Short Effective Intervals Containing Primes and a property of the Riemann Zeta Function $\zeta\left(\frac{1}{2}+it\right)$

Kang-Ho. Kim

E-mail addresses:shypinguo202108@163.com

Department of Mathematics, Kim Chol Ju University of Education, Pyongyang, DPR Korea,

Abstract

In this paper, we prove the existence of primes in the interval $]x, x+2\sqrt{x}]$ by inducing an inequality which defines the lower bound of number of primes in the interval $]x, x+2\sqrt{x}]$ and suggest an opinion for truth of the Lindelöf hypothesis based on the existence of primes in the interval $]x, x+2\sqrt{x}]$ with the success of Ingham's preceded work.

Keywords: Distribution of primes, Sieve method

MSC(2010):11A41,11N05,11N35

1 Introduction

We consider the existence of primes in the interval $]x, x+2\sqrt{x}]$. The history of this problem seems to start in 1845 when Bertrand conjectured after numerical trials that the interval $]n, 2n-3]$ contains a prime for $n \geq 4$. This was proved by Chebycheff in 1852 in a famous work where he got the first good quantitative estimates for the number of primes less than a given bound, say x. In 2003, Ramare and Saouter [19] proved that every interval $]x(1-\Delta^{-1}), x]$ contains a prime number with Δ=28314000 provided $x \geq 10726905041$. Also, they did mention the following: Under the Riemann hypothesis, the interval $]x-\frac{8}{5}\sqrt{x}logx, x]$ contains a prime for $x \geq 2$. Of course, these results are still far from the conjecture Cramér [5] made in 1936 on probabilistic grounds: the interval $[x-Klog^2x, x]$ contains a prime for any K>1 and $x \geq x_0(K)$.

On the other hand, it is easy to see that the problem about the difference between consecutive primes is similar to the problem to determine the size of intervals to contain at least one prime. In 1905, Lindelöf(1827-1908)[4,13] hypothesized that if the Riemann zeta function is

$$\zeta\left(\frac{1}{2}+it\right) = O(|t|^c) \quad (1)$$

as $t \to \infty$, (where c is a positive constant) then $c \to 0$. In 1937, Ingham [11] proved

$$p_{n+1} - p_n < p_n^{\Theta},\ \frac{1+4c}{2+4c} < \Theta < 1 \quad (2)$$

for all n∈N when he supposed that (1) is true. If the Lindelöf hypothesis is true (that is to say, if (1) holds with an arbitrarily small c), then $p_{n+1}-p_n<p^{\Theta}_n$ is true with $\Theta=\frac{1}{2}+\epsilon$.

This may be compared with Cramér's theorem[5] that, if the Riemann hypothesis is true, then $p_{n+1}-p_n=O\left(p_n^{\frac{1}{2}}logp_n\right)$, or Ramare and Saouter [19]'s theorem that, under the Riemann hypothesis, the interval $]x-\frac{8}{5}\sqrt{x}logx,x]$ contains a prime for $x\geq 2$.

Montgomery [15] showed that the interval $]x,x+x^{\Theta}]$ contains at least one prime number for every $\Theta>\frac{3}{5}$ and $x\geq x_0$, Huxly [10] showed $\Theta>\frac{7}{12}$, Iwaniec and Jutila [12] obtained $\Theta>\frac{5}{9}$, Heath-Brown and Iwaniec [9] had $\Theta>\frac{11}{20}$. The more important one is the result of Baker and Harman[21] who proved $\Theta>0.535$. Especially, in 2001, analytical means combined with sieve methods (and the joint efforts of Baker et al.[1]) ensures us that each of the intervals $]x,x+x^{0.525}]$ for $x\geq x_0$ contains at least one prime.

Dorin Andrica conjectured[18] that, for all natural n, $\sqrt{p_{n+1}}-\sqrt{p_n}<1$. Andrew Granville[7] introduced Zhang's result[22] for small gaps between primes as following:

$$\lim_{n} inf\,(p_{n+1}-p_n)\leq 70000000. \tag{3}$$

By the GPY(Goldston, Pintz, and Yildirim) sieve method, in 2018 [14], James Maynard represented new results for small gaps and large gaps between primes as followings:

$$\sup_{p_n\leq X}(p_{n+1}-p_n)\geq C\cdot\frac{logX\cdot loglogX\cdot loglogloglogX}{logloglogX} \tag{4}$$

$$\lim_{n} inf\,(p_{n+1}-p_n)\leq 246, \tag{5}$$

where C is a positive constant. Of course, these don't prove that some interval $]x,y(x)]$ for $x\geq 1$ contains at least one prime. However, it is one of the best results about the difference between consecutive primes. Almost all these results are proved by using the sieve method.

