# Fermat's Last Theorem and Pythagorean Triples By Michael Griffin, MLS 


#### Abstract

Fermat's Last Theorem is investigated on the set of Pythagorean triples using the ancient Greek formulas of Pythagoras, Euclid, and Plato. These are formulas used to derive natural number solutions of the Pythagorean theorem. Since Euclid's formula makes all possible triples, proofs that Euclid's cannot work at higher powers thus prove Fermat's on the set of triples.


## Prologue: Curves and Fermat

In Fermat's Last Theorem, $A^{N}+B^{N}=C^{N}$ has infinitely many solutions for $N$ but the highest whole number solution is 2 , as with the sides of a right triangle in the Pythagorean theorem. These solutions are called Pythagorean triples; for example, the first ABC is 345 since $9+16=25$. Although it has been proved by modern methods, Fermat's Last Theorem will continue to draw anyone curious as to a method Fermat himself might have used to prove it. To that end, a few idle investigations are reported here that make some sort of proofs on the set of Pythagorean triples. While these results might make for some interesting math, to show just how trivial they are, we will look at some graphs that show the obvious nature of why any Pythagorean triple would not work at a higher power.

We begin with a graph, such that the general Pythagorean theorem is converted into a unit circle here by division:
If $A^{2}+B^{2}=C^{2}$ then $\left(A^{2}+B^{2}=C^{2}\right) / C^{2}$ and $A^{2} / C^{2}+B^{2} / C^{2}=C^{2} / C^{2}$
and $A^{2} / C^{2}+B^{2} / C^{2}=1$ so we define $A^{2} / C^{2}=X^{2}$ and $B^{2} / C^{2}=Y^{2}$
So we have a unit circle of $X^{2}+Y^{2}=1$ shown below.
The natural number solutions for Pythagorean triples will lie on the unit circle, converted into rational number coordinates, such as $(3 / 5,4 / 5)$ or $(5 / 13,12 / 13)$.
It is easy enough to show that these points and their triples will stay on the unit circle and not appear in higher curves.
With the same kind of division process, we can turn higher powers into unit equations and graph them. So $A^{3}+B^{3}=C^{3}$ becomes $X^{3}+Y^{3}=1, A^{4}+B^{4}=C^{4}$ becomes $X^{4}+Y^{4}=$ 1, $A^{5}+B^{5}=C^{5}$ becomes $X^{5}+Y^{5}=1$, etc. Using the online Desmos graphing calculator we can show these equations and their curves (fig 1). Even powers give closed figures starting with a circle $\mathrm{N}=2$ whereas odd powers have unbounded lines that in the first quadrant curve out near the even powers.


Figure 1

We zoom in on the first quadrant because it would be the positive values to fit Fermat's theorem (fig 2). Zooming in also shows that as N increases the curvature moves further out from the circle in an orderly progression. The straight line is our beginning baseline when $\mathrm{N}=1$ so $\mathrm{X}+\mathrm{Y}=1$.


Figure 2

The curves never intersect except at the points $(0,1)$ and $(1,0)$. So any natural number Pythagorean triples never happen on higher power curves, which is a visual depiction that these triples would never provide solutions to Fermat's Last Theorem. All of these graphs simply serve as alternatives of the math to come.

## Pythagoras's Formula

The first part of this work is attributed to Pythagoras.
In $A^{2}+B^{2}=C^{2}$ Pythagoras found that you can always generate solutions from $A^{2}=B+C$ where $C-B=1$
This is almost a midpoint formula since the length of $A^{2}$ is divided by a midpoint which is added to length $B$ to make length $C$. The clear property of this type of Pythagorean midpoint triple is that every single odd digit is the beginning of a set of Pythagorean triples. $3,5,7, \ldots$ etc all fit the same formula that generates triples. We get $(3,4,5)$ $(5,12,13)(7,24,25) \ldots$ etc.

