Two Proofs of Riemann Hypothesis by Vector Properties of Riemann Zeta Function

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Abstract: The Riemann zeta function(RZF) *ζ(s)* is useful in number theory for studing properties of prime numbers. The Dirichlet eta function(DEF) *η(s)* is modification of RZF. In this thesis, we treat each term of RZF and DEF as a vector. From the geometric properties of vectors, we got clues of proof from the fact that, in a complex variable $s = \alpha + i\beta$, α only affects the magnitude of each vector and β affects only the argument of each vector, independently. So, each vector with same n are parallel to each other, regardless of the value of α. This parallel property implies a very strict geometric restriction which lead to two successful proofs of Riemann Hypothesis(RH). One proof is from the contradictions which come from the trajectories of RZF, and the other proof is by applying Chauchy integral theorem to the trajectory of RZF. We tried to provide sufficient graphs and videos for the understanding of the vector geometry properties of RZF and DEF. In appendix, we provided the source programs for analyzing vectors and suggested two other possible proofs of RH for further studies.

1. Introduction

RZF [1][2][3][4] $\zeta(s)$ is a function of a complex variable $s = \alpha + i\beta$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \quad (1.1)$$

Equation (1.1) converges only when Re(s) > 1. RH [5] states that all the non-trivial zeros of RZF are of the form $s = 0.5 + i\beta$. which is called the critical line.

DEF [6] $\eta(s)$ gives an equation for calculating *ζ(s)* in the region 0 < Re(*s*) = α < 1.

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s) \quad (1.2)$$

In (1.1) and (1.2), let each term of RZF and DEF as $f_n(s)$ and $g_n(s)$, then,

$$f_n(s) = \frac{1}{n^s} = e^{-\alpha ln n} e^{-i\beta ln n} = r_n e^{i\theta_n} = u_n + iv_n \quad (1.3)$$

$$r_n = e^{-\alpha ln n} \quad (1.4)$$

$$\theta_n = -\beta ln n \quad (1.5)$$

$$u_n = r_n cos\theta_n \quad (1.6)$$

$$v_n = r_n sin\theta_n \quad (1.7)$$

$$f_1(s) = \frac{1}{1^s} = e^{-\alpha ln 1} e^{-i\beta ln 1} = 1 \quad (1.8)$$

$$f_2(s) = \frac{1}{2^s} = e^{-\alpha ln 2} e^{-i\beta ln 2} = e^{-\alpha ln 2}(cos\beta ln 2 - i sin\beta ln 2) \quad (1.9)$$

$$\zeta(s) = \sum_{n=1}^{\infty} f_n(s) = \sum_{n=1}^{\infty} e^{-\alpha \ln n} e^{-i\beta \ln n}$$

$$= 1 + \sum_{n=2}^{\infty} e^{-\alpha \ln n} e^{-i\beta \ln n} \quad (1.10)$$

$$g_n(s) = \frac{(-1)^{n+1}}{n^s} = (-1)^{n+1} e^{-\alpha \ln n} e^{-i\beta \ln n} = (-1)^{n+1} f_n(s) \quad (1.11)$$

$$\eta(s) = \sum_{n=1}^{\infty} g_n(s) = \sum_{n=1}^{\infty} (-1)^{n+1} e^{-\alpha \ln n} e^{-i\beta \ln n}$$

$$= 1 + \sum_{n=2}^{\infty} (-1)^{n+1} e^{-\alpha \ln n} e^{-i\beta \ln n} \quad (1.12)$$

We can see the following roles of α and β.

① α determines the magnitude of $f_n(s)$ or $g_n(s)$, $e^{-\alpha \ln n}$.

② β determines the argument of $f_n(s)$ or $g_n(s)$, $-\beta \ln n$.

③ α and β are independent.

Table 1 shows some examples of α vs r_n relationships.

Table 1. Radius of $f_n(s)$ and $f_2(s)$ for some α.