As it was described in [6] and [21], the prototype of sieve method is Eratosthenes (276-194 BCE) sieve. It is the following formula(see [6]) for the number of primes.

$$\pi(x)-\pi(\sqrt{x})+1=[x]-\sum_{p\leq\sqrt{x}}\left[\frac{x}{p}\right]+\sum_{p_1<p_2\leq\sqrt{x}}\left[\frac{x}{p_1\cdot p_2}\right]-$$

$$-\sum_{p_1<p_2<\cdot p_3\leq\sqrt{x}}\left[\frac{x}{p_1\cdot p_2\cdot p_3}\right]+\cdots+(-1)^{\pi(\sqrt{x})}\left[\frac{x}{p_1\cdot p_2\cdot\cdots\cdot p_{\pi(\sqrt{x})}}\right]=\sum_{d_{p|d\Rightarrow p\leq\sqrt{x}}}\mu(d)\left[\frac{x}{d}\right] \tag{6}$$

(Here, as usual, $\pi(x)$ denotes the prime counting function

$$\pi(x)=\sum_{p\leq x}1 \tag{7}$$

and, throughout, the letter p will always be a prime. As usual, the Möbius function $\mu(d)$ is $(-1)^{\nu}$ when d is the product of $\nu\geq 0$ distinct primes and is zero if d has a repeated prime factor.) for Eratosthenes sieve which used the representation method of the Möbius function $\mu(d)$.

In the first half of 20^{th} century, by Brun, Vinogradov, Selberg, etc., the basis of modern sieve method, that uses the notion of "weight" with the Möbius function $\mu(d)$, was found and, better successes on the distribution state of primes were achieved.

In this paper, we represent the model of Eratosthenes sieve in a different way with the absence of the Möbius function $\mu(d)$.

The representation and method that use the Möbius function have achieved great success in the field of Number theory and is being used as very importantly as before.

However, in this paper, we will use the simpler function of 2 variables $g(x,k)$ instead of right side of (6) to represent the model for Eratosthenes sieve for the convenience of discussion(see (9) and Remarks for more details.).

Also, we will apply Eratosthenes sieve straightly for the integers in the interval $]x, x+2\sqrt{x}]$. (Of course, this isn't just doctrinaire method that apply Eratosthenes sieve for the integers in the interval $]1,2\sqrt{x}]$. So, strictly and generally $g(x,k)-g(y,k) \neq g(x-y,k)$).

As a result, in this paper, we prove the existence of primes in this interval by inducing an inequality which defines the lower bound of number of primes in the interval $]x, x+2\sqrt{x}]$ and suggest an opinion for truth of the Lindelöf hypothesis based on it with the famous success of Ingham's preceded work.

Theorem 1.1. *Each of the intervals* $]x, x+2\sqrt{x}]$ *for* $x \geq 1$ *contains at least one prime and the lower bound of number* $T\left(= \pi(x+2\sqrt{x}) - \pi(\sqrt{x})\right)$ *of primes is determined by the following formula*

$$T \geq \pi(2\sqrt{x}) - \pi(\sqrt{x}) - \pi\left(\frac{\sqrt{x}}{2}\right) - \left[\pi_2(2\sqrt{x}+4) - \pi_2(\sqrt{x})\right] - 1 \quad (8)$$

2 A new expression and some lemmas

Here, we recall some well-known definitions and lemmas.

Let $[x]$ denote the integral part of x, and $\pi(x)$ denote the number of primes not exceeding x for $x \geq 1$ (see (7)). And we denote by $\pi_2(x)$ the number of primes(twin primes) $p \leq x$ such that $p+2$ is also a prime.

Lemma 2.1. *n : composite number*$\Rightarrow \exists p \leq \sqrt{n}, p|n$.

Proof. It is trivial from the fundamental theorem of elementary theory of numbers.

Lemma 2.2. *There is certainly only one multiple of λ among consecutive λ integers.*

Proof. Omitted.