The same cannot be said for the second digit B. We cannot say that every even digit B is part of a set of triples, but there is something we can say. There is a clear relation between subsequent $B$ digits proceeding off of an increasing sequence of triples: the next $B$ digit that fits a triple pattern is $4 n$ more than the previous:
$B_{n}=B_{n-1}+4 n$
Where n is the ordinal value of that triple. So after 7 is 9 for the 4th triple, $n=4,24+4(4)$ $=24+16=40$ so the next triple is $(9,40,41)$ and $81+1600=1681=41$ squared.

It doesn't take much work to see this pattern.
Set up a row of the first 6 even members of Pythagorean midpoint triples:
4, 12, 24, 40, 60, 84
Then make a row of the increasing distance between members:
$8,12,16,20,24$
The common factor is 4 so divide it out:
2, 3, 4, 5, 6
Which gives the n-ordinal value of that member's triple.
So each next $B$ value is $4 n$ more than the previous one.
Again, $\mathrm{Bn}_{\mathrm{n}}=\mathrm{B}_{\mathrm{n}-1}+4 \mathrm{n}$ in formula notation, however that is as far as that goes for now.

We shall set it aside and instead make a generalization of the Pythagorean midpoint formula for triple solutions ( $A, B, C$ ) where $N$ is any positive whole digit:
$A^{N}=B^{N-1}+C^{N-1}$
As with Fermat's theorem, this is easily true if we make no restrictions on the values of $A, B, C$, or $N$. For the sake of argument and the rules of the game, we make the unfounded assumption that $A, B, C$, and $N$ are natural numbers only.
This unfounded postulate leads to a result that there are no solutions for this kind of Pythagorean midpoint triple for N greater than 2:
If $A^{3}+B^{3}=C^{3}$ then
$A^{3}=B^{2}+C^{2}$ and substituting into the first equation:
$\mathrm{B}^{2}+\mathrm{C}^{2}+\mathrm{B}^{3}=\mathrm{C}^{3}$ rearranging:
$B^{2}+B^{3}=C^{3}-C^{2}$ and factoring:
$B^{2}(1+B)=C^{2}(C-1)$
For the smallest pair of $C-B=1 \quad(B, C)$ is $(1,2)$
$1^{2}(1+1)=2^{2}(2-1)$
$1(2)=4(1)=42=4$ false

The larger $(B, C)$ gets, the worse the inequality gets:
Take the next pair $(2,3)$
$2^{2}(1+2)=3^{2}(3-1)$
$4(3)=9(2)=18 \quad 12=18$ false
The inequality has grown from a difference of $4-2=2$ to $18-12=6$
And it will just get bigger.

Also the larger N gets, the worse the inequality gets:
Instead of $\mathrm{N}=3$ let $\mathrm{N}=4$
If $\mathrm{A}^{4}+\mathrm{B}^{4}=\mathrm{C}^{4}$ then
$A^{4}=B^{3}+C^{3}$ and substituting into the first equation:
$B^{3}+C^{3}+B^{4}=C^{4}$ rearranging:
$B^{3}+B^{4}=C^{4}-C^{3}$ and factoring:
$B^{3}(1+B)=C^{3}(C-1)$
For the smallest pair of $C-B=1 \quad(B, C)$ is $(1,2)$
$1^{3}(1+1)=2^{3}(2-1)$
$1(2)=8(1)=8 \quad 2=8$ false

The inequality has grown from a difference of $4-2=2$ to $8-2=6$ And it will just get bigger.
So as pairs $(B, C)$ or the power $N$ increases, so does the inequality.

So if $C-B=1$ there is no $A, B, C$ for $N>2$ such that $A^{N}=B^{N-1}+C^{N-1}$
So there is no $A, B, C$ for $N>2$ such that
$A^{N}+B^{N}=C^{N}$
QED
This only applies to Pythagorean midpoint triples based on the unfounded postulate: $\mathrm{A}^{\mathrm{N}}=\mathrm{B}^{\mathrm{N}-1}+\mathrm{C}^{\mathrm{N}-1}$ Other sections will address formulas made by Euclid and Plato.

## Euclid's Formula

Note that $A^{2}+B^{2}=C^{2}$ is easily possible if we make no restrictions on the values of $A, B$, or C. Recalling the rules of the game, $A, B, C$, and $N$ in Fermat's, are natural numbers only.

