)^3 = -\pi^3(a - \delta^3) \tag{7}$$

Finally, the final formula holds for any combination of the two possible values of m and of the three possible cubic roots of one. That is, the solution of equation (1) is simply:

$$\pi\left(\delta + \xi\sqrt[3]{a - \delta^3}\right) = -\Delta \Rightarrow x = m - \frac{\Delta}{\delta + \xi\sqrt[3]{a - \delta^3}}, \text{ where } \xi^3 = 1 \tag{8}$$

The values of Δ and δ in turn are a function of either value of m:

$$\begin{cases} \Delta = \sqrt[3]{a\,m^3 + b\,m^2 + c\,m + d} \\ \delta = \dfrac{3\,a\,m^2 + 2\,b\,m + c}{3\,\Delta^2} \end{cases}$$

Note equation (8) allows a to be zero generally (one just needs to choose $\xi \neq 1$). That is, this formula is more general than Bhaskara's or Cardano's, where a can't be zero. In fact, if a is zero, it can be demonstrated that (8) reduces to Bhaskara's formula.

References

[1] Katz, Victor *A History of Mathematics, Boston: Addison Wesley. p. 220* 2004 ISBN 9780321016188.