Corollary 2.3. *Let suppose* $\Lambda > \lambda \geq 1$ *and* $\Lambda, \lambda \in \mathrm{N}$. *Among consecutive λ positive integers, there is no multiple of Λ, or if there exists multiple of Λ, there is only one.*

Now, let $g(x,k)$ be expressed as follows.

$$g(x,k) := [x] - \sum_{1\leq i\leq k}\left[\frac{x}{p_i}\right] + \sum_{1\leq i1<i2\leq k}\left[\frac{x}{p_{i1}\cdot p_{i2}}\right] -$$

$$-\sum_{1\leq i1<i2<i3\leq k}\left[\frac{x}{p_{i1}\cdot p_{i2}\cdot p_{i3}}\right] + \cdots + (-1)^k\left[\frac{x}{p_1\cdot p_2\cdot\cdots\cdot p_k}\right], \quad (9)$$

Where p_j is j^{th} prime, $x \geq 1$ and $k = \pi(\sqrt{x})$.

Remark 1(The meaning of $g(x,k)$)

In fact, the right−hand side of (9) is the right−hand side of (6). Thus, the value of the function $g(x,k)$ is equal to the number of positive integers which are remained excluding the multiples of each distinct prime $p_1, p_2, \cdots, p_j, \cdots, p_k (\leq \sqrt{x})$ among the consecutive $[x]$ positive integers from "1". It is described in detail in [6,8,21].

Now, if we consider $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$, then

$$g(x,k) = \pi(x) - \pi(\sqrt{x}) + 1 = \text{the right-hand side of (6), for } k = \pi(\sqrt{x}). \qquad (10)$$

Remark 2(The generalization of $g(x,k)$)

Now, in case that $\pi(\sqrt{x}) < k \leq \pi(x), k \in N$.

$$g(x,k) = \pi(x) - k + 1.$$

Thus,

$$g(x,k) = \pi(x) - k + 1, \left(\pi(\sqrt{x}) \leq k \leq \pi(x), k \in N\right) \qquad (11)$$

In fact, in (10), it was $k = \pi(\sqrt{x})$. The integers which are not the multiples of any of the primes $p(\leq \sqrt{x})$ in the interval $[1, x]$ are number "1" and just the prime numbers p with $\sqrt{x} < p \leq [x]$ by Lemma 2.1. So (6) (or (9)) is composed.

Therefore, this imply followings.

If $k = \pi(\sqrt{x}) + 1$, thus, we consider for the multiples of $p_{\pi(\sqrt{x})+1}$, then the multiple of $p_{\pi(\sqrt{x})+1}$ is only one prime number(:$p_{\pi(\sqrt{x})+1}$) in the interval $[1, x]$.

Then $g(x, \pi(\sqrt{x}) + 1) = \pi(x) - (\pi(\sqrt{x}) + 1) + 1$.

By the as principle as above, if $k = \pi(\sqrt{x}) + 2$, then $g(x,k) = \pi(x) - (\pi(\sqrt{x}) + 2) + 1$, $\cdots$, and if $k = \pi(x)$, then $g(x,k) = 1$.

Above "if $k = \pi(x)$, then $g(x,k) = 1$" mean that number of the integer("1") which are not the multiples of any of the primes $p(\leq \sqrt{x})$ in the interval $[1, x]$.

As result, we obtain (11).

Lemma 2.4. *Let's suppose we excluded the multiples of primes not exceeding* $\sqrt{x} + 1$ *in the interval* $]x, x + 2\sqrt{x}]$. *Then, in the interval* $]x, x + 2\sqrt{x}]$, *there are no composite numbers.*

Proof. Considering $(\sqrt{x})^2 < x + 2\sqrt{x} < (\sqrt{x} + 1)^2$, it is trivial by Lemma 2.1.

3 Proof of Theorem 1.1

There are $[2\sqrt{x}]$ or $[2\sqrt{x} + 1]$ consecutive integers in the interval $]x, x + 2\sqrt{x}]$.

At this time, the case that there are $[2\sqrt{x}]$ consecutive integers in the interval $]x, x + 2\sqrt{x}]$ is a foundation, so we will lay emphasis on obtain T in case that there are $[2\sqrt{x}]$ integers in the interval $]x, x + 2\sqrt{x}]$.

By Lemma 2.4, we must obtain the number of integers which are not the multiples of any of the primes $p(\leq \sqrt{x} + 1)$ in the interval $]x, x + 2\sqrt{x}]$.

Remark 3 In fact, by Corollary 2.3, the number of multiples of any integer "A" in the interval $]x, x + 2\sqrt{x}]$ is at most one more than the number of multiples of "A" in the interval $[1, 2\sqrt{x}]$. And we must take care that although any composite number gets contained in the interval $]x, x + 2\sqrt{x}]$ one more than the interval $[1, 2\sqrt{x}]$, it is just a multiple of some a prime $p(\leq \sqrt{x} + 1)$ by Lemma 2.1.