Euclid is credited with a formula to generate all possible Pythagorean triples: for any natural numbers $m$ and $n, m>n$, a triplet of $(A, B, C)$ results from $\left(m^{2}-n^{2}, 2 m n, m^{2}+n^{2}\right)$. For example, $(m, n)$ of $(2,1)$ makes $(3,4,5)$, the first such triple, for $9+16=25$.

Let us begin by expanding Euclid's formula in the power of two:
$\left(m^{2}-n^{2}\right)^{2}+(2 m n)^{2}=\left(m^{2}+n^{2}\right)^{2}$ is $A^{2}+B^{2}=C^{2}$ and we get:
$m^{4}-2 m^{2} n^{2}+n^{4}+4 m^{2} n^{2}=m^{4}+2 m^{2} n^{2}+n^{4}$
Cancelling terms leaves $-2 m^{2} n^{2}+4 m^{2} n^{2}=2 m^{2} n^{2}$ or $4 m^{2} n^{2}=4 m^{2} n^{2}$
A trivial result that we see has any possible solutions for $m$ and $n$.

When we go to higher powers we get instead binomial expansions, with alternating +/terms on the left, and only positive terms canceling on both sides of the equation.
If $\mathrm{N}=3$
$\left(m^{2}-n^{2}\right)^{3}+(2 m n)^{3}=\left(m^{2}+n^{2}\right)^{3}$ is $A^{3}+B^{3}=C^{3}$ and we get:
$\left(m^{2}-n^{2}\right)\left(m^{4}-2 m^{2} n^{2}+n^{4}\right)+8 m^{3} n^{3}=\left(m^{2}+n^{2}\right)\left(m^{4}+2 m^{2} n^{2}+n^{4}\right)$ then distributing:
$m^{6}-2 m^{4} n^{2}+m^{2} n^{4}-n^{2} m^{4}+2 m^{2} n^{4}-n^{6}+8 m^{3} n^{3}=m^{6}+2 m^{4} n^{2}+m^{2} n^{4}+n^{2} m^{4}+2 m^{2} n^{4}+n^{6}$
Cancelling identical terms on both sides leaves:
$-2 m^{4} n^{2}-n^{2} m^{4}-n^{6}+8 m^{3} n^{3}=2 m^{4} n^{2}+n^{2} m^{4}+n^{6}$

Except for the term $8 m^{3} n^{3}$ everything on the left is just the negative of what is on the right so let us combine like terms:
$8 m^{3} n^{3}-4 m^{4} n^{2}-2 n^{2} m^{4}-2 n^{6}=0$ and then we can divide all by 2
$4 m^{3} n^{3}-2 m^{4} n^{2}-n^{2} m^{4}-n^{6}=0$
then what remains after expansion and cancellation is:
$4 m^{3} n^{3}-3 m^{4} n^{2}-n^{6}=0$ and $m=n$ is the solution of this. That is not allowed since the only resulting triple would be $\left(0,2 m^{2}, 2 m^{2}\right)$ which also shows why we need $m>n$.
$\mathrm{N}=3$ is the first case where N is odd. What about when N is even? Let's examine the expansion of the A term with its subtraction components if $\mathrm{N}=4$ :
$\left(m^{2}-n^{2}\right)^{4}=\left(m^{2}-n^{2}\right)^{2}\left(m^{2}-n^{2}\right)^{2}=\left(m^{4}-2 m^{2} n^{2}+n^{4}\right)\left(m^{4}-2 m^{2} n^{2}+n^{4}\right)=$ $m^{8}-2 m^{6} n^{2}+m^{4} n^{4}-2 m^{6} n^{2}+4 m^{4} n^{4}-2 m^{2} n^{6}+m^{4} n^{4}-2 m^{2} n^{6}+n^{8}$
The positive terms would be cancelled with their matches in the C term on the other side of the equation, leaving the negative terms: $-4 m^{6} n^{2}-4 m^{2} n^{6}$
If we bring the remaining $C$ terms over to the left side and write out the full equation we now have:
$(2 m n)^{4}-8 m^{6} n^{2}-8 m^{2} n^{6}=0 \quad$ Then dividing by 8 :
$2 m^{4} n^{4}-m^{6} n^{2}-m^{2} n^{6}=0 \quad$ And once again $m=n$ is the natural number solution.