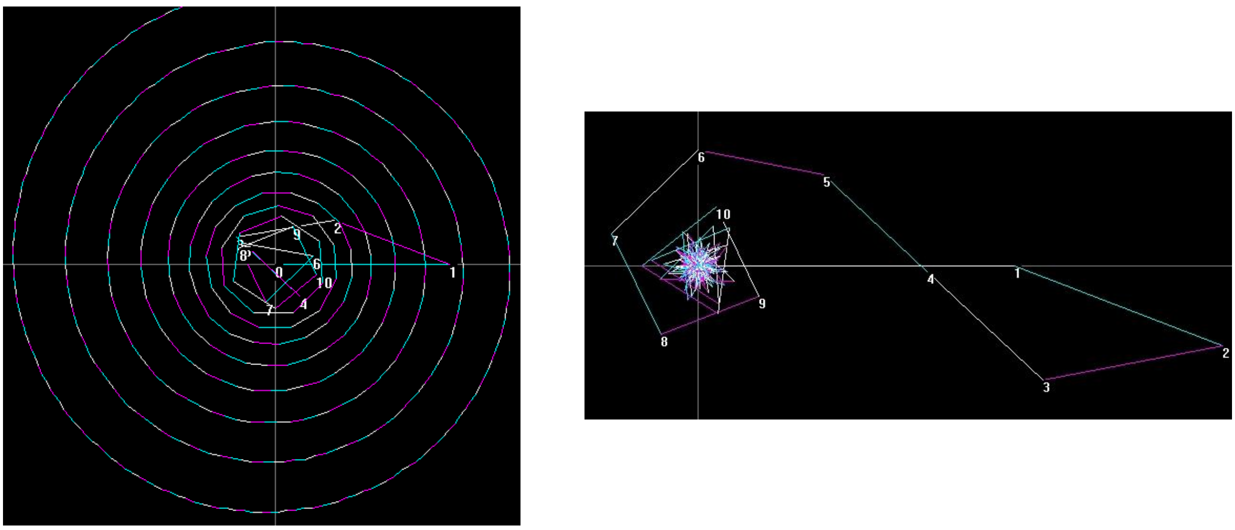
α	0	1/3	1/2	2/3	1
r_n	1	$(1/n)^{1/3}$	$(1/n)^{1/2}$	$(1/n)^{2/3}$	$(1/n)^{1}$
r_2	1	0.79	0.71	0.63	0.5