At first, we will obtain $\widehat{T}$, the number of integers which are not the multiples of any of primes $p_\alpha \left(2 \le p_\alpha \le \frac{\sqrt{x}}{2}\right)$ in the interval $]x, x + 2\sqrt{x}]$.

1). $\widehat{T}$, the number of integers which are not the multiples of any of primes p_α in the interval $]x, x + 2\sqrt{x}]$

By the function $g(2\sqrt{x}, k)$, the multiples of each prime p_γ ($\gamma = 1,2,\cdots,k$) in the interval $[1, 2\sqrt{x}]$ are excluded from the interval $[1, 2\sqrt{x}]$ without any error.

However, by Corollary 2.3, we consider the fact that the number of multiples of each prime p_γ among $[2\sqrt{x}]$ consecutive integers in the interval $]x, x + 2\sqrt{x}]$ is at most one more than the number of multiples of each prime p_γ among $[2\sqrt{x}]$ consecutive integers in the interval $[1, 2\sqrt{x}]$.

That is to say, we consider the fact that the number of multiples of prime $(p_1 =)2$ among $[2\sqrt{x}]$ consecutive integers in the interval $]x, x + 2\sqrt{x}]$ is at most one more than the number of multiples of prime $(p_1 =)2$ among $[2\sqrt{x}]$ consecutive integers in the interval$[1, 2\sqrt{x}]$, that the number of multiples of prime $(p_2 =)3$ among $[2\sqrt{x}]$ consecutive integers in the interval $]x, x + 2\sqrt{x}]$ is at most one more than the number of multiples of prime $(p_2 =)3$ among $[2\sqrt{x}]$ consecutive integers in the interval$[1, 2\sqrt{x}]$ and further, that the number of multiples of prime $\left(p_{\pi\left(\frac{\sqrt{x}}{2}\right)}\right)$ among $[2\sqrt{x}]$ consecutive integers in the interval $]x, x + 2\sqrt{x}]$ is at most one more than the number of multiples of prime $\left(p_{\pi\left(\frac{\sqrt{x}}{2}\right)}\right)$ among $[2\sqrt{x}]$ consecutive integers in the interval $[1, 2\sqrt{x}]$.

On the other hand, the number of integers which are not multiples of any of $2,3,\cdots,p_{\pi\left(\frac{\sqrt{x}}{2}\right)}$ among $[2\sqrt{x}]$ consecutive integers in the interval $[1, 2\sqrt{x}]$ is $g\left(2\sqrt{x}, \pi\left(\frac{\sqrt{x}}{2}\right)\right)$ by (11).

So, by Remark 3**,** $\widehat{T}$, the number of integers which are not multiples of any of $2,3,\cdots,p_{\pi\left(\frac{\sqrt{x}}{2}\right)}$ among $[2\sqrt{x}]$ consecutive integers in the interval $]x, x + 2\sqrt{x}]$ is at least

$$g\left(2\sqrt{x}, \pi\left(\frac{\sqrt{x}}{2}\right)\right) - \pi\left(\frac{\sqrt{x}}{2}\right). \qquad (12)$$

Thus, for the lower bound of $\widehat{T}$,

$$\widehat{T} \ge g\left(2\sqrt{x}, \pi\left(\frac{\sqrt{x}}{2}\right)\right) - \pi\left(\frac{\sqrt{x}}{2}\right) \qquad (13)$$

Therefore, 1) is done.

2). The number of integers which are not multiples of any of primes $p_\beta \left(\frac{\sqrt{x}}{2} < p_\beta \le \sqrt{x} + 1\right)$.

In 1), we obtained the lower bound of $\widehat{T}$, the number of integers which are not multiples of the primes $p_\alpha\left(2 \le p_\alpha \le \frac{\sqrt{x}}{2}\right)$.

Now, starting from (13), we are going to obtain the lower bound of T, the number of integers which are not multiples of any of the primes $p_{\pi\left(\frac{\sqrt{x}}{2}\right)+1}, p_{\pi\left(\frac{\sqrt{x}}{2}\right)+2}, \cdots, p_{\pi(\sqrt{x})}, p_{\pi(\sqrt{x})+1}$ in the interval $]x, x + 2\sqrt{x}]$.

Considering Lemma 2.4, we are going to obtain the lower bound of the number of integers which are not multiples of any of the primes $p_{\pi\left(\frac{\sqrt{x}}{2}\right)+1}, p_{\pi\left(\frac{\sqrt{x}}{2}\right)+2}, \cdots, p_{\pi(\sqrt{x})}$, and then are going to obtain the number of integers which are not multiples of any of a prime $p_{\pi(\sqrt{x})+1}$ in the interval $]x, x+2\sqrt{x}]$.