In general, a binomial expansion and cancellation leaves:
$(2 \mathrm{mn})^{\mathrm{N}}=2(+$ - leftover terms) and so $\mathrm{m}=\mathrm{n}$ is the solution.
By intuition, any higher powers would result in the same balancing of $+/$ - terms and a final result with $m=n$ the only solution. So if $\mathrm{N}>2$ there is $\mathrm{no} \mathrm{m}>\mathrm{n}$ solution to Euclid's formula.
So there is no $A, B, C$ for $N>2$ such that
$\mathrm{A}^{\mathrm{N}}+\mathrm{B}^{\mathrm{N}}=\mathrm{C}^{\mathrm{N}}$
QED
A more rigorous proof might need to show that $\mathrm{m}=\mathrm{n}$ is the only solution and that no leftover terms with mismatched exponents could still match values other than $(2 \mathrm{mn})^{\mathrm{N}}$. Also, it is an unfounded assumption that just because Euclid's formula applies to $\mathrm{N}=2$ it should generalize to all powers. What this exercise shows is that it cannot.

## Plato's Formula

Plato has a different formula to generate triples from every even digit B :
$A, B, C,=(B / 2)^{2}-1, B,(B / 2)^{2}+1 \quad$ and $C-A=2$ always
So if $B$ equals 2 we get ( $0,2,2$ ) and if $B$ equals 4 we once again get $(3,4,5)$.
Plato's formula does not give all possible triples or even all triples using even numbers, for example it omits $(5,12,13)$. Since it is based on even numbers, we shall see how Plato's formula is a restricted version of Euclid's formula:

First, since $B$ is even we can rewrite it as $2 X$ and $X$ is any natural number. Substituting into Plato's formula we get:
$A, B, C,=(2 X / 2)^{2}-1,2 X,(2 X / 2)^{2}+1=X^{2}-1,2 X, X^{2}+1$
Comparing this to Euclid's formula shows that $m$ equals $X$ and $n$ equals 1 .

In Fermat's theorem of $A^{N}+B^{N}=C^{N}$ we now have: $\left(X^{2}-1\right)^{N}+(2 X)^{N}=\left(X^{2}+1\right)^{N}$ This is an unfounded assumption that just because Plato's formula applies to $N=2$ it should generalize to all powers.

Let us begin by expanding Plato's formula in the power of two:
$\left(X^{2}-1\right)^{2}+(2 X)^{2}=\left(X^{2}+1\right)^{2}$ is $A^{2}+B^{2}=C^{2}$ and we get:
$X^{4}-2 X^{2}+1+4 X^{2}=X^{4}+2 X^{2}+1$
Cancelling terms leaves $-2 X^{2}+4 X^{2}=2 X^{2}$ or $4 X^{2}=4 X^{2}$
A trivial result that we see has any possible solutions for $X$.