2. Vector Trace Analysis

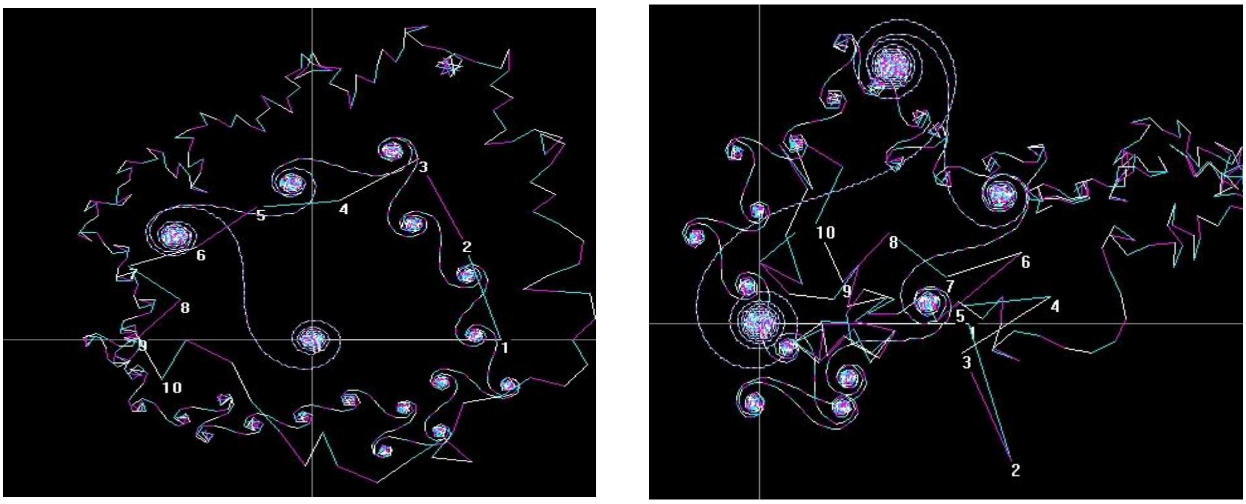
2.1 Trace Graphs

In this study, we deal each $f_n(s)$ or $g_n(s)$ as a vector in the complex plane and by tracing the vector sum we can grasp the graphical properties of RZF and DEF. We used PureBasic [7] free version to plot the trace. The source program and some videos are given in appendix A and C. Figure 1 shows some vector trace graphs.

Figure 1. Sample vector trace graphs.



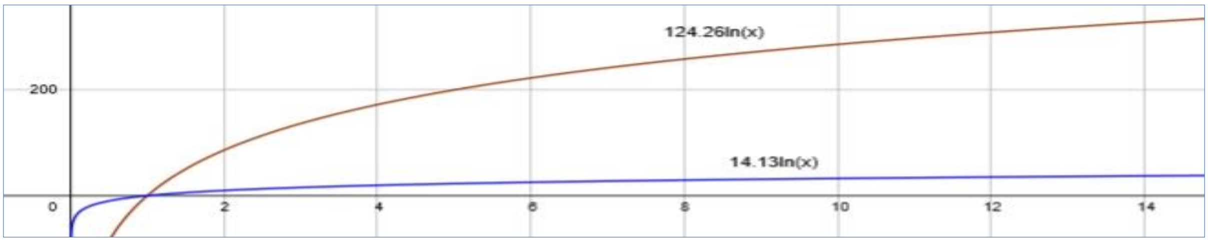
(a) RZF: α = 0.5, β = 14.134725141734 (b) DEF: α = 0.5, β = 14.134725141734



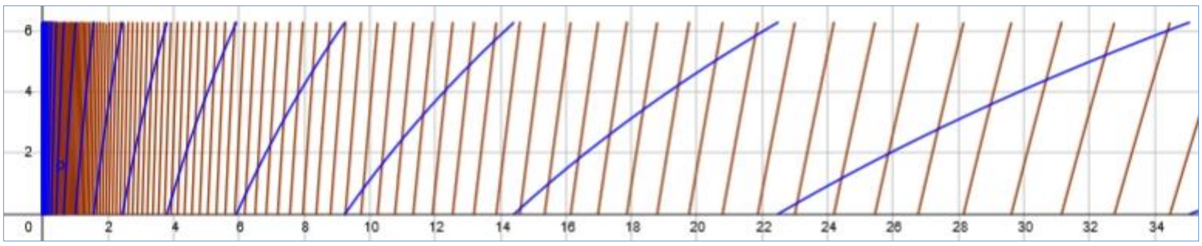
(c) RZF: $\alpha = 0.5$, $\beta = 24499.249265478$ (d) DEF: $\alpha = 0.5$, $\beta = 24499.249265478$

From Figure 1, we can see that when β is small, graphs are simpler than when β is large. In Figure 1 (c) and (d), you can see the lumps where vectors spiral in and spiral out. This phenomena occurs because of the argument change patterns of vectors, as in Figure 2.

Figure 2. Example argument graphs.



(a) $\alpha = 0.5$, $\beta = 14.13$(blue), $\beta = 124.26$(brown)



(b) mod(arg, 2π), $\alpha = 0.5$, $\beta = 14.13$(blue), $\beta = 124.26$(brown)

There are four types of vector movement.

① **Zigzag:** Vectors zigzag when arg(v), the argument of vector v, changes abruptly.

② **Spiral in:** Vectors shrink to a point. It occurs when sequence of vectors with $\arg(v_{n+1}) - \arg(v_n) > 90°$ are prevalent.