$2\sqrt{x}$, the size of the interval $]x, x+2\sqrt{x}]$ divided by $\frac{\sqrt{x}}{2}$ gives 4. Thus $\frac{2\sqrt{x}}{\frac{\sqrt{x}}{2}} = 4$, so there are at most 4 multiples of $p_{\pi\left(\frac{\sqrt{x}}{2}\right)+1} \left(> \frac{\sqrt{x}}{2}\right)$ in the interval $]x, x+2\sqrt{x}]$.

And the factors $F_1, F_2, F_3, F_4 \left(x < F_i \cdot p_{\pi\left(\frac{\sqrt{x}}{2}\right)+1} \le x + 2\sqrt{x}\right)$ of these 4 multiples $\left(F_i \cdot p_{\pi\left(\frac{\sqrt{x}}{2}\right)+1}, i = 1,2,3,4\right)$ of $p_{\pi\left(\frac{\sqrt{x}}{2}\right)+1}$ are the consecutive integers, and at most 2 integers among them are odd integers.

If these odd integers (2 factors) are composite numbers, then they trivially are composed including prime factors that are smaller than $\sqrt{\left(x+2\sqrt{x}\right)/\frac{\sqrt{x}}{2}} \left(< \frac{\sqrt{x}}{2} \text{ for } x > 48 + 32 \cdot \sqrt{2}\right)$ by Lemma 2.1.

Because,

$$F_i \le \frac{x+2\sqrt{x}}{p_{\pi\left(\frac{\sqrt{x}}{2}\right)+1}} < \frac{x+2\sqrt{x}}{\frac{\sqrt{x}}{2}},$$

So

$$\sqrt{F_i} \le \sqrt{\frac{x+2\sqrt{x}}{p_{\pi\left(\frac{\sqrt{x}}{2}\right)+1}}} < \sqrt{\frac{x+2\sqrt{x}}{\frac{\sqrt{x}}{2}}} < \frac{\sqrt{x}}{2}$$

for, $x > 48 + 32 \cdot \sqrt{2}$, then the composite numbers $F_i \cdot p_{\pi\left(\frac{\sqrt{x}}{2}\right)+1}$ are the multiples of at least a prime $\left(\le \sqrt{F_i}\right)$ by Lemma 2.1.

Therefore, it means that the composite numbers(multiples of $p_{\pi\left(\frac{\sqrt{x}}{2}\right)+1}$) have already been calculated in the process of obtaining $\widehat{T}$.

Of course, if 2 integers among the factors F_1, F_2, F_3, F_4 are the even numbers, then the multiples$\left(F_i \cdot p_{\pi\left(\frac{\sqrt{x}}{2}\right)+1} \right)$ of them also have already been calculated in the process of obtaining $\widehat{T}$.

Thus, if we suppose that there are as few prime numbers as possible in the interval $]x, x+2\sqrt{x}]$, we must suppose that such factors (odd integers F_i) are the primes.

And then, such factors are consecutive odd integers, so they are twin primes

$$\left(\sqrt{x} \le \frac{x}{p_{\pi(\sqrt{x})}} \le \text{twin primes} \le \frac{x+2\sqrt{x}}{p_{\pi\left(\frac{\sqrt{x}}{2}\right)+1}} < \frac{x+2\sqrt{x}}{\frac{\sqrt{x}}{2}} = 2\sqrt{x} + 4\right)..$$

As a result, it implies that all $\left(\pi_2\left(2\sqrt{x}+4\right) - \pi_2\left(\sqrt{x}\right)\right)$ twin primes which are contained in the interval $[\sqrt{x}, 2\sqrt{x}+4]$ are individually combined with each $\pi_2\left(2\sqrt{x}+4\right) - \pi_2\left(\sqrt{x}\right)$ primes among the primes $p_\beta \left(\frac{\sqrt{x}}{2} < p_\beta \le \sqrt{x}\right)$ to form at most $2 \cdot \left[\pi_2\left(2\sqrt{x}+4\right) - \pi_2\left(\sqrt{x}\right)\right]$ composite numbers in the interval $]x, x+2\sqrt{x}]$.