When we go to higher powers we get instead binomial expansions, with alternating +/terms on the left, and only positive terms canceling on both sides of the equation.
If $\mathrm{N}=3$
$\left(X^{2}-1^{2}\right)^{3}+(2 X 1)^{3}=\left(X^{2}+1^{2}\right)^{3}$ is $A^{3}+B^{3}=C^{3}$ and we get:
$\left(X^{2}-1^{2}\right)\left(X^{4}-2 X^{2} 1^{2}+1^{4}\right)+8 X^{3} 1^{3}=\left(X^{2}+1^{2}\right)\left(X^{4}+2 X^{2} 1^{2}+1^{4}\right)$ then distributing:
$X^{6}-2 X^{4} 1^{2}+X^{2} 1^{4}-1^{2} X^{4}+2 X^{2} 1^{4}-1^{6}+8 X^{3} 1^{3}=X^{6}+2 X^{4} 1^{2}+X^{2} 1^{4}+1^{2} X^{4}+2 X^{2} 1^{4}+1^{6}$
Cancelling identical terms on both sides leaves:
$-2 X^{4} 1^{2}-1^{2} X^{4}-1^{6}+8 X^{3} 1^{3}=2 X^{4} 1^{2}+1^{2} X^{4}+1^{6}$
Except for the term $8 X^{3} 1^{3}$ everything on the left is just the negative of what is on the right so let us combine like terms:
$8 X^{3} 1^{3}-4 X^{4} 1^{2}-2\left(1^{2} X^{4}\right)-2\left(1^{6}\right)=0$ and then we can divide all by 2
$4 X^{3} 1^{3}-2 X^{4} 1^{2}-1^{2} X^{4}-1^{6}=0$
then what remains after expansion and cancellation is:
$4 X^{3}-3 X^{4}-1=0$ and $X=1$ is the only real solution of this. That is not allowed since the only resulting triple would be $(0,2,2)$
$\mathrm{N}=3$ is the first case where N is odd. What about when N is even? Let's examine the expansion of the $A$ term with its subtraction components if $N=4$ :
$\left(X^{2}-1^{2}\right)^{4}=\left(X^{2}-1^{2}\right)^{2}\left(X^{2}-1^{2}\right)^{2}=\left(X^{4}-2 X^{2} 1^{2}+1^{4}\right)\left(X^{4}-2 X^{2} 1^{2}+1^{4}\right)=$
$X^{8}-2 X^{6} 1^{2}+X^{4} 1^{4}-2 X^{6} 1^{2}+4 X^{4} 1^{4}-2 X^{2} 1^{6}+X^{4} 1^{4}-2 X^{2} 1^{6}+1^{8}$
The positive terms would be cancelled with their matches in the $C$ term on the other side of the equation, leaving the negative terms: $-4 X^{6} 1^{2}-4 X^{2} 1^{6}$
If we bring the remaining $C$ terms over to the left side and write out the full equation we now have:
$(2 X 1)^{4}-8 X^{6} 1^{2}-8 X^{2} 1^{6}=0 \quad$ Then dividing by 8 :
$2 X^{4} 1^{4}-X^{6} 1^{2}-X^{2} 1^{6}=0 \quad$ or $\quad 2 X^{4}-X^{6}-X^{2}=0$
And once again $X=1$ is the natural number solution.

In general, a binomial expansion and cancellation leaves:
$(2 X)^{N}=2(+/-$ leftover terms) and so $X=1$ is the solution.
By intuition, any higher powers would result in the same balancing of $+/$ - terms and a final result with $\mathrm{X}=1$ the only real solution. So if $\mathrm{N}>2$ there are no distinct natural number (A,B,C) solutions to Plato's formula.
So there is no $A, B, C$ for $N>2$ such that
$A^{N}+B^{N}=C^{N}$
QED
This only uses triples with an even $B$ and where $C-A=2$.
What this exercise shows is that Plato's formula cannot generalize to higher powers if it is limited to natural numbers.

## Euclid's Formula Again

Let us expand Euclid's formula with an unfounded assumption that just because Euclid's formula applies to $\mathrm{N}=2$ it should generalize to all powers. What this exercise will show is that it still cannot.

So for $A^{N}+B^{N}=C^{N}$ we use $\left(m^{N}-n^{N}\right)^{N}+(2 m n)^{N}=\left(m^{N}+n^{N}\right)^{N}$.
First, to see that $A^{N}+B^{N}=C^{N}$ has infinitely many solutions for $N$, look at the drawing below (fig 3). The height of the curve $C^{N}$ can be arbitrarily divided into any two parts that will satisfy $A^{N}+B^{N}$.