③ **Spiral out:** Vectors are inverted from spiral in and spiral out. It occurs when sequence of vectors with $\arg(v_{n+1}) - \arg(v_n) < 90°$ are prevalent.

④ **Smooth moving:** Vectors move smoothly.

2.2 Trace of α and β

We used GeoGebra [8] to trace RZF with respect to α and β. GeoGebra has zeta function and we set the following parameter and function, and animated α and β.

$$s = \alpha + i\beta$$

$$w = zeta(\alpha + i\beta)$$

Graphs for the following cases are drawn.

(a) $\alpha = 0.5$, $14.13 \leq \beta \leq 32.94$.

(b) $\alpha = 0.25$, $14.13 \leq \beta \leq 32.94$.

(c) $\alpha = 0.75$, $14.13 \leq \beta \leq 32.94$.

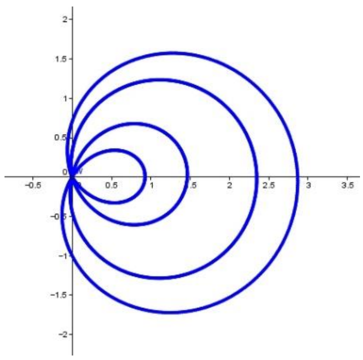
(d) $0 \leq \alpha \leq 1$, $\beta = 14.13$.

(e) $0 \leq \alpha \leq 1$, $\beta = 124.26$.

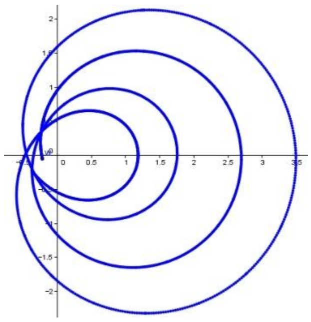
(f) $0 \leq \alpha \leq 1$, $\beta = 294014.13$.

Figure 3 shows above 6 graphs.

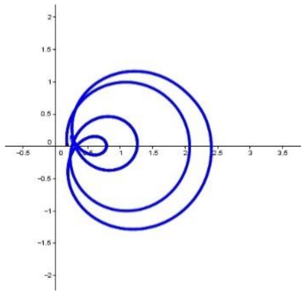
Figure 3. Trace of RZF with respect to α and β.



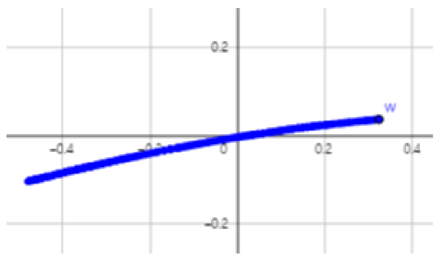
(a) $\alpha = 0.5$, $14.13 \leq \beta \leq 32.94$.



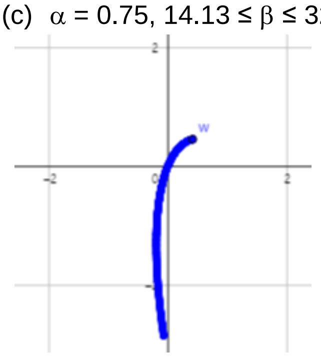
(b) $\alpha = 0.25$, $14.13 \leq \beta \leq 32.94$.



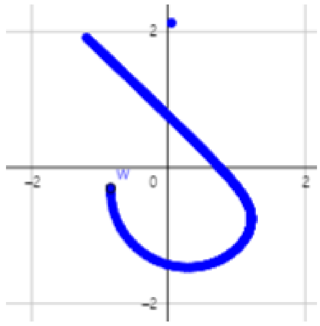
(c) $\alpha = 0.75$, $14.13 \leq \beta \leq 32.94$.



(d) $0 \leq \alpha \leq 1$, $\beta = 14.13$.



(e) $0 \leq \alpha \leq 1$, $\beta = 124.26$



(f) $0 \leq \alpha \leq 1$, $\beta = 294014.13$

From the graphs in Figure 3, we can see the followings.