So, except such composite numbers$\left(\pi_2(2\sqrt{x}+4)-\pi_2(\sqrt{x})\right)$ in the interval $]x, x+2\sqrt{x}]$ and if there are more other composite numbers in the interval $]x, x+2\sqrt{x}]$, then they are $\pi(\sqrt{x})-\pi\left(\frac{\sqrt{x}}{2}\right)-\left[\pi_2(2\sqrt{x}+4)-\pi_2(\sqrt{x})\right]$ composite numbers that are made more by prime numbers $p_\beta\left(\frac{\sqrt{x}}{2}<p_\beta\leq\sqrt{x}\right)$ that are remained by excluding $\pi_2(2\sqrt{x}+4)-\pi_2(\sqrt{x})$ among $\pi(\sqrt{x})-\pi\left(\frac{\sqrt{x}}{2}\right)$ prime numbers $p_\beta\left(\frac{\sqrt{x}}{2}<p_\beta\leq\sqrt{x}\right)$.

Now, considering $\pi(\sqrt{x})\leq\pi(\sqrt{x}+1)\leq\pi(\sqrt{x})+1$, Lemma 2.4 and Lemma 2.1, we obtain the following with accompanied by "-1".

$$T\geq g\left(2\sqrt{x},\pi\left(\frac{\sqrt{x}}{2}\right)\right)-\pi\left(\frac{\sqrt{x}}{2}\right)-2\cdot\left[\pi_2(2\sqrt{x}+4)-\pi_2(\sqrt{x})\right]-$$

$$-\left\{\left[\pi(\sqrt{x})-\pi\left(\frac{\sqrt{x}}{2}\right)\right]-\left[\pi_2(2\sqrt{x}+4)-\pi_2(\sqrt{x})\right]\right\}-1.$$

Thus,

$$T\geq g\left(2\sqrt{x},\pi\left(\frac{\sqrt{x}}{2}\right)\right)-\pi\left(\frac{\sqrt{x}}{2}\right)-\left[\pi_2(2\sqrt{x}+4)-\pi_2(\sqrt{x})\right]-\left[\pi(\sqrt{x})-\pi\left(\frac{\sqrt{x}}{2}\right)\right]-1.$$

Therefore, by (11), we have

$$T\geq\pi(2\sqrt{x})-\pi(\sqrt{x})-\pi\left(\frac{\sqrt{x}}{2}\right)-\left[\pi_2(2\sqrt{x}+4)-\pi_2(\sqrt{x})\right]. \tag{14}$$

Heretofore, we saw the case there are $[2\sqrt{x}]$ integers in the interval $]x, x+2\sqrt{x}]$.

In case there are $[2\sqrt{x}+1]$ integers in the interval $]x, x+2\sqrt{x}]$,considering Lemma 2.4 and Lemma 2.1, we have the right-hand of (8) for T by the right-hand of (14) with accompanied by "-1". Because one(=$[2\sqrt{x}+1]-[2\sqrt{x}]$) integer may be a composite number.

On the other hand, Theorem 1 of [20] is

$$\frac{x}{logx}\left(1+\frac{1}{2logx}\right)<\pi(x)<\frac{x}{logx}\left(1+\frac{3}{2logx}\right)\ \text{for}\ 59\leq x. \tag{15}$$

And for $\pi_2(x)$, [18] introduced the result of Halberstam and Richert:

$$\pi_2(x)\leq 2\cdot C\cdot\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\frac{x}{(logx)^2}, \tag{16}$$

Where $2\cdot\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)=1.32032\cdots$.

And, the best values obtained for C have been:$C=3.5$ by Bombieri, Friedlander and Iwaniec (1986), or $C=3.13$ by S.Lou(private communication)[18]. Therefore,

$$T\geq\pi(2\sqrt{x})-\pi(\sqrt{x})-\pi\left(\frac{\sqrt{x}}{2}\right)-\left[\pi_2(2\sqrt{x}+4)-\pi_2(\sqrt{x})\right]-1=$$

$$=\pi(2\sqrt{x})-\pi(\sqrt{x})-\pi\left(\frac{\sqrt{x}}{2}\right)-\pi_2(2\sqrt{x}+4)+\pi_2(\sqrt{x})-1>$$

(Now, considering $\pi_2(\sqrt{x})>1$ for $x>48+32\cdot\sqrt{2}$)

$$>\pi(2\sqrt{x})-\pi(\sqrt{x})-\pi\left(\frac{\sqrt{x}}{2}\right)-\pi_2(2\sqrt{x}+4)>$$

$$>\frac{2\sqrt{x}}{log(2\sqrt{x})}\left(1+\frac{1}{2\cdot log(2\sqrt{x})}\right)-\frac{\sqrt{x}}{log(\sqrt{x})}\left(1+\frac{3}{2\cdot log(\sqrt{x})}\right)-$$

$$-\frac{\sqrt{x}/2}{log(\sqrt{x}/2)}\left(1+\frac{3}{2\cdot log(\sqrt{x}/2)}\right)-2\cdot C\cdot\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\cdot\frac{2\sqrt{x}+4}{\left(log(2\sqrt{x}+4)\right)^2}>$$

$> 0,$ for $x \geq e^{43.3181}$, with $C = 3.13$.