Figure 3

The question is now are there any values for $m, n$ that will satisfy:
$A^{N}=\left(m^{N}-n^{N}\right)^{N}$ and $B^{N}=(2 m n)^{N}$ and $C^{N}=\left(m^{N+}+n^{N}\right)^{N}$.
The previous section showed that if we do a binomial expansion for any power of N and then cancel like terms on both sides of the equation what we will have remaining is $B^{N}=(2 m n)^{N}=2(+/-$ leftover terms) and so $m=n$ is the solution.
Putting this solution into Euclid's formula we see:
$A^{N}=\left(m^{N}-m^{N}\right)^{N}=0$ and $B^{N}=(2 m m)^{N}=\left(2 m^{2}\right)^{N}$ and $C^{N}=\left(m^{N+}+m^{N}\right)^{N}=\left(2 m^{N}\right)^{N}$
So $0+B^{N}=C^{N}$ or $\left(2 m^{2}\right)^{N}=\left(2 m^{N}\right)^{N}$ so $2^{N} m^{2 N}=2^{N}\left(m^{N}\right)^{N}$
Cancelling the $2^{N}$ on both sides leaves $\mathrm{m}^{2 \mathrm{~N}}=\left(\mathrm{m}^{\mathrm{N}}\right)^{\mathrm{N}}$
And the only solution for the exponent is $\mathrm{N}=2$
So If we define $A, B, C$ according to Euclid's formula, there is no $A, B, C$ for $N>2$ such that $A^{N}+B^{N}=C^{N}$

## QED

So we see that altering Euclid's formula does not change the result when we still used the only solution of an invalid triple where the A term is zero.

We make an intuitive leap to conclude that if the Greek formulas comply with Fermat's, then the solution sets of these formulas will always comply also.
While these investigations have established Fermat's Last Theorem for all Pythagorean triples, that leaves out all other possible combinations of numbers, that is all ABC's that do not fit Euclid's formula. These would differ from Euclid's values as:
$A=m^{2}-n^{2}$ plus or minus $0,1,2, \ldots$
$B=2 m n$ plus or minus $0,1,2, \ldots$
$C=m^{2}+n^{2}$ plus or minus $0,1,2, \ldots$

If we kept the same factoring of these according to Euclid's formula there would be extra bits resulting in
$B^{N}=(2 m n)^{N}=2(+/-$ leftover terms $)$ plus or minus $1,2, \ldots \quad$ and so $m=n$ is NOT the solution. Also, since the ABC's do not fit Euclid's formula there is no solution for $\mathrm{N}=2$. We'd be starting all over, with an infinite endless supply of unproved triples, minus the set of Pythagorean triples.
Does Euclid's formula cover all possible solutions? It does generate all Pythagorean triples if a $K$ factor is put onto its formula for $A B C$. For example, if $K$ is equal to 2 then $(3,4,5)$ and $(5,12,13)$ become $(6,8,10)$ and $(10,24,26)$. And if we interpret Euclid's derivation of his formula as a proof of all possible solutions for $\mathrm{N}=2$ then in fact to prove Fermat's is it sufficient to show that Euclid's has no solutions beyond $\mathrm{N}=2$ ? That has now been shown.

It would be sufficient if it could be shown that all solutions of $A^{N}+B^{N}=C^{N}$ must comply with Euclid's. That is easy for many if not all triples, so it is not enough. We can only pursue it as another possible proof just on the set of Pythagorean triples. The next section will discuss that.

## Euclid's Formula Yet Again

To repeat, does Euclid's formula cover all possible solutions of Fermat's? It may not matter since it probably cannot be made to cover all possible non-Pythagorean triples anyway. Let's at least explore its root properties.

In general, if $A, B, C, N, k, m, n$ are real numbers then $m$ and $n$ can always be chosen such that the Nth root $C=N \sqrt{ }\left(\left(m^{2}-n^{2}\right)^{N}+(2 m n)^{N}\right)$. While this may not cover all possible, there are infinitely many solutions in real numbers, just not integers or natural numbers. That is not necessary to comply with Euclid's formula. Restricting these to natural numbers results in Pythagorean triples. If $\mathrm{N}=2$ the result for the Nth root C is Euclid's definition for the C term, $m^{2}+n^{2}$.
$\sqrt{ }\left(\left(m^{2}-n^{2}\right)^{2}+(2 m n)^{2}\right)=\sqrt{ }\left(\left(m^{4}-2 m^{2} n^{2}+n^{4}\right)+\left(4 m^{2} n^{2}\right)\right)=\sqrt{ }\left(m^{4}+2 m^{2} n^{2}+n^{4}\right)=\sqrt{ }\left(m^{2}+n^{2}\right)^{2}$

If N is beyond 2 we do not get Euclid's C term and Fermat's Last Theorem means at least one of $A, B, C$ is not an integer. How can we show that?