(a) When $\alpha = 0.5$, graphs have zeros.

(b) When $\alpha = 0.25$, graph swells with some biase to the left, because the magnitude of each vector $e^{-\alpha lnn}$ becomes large.

(c) When $\alpha = 0.75$, graph shrinks with some biase to the right, because the magnitude of each vector $e^{-\alpha lnn}$ becomes small.

(d) When $\beta = 14.13$, as α moves between 0 and 1, an open curve which passes origin at $\alpha = 0.5$ is drawn.

(e) When $\beta = 124.26$, as α moves between 0 and 1, an open curve which passes origin at $\alpha = 0.5$ is drawn.

(f) When $\beta = 290414.13$, as α moves between 0 and 1, an open curve which does not cross the origin is drawn.

3. Symmetry Property of the Zeros of RZF

It is well known that the following two consequences are true, where $\xi(s)$ is Riemann's Xi-function [9].

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{\frac{-s}{2}} \tag{3.1}$$

$$\xi(s) = \xi(1-s) \tag{3.2}$$

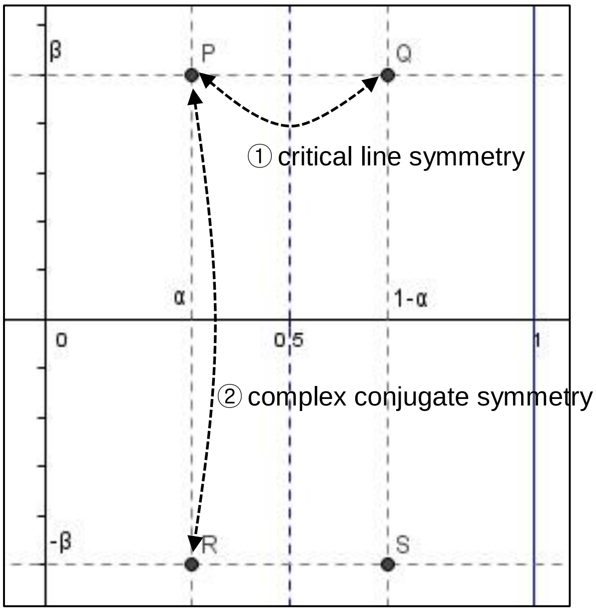
$$\zeta(\overline{s}) = \overline{\zeta(s)} \tag{3.3}$$

Equations (3.2) and (3.3) imply two kinds of symmetry of RZF zeros, as in Figure 4.

① **Critical line symmetry:** Symmetry of *s* and 1 - *s*, so, *α* and 1 – *α* symmetry along the crtical line.

② **Complex conjugate symmetry:** Symmetry of $\zeta(\overline{s}) = \overline{\zeta(s)}$ along the *x* axis.

Figure 4. Zero symmetries of RZF.



Suppose that there exists critical line symmetry zeros at P(α, β) and Q(1 - α, β), then, a closed trajectory must be drawn by the following 3 steps, as in Figure 5.

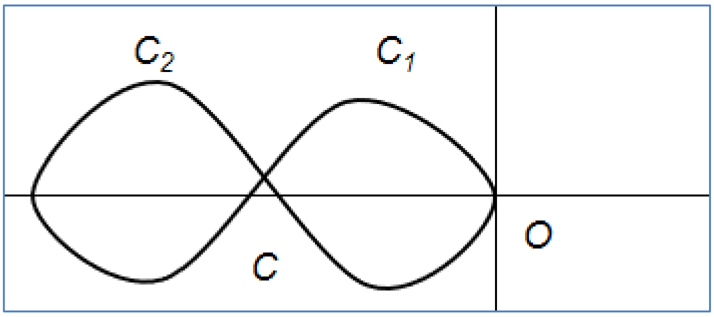
① **Initial state at (α, β):** At P(α, β), 0 < α < 1/2, trajectory remains at origin O.

② **Movement to (1/2, β):** Trajectory will leave origin and will reach somewhere on C.