And, if we consider the result (for $x \in$ N) that is concluded from the following Code(Mathematica) and $[2\sqrt{x}] = \begin{cases} 2[\sqrt{x}], & \sqrt{x}-[\sqrt{x}] < \frac{1}{2} \\ 2[\sqrt{x}]+1, & \sqrt{x}-[\sqrt{x}] \geq \frac{1}{2} \end{cases}$, we obtain the result that the interval $]x, x+2\sqrt{x}]$ for $1 \leq x < e^{43.3181} (= 6498480147552563678.1040339\cdots)$ contains at least one prime.

Code.

```
(x=1;Label[begin];b=2*Sqrt[x]//N;c=PrimePi[x];d=PrimePi[x+b];e=d-c;
 Print[SequenceForm["T=",e]];x+=1;If[x<6498480147552563679,Goto[begin]])
```

Therefore, Theorem 1.1 is fully proved.■

4 Andrica' conjecture and Lindelöf hypothesis

The idea of Theorem 1.1 shows the following.

First, by the PNT(Prime Number Theorem), $T \to \infty$ *as* $x \to \infty$.

In detail,

$$C_1\cdot\frac{\sqrt{x}}{logx} < \pi(x+2\sqrt{x})-\pi(\sqrt{x}) < C_2\cdot\frac{\sqrt{x}}{logx}, \textit{for some } C_2 > C_1 > 0. \quad (17)$$

Second, if we put $x = p_n$*, we have*

$$\forall n \in N, \exists p_{n+1}-p_n < p_n^{\Theta}, \left(\Theta = \frac{1}{2}+\epsilon(p_n), \exists\epsilon(p_n)\,\middle|\,0 \leq \epsilon(p_n) \leq \frac{1}{log_2 p_n}\right) \quad (18)$$

In fact,

$$\forall n \in N, \exists p_{n+1} \in \left]p_n, p_n+2p_n^{\frac{1}{2}}\right] \text{(by Theorem 1.1)} \Rightarrow$$

$$\Rightarrow p_{n+1} < p_n+2p_n^{\frac{1}{2}}\left(\text{because } p_n+2p_n^{\frac{1}{2}} \neq \text{integer}, p_{n+1} = \text{integer}\right) \Rightarrow$$

$$\Rightarrow p_{n+1}-p_n < p_n^{\frac{1}{2}+\frac{1}{log_2 p_n}}$$

Also,

$$\forall n \in N, \exists p_{n+1} \in \left]p_n, p_n+2p_n^{\frac{1}{2}}\right] \text{(by Theorem 1.1)} \Rightarrow \sqrt{p_{n+1}}-\sqrt{p_n} < 1 \text{ for } n \geq 1.$$

Thus, following proposition does hold good.

Proposition 4.1. *Andrica' conjecture is true.*

And, if we choose $\epsilon(p_n)$ that satisfies $0 \leq \epsilon(p_n) \leq \frac{1}{log_2 p_n}$, we have (18).

This shows the following results.

1) $\frac{1}{2} < \Theta < 1$ for $n \geq 3$.

2) $\epsilon(p_1) = 0$ for $n = 1 \Rightarrow\ 3\ \in]2,2\ +\ \sqrt{2}]$,

therefore , $p_{n+1} \in \left]p_n, p_n + p_n^{\Theta}\right]$ and $\frac{1}{2} \leq \Theta < 1$.

3) $\epsilon(p_2) = \log_3 2 - \frac{1}{2}\left(< 0.131 < \frac{1}{\log_2 3}\right)$ for $n = 2 \Rightarrow$

$\Rightarrow\ 5\ \in]3,3\ +\ 3^{\frac{1}{2}+\epsilon(p_2)}]$, therefore $p_{n+1} \in \left]p_n, p_n + p_n^{\Theta}\right]$, and $\frac{1}{2} < \Theta < 1$.

4) $\epsilon(p_n) \to 0$ as $n \to \infty$, therefore $\Theta \to \frac{1}{2}$ as $n \to \infty$.

Proposition 4.2. *The Lindelöf hypothesis is true.*

Proof. If (2) is true, we have

$$c \to 0 \text{ as } n \to \infty \tag{19}$$

from 1), 2), 3) and 4).