Consider the Nth roots where N is greater than 2 . Higher roots would be smaller than the second root which is Euclid's $\mathrm{m}^{2}+\mathrm{n}^{2}$. Since they are the C term, higher roots are bigger than the $A$ or $B$ terms. On a number line the terms would arrange as $A, B$, Nth root, Euclid's C. For the $A$ and $B$ terms to be natural numbers so must the $m$ and $n$ terms be.

So we are looking for the Nth root C term not to be natural. Its possible domain on the number line goes from the B term to Euclid's C. Because the Nth root is less than Euclid's C we can use some properties of Pythagorean triples. Previous sections discussed the three Greek formulas so we will make a case for each:

If Euclid's $C$ fits Pythagoras' formula then $C-B=1$ and so the domain of the Nth root is not large enough after $B$ to make a natural number.

If Euclid's C fits Plato's formula then $\mathrm{C}-\mathrm{B}=2$ and so the domain of the Nth root is not large enough after $B$ to make a natural number except for $B+1$. In that case $C-B=1$ and we would have a Pythagorean triple for the second power.

If Euclid's $C$ fits Euclid's formula, the domain of the Nth root is less than $C-B$ which is $m^{2}+n^{2}-2 m n$ which factors as $(m-n)^{2}$. Or if $A$ is the larger term, the domain of the Nth root is less than C-A which is $m^{2}+n^{2}-\left(m^{2}-n^{2}\right)$ which is $2 n^{2}$. For example, for $m, n=5,2$ then $A B C=21,20,29$ and the domain of the Nth root is less than 8 . We have room for natural numbers now, although we can make more restrictions. Since the $m$ and $n$ terms must be natural numbers, so are the domains $(m-n)^{2}$ and $2 n^{2}$, so the domain boundaries are only square numbers $1,4,9,16,25, \ldots$ Higher roots of square numbers are usually not natural numbers. An exception example is like when 2 is the 4th root of 16. Of course, the actual $C$ term is below $A$ or $B$ plus $1,4,9,16,25, \ldots$ and each $A$ or $B$ term is a natural perfect Nth root of $A^{N}$ or $B^{N}$. However, there can also be Nth roots of combinations of natural numbers like $\left(B+X^{2}\right)$. For example, the third root of $\left(2+5^{2}\right)=$ $(2+25)=27$ is 3 , a natural number. So, unfortunately, our Nth root $C$ term may be natural. Euclid's formula is not concluding easily like the other Greek formulas. This method is not as productive as the prior ways to show Fermat's on Pythagorean triples, using Euclid's formula.

## Epilogue: More Curves and Fermat

A corollary of Fermat's theorem is that any solutions would have to have coordinates on Desmos curves (as shown in the Prologue) that could be converted into natural number triples.

This does happen on the unit circle for $\mathrm{N}=2$. Note the two coordinate points $(0.6,0.8)$ and $(0.8,0.6)$. As fractions these are $(3 / 5,4 / 5)$ and $(4 / 5,3 / 5)$ which convert into the triple 345.
Another result occurs below the baseline using fractional values of N. For example, the square root is $\mathrm{N}=1 / 2$ (fig 4).


Figure 4

Zooming into the first quadrant for this (fig 5 ) we can see that the point $(0.25,0.25)$ has a solution ( $0.5,0.5$ ) on the $\mathrm{N}=1$ baseline which would convert into the triple 112.


Figure 5

That is not quite fitting the rules since ABC are supposed to be distinct. The points on the baseline $(0.25,0.75)$ and $(0.75,0.25)$ would become distinct as 134 or 314 . Still, we might generalize Fermat's theorem as that any natural number solutions would fit the condition of $1 / 2 \leq \mathrm{N} \leq 2$. Higher roots would have the same unsolvable problem as higher powers. Or at least they seem like they should. And the principle would hold that any convertible solution points on the square root curve would never occur on the higher roots curves. Now this is getting away from the Greek formulas which only dealt with natural numbered powers. These graphs have simply served as alternative visuals for all of that.

## Sources

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