In fact, the result of [11] is

$$\zeta\left(\frac{1}{2} + it\right) = O(|t|^c), (t \to \infty) \Rightarrow \forall n \in N, \exists p_{n+1} - p_n < p_n^{\Theta}, \frac{1+4c}{2+4c} < \Theta < 1.$$

Now, from 1), 2), 3) and 4), if we consider the relation $t \leq x$ (see 257page of [11]),

$$\forall n \in N, \exists p_{n+1} - p_n < p_n^{\Theta}, \frac{1 + 4c}{2 + 4c} < \Theta < 1 \Rightarrow$$

$$\Rightarrow c \to 0 \text{ as } n \to \infty (\text{because } t \to \infty \Rightarrow x \to \infty \Leftrightarrow n \to \infty).$$

Thus, $\zeta\left(\frac{1}{2} + it\right) = O(|t|^c), (t \to \infty) \Rightarrow c \to 0$. ■

Acknowledgment

Thanks are due to the referee for his careful reading of the paper.

References

[1] R.Baker, G. Harman, J. Pintz, The difference between consecutive primes, II. Proc. London Math. Soc. (3) 2001, to appear.

[2] E. Bank, L. Bary-Soroker, Prime polynomial values of linear functions in short intervals, J. Number Theory. 151(2015) 263-275.

[3] E. Bank, L. Bary-Soroker, L. Rosenzweig, Prime polynomials in short intervals and in arithmetic progressions, Duke Math. J. 164(2)(2015) 277-295.

[4] Octavia Couture, Unsolved problems in Mathematics, Chapter 12 Lindelöf hypothesis and growth of the zeta function, White Word Publications. Delhi-110002. First Edition. 2012. ISBN 978-81-323-4253-3.

[5] H. Cramér, On the other of magnitude of the difference between consecutive prime numbers, Acta Arith. 2(1936) 23-46.

[6] J. B. Friedlander, D. R. Heath-Brown, H. Iwaniec, J. Kaczorowski, Analytic Number Theory, Springer-Verlag. Berlin, Heidelberg. Pub. (2002) 1-50.

[7] Andrew Granville, Primes in intervals of bounded length, Bull. Amer. Math. Soc.(N.S) (2015) 52.2, pp.171-222. MR:3312631(cit. on p. 356).

[8] F. Gross, On the sieve of Eratosthenes, The American Mathematical Monthly. Vol. 70, No.10 (Dec., 1963) pp.1079-1081.

[9] D. R. Heath-Brown and H. Iwaniec, On the difference between consecutive primes, Invent. Math. 55(1979) 49-69.

[10] M. N. Huxley, On the difference between consecutive primes, Invent. Math. 15(1972) 164-170.

[11] A. E. Ingham, On the difference between consecutive primes, Quart. J. Math. (Oxford) 8(1937) 255-266.

[12] H. Iwaniec and M. Jutila, Primes in short intervals, Arkiv. Mat. 17(1979) 167-176.

[13] K. Ito, Encyclopedic Dictionary of Mathematics. second edition. 468p. Mit Press. Cambridge, Massachusetts, and London, England. 1993.

[14] J. Maynard, Gaps between primes, Proc. Int. Cong. of Math-2018. Rio de Janeiro, Vol. 1 (343-360).

[15] H. L. Montgomery, Zeros of L-functions, Invent. Math. 8(1969) 346-354.

[16] H. L. Montgomery and Robert C. Vaughan, Multiplicative number theory:1.Classical Theory, Cambridge studies in advanced mathematics, Cambridge University Press 199-227, 326-344.

[17] D. H. J. Polymath, Variants of the Selberg sieve, and bounded intervals containing many primes, Res. Math. Sci. 1, Art. 12, 83. MR:3373710(cit. on pp.344,351,355).

[18] P. Ribenboim, The little book of bigger primes (ISBN 0387201696).(Springer 2004) 191-197.

[19] O. Ramare, Y. Saouter, Short effective intervals containing primes, J. Number Theory. 98(2003) 10-33.

[20] J. B. Rosser, L. Schoenfeld, Approximate formulas for some functions of prime numbers, Illinois J. Math 6(1962), 64-94.

[21] G. Tenenbaum, M. M. France, The prime numbers and their distribution, Student Mathematical Library. Vol. 6 (2000) 19-49.

[22] Yitang Zhang(2014), Bounded gaps between primes, Ann of Math.(2) 179.3, pp. 1121-1174. MR:3171761(cit. on p. 346).