③ **Movement to (1 - α, β):** Trajectory will come back to O, the final state.

So, the trajectory drawn while moving domain $\alpha \leq x \leq 1 - \alpha$, will form a logically closed contour C, as in Figure 5. Here, a logically closed contour means that, C may cross itself, resulting multiple loops C_1, C_2, …, C_i.

Figure 5. Contour for $\alpha \leq x \leq 1-\alpha, 0 \leq \alpha \leq 1/2$.



4. Two Proofs of RH

4.1. Proof by the Last Vector Contradiction

We can rewrite (1.10) as follows.

$$\zeta(s) = \sum_{n=1}^{\infty} f_n(s) = \sum_{n=1}^{\infty} e^{-\alpha \ln n} e^{-i\beta \ln n}$$

$$= 1 + \sum_{n=2}^{\infty} e^{-\alpha \ln n} e^{-i\beta \ln n} \tag{1.10}$$

$$= 1 + \sum_{n=3}^{\infty} e^{-\alpha \ln n} e^{-i\beta \ln n} + e^{-\alpha \ln 2} e^{-i\beta \ln 2} \tag{4.1}$$

$$= \overrightarrow{A_\alpha} + \overrightarrow{B_\alpha} + \overrightarrow{C_\alpha} \tag{4.2}$$

$$\overrightarrow{A_\alpha} = (1,0) \tag{4.3}$$

$$\overrightarrow{B_\alpha} = \sum_{n=3}^{\infty} e^{-\alpha \ln n} e^{-i\beta \ln n} \tag{4.4}$$

$$\overrightarrow{C_\alpha} = e^{-\alpha \ln 2} e^{-i\beta \ln 2} \tag{4.5}$$

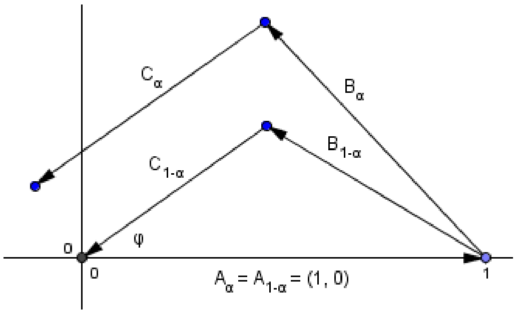
Definition 4.1. Last vector: Vector $\overrightarrow{C_\alpha}$ in (4.2) and (4.5).

Definition 4.2. Triangle vector set(TVS): Set of three vectors in (4.2), $V_\alpha = \{\overrightarrow{A_\alpha}, \overrightarrow{B_\alpha}, \overrightarrow{C_\alpha}\}$ or $V_{1-\alpha} = \{\overrightarrow{A_{1-\alpha}}, \overrightarrow{B_{1-\alpha}}, \overrightarrow{C_{1-\alpha}}\}$ or for any value $x, 0 < x < 1$, $V_x = \{\overrightarrow{A_x}, \overrightarrow{B_x}, \overrightarrow{C_x}\}$.

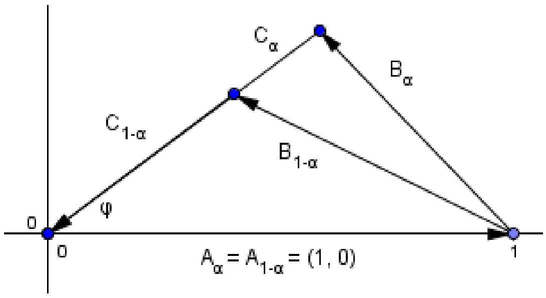
Lemma 4.3. If $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) = 0, 0 < \alpha < 0.5$, then, the two last vectors $\overrightarrow{C_\alpha}$ and $\overrightarrow{C_{1-\alpha}}$ must be on the same line.

Proof. Figure 6 shows example TVSs $V_\alpha = \{\overrightarrow{A_\alpha}, \overrightarrow{B_\alpha}, \overrightarrow{C_\alpha}\}$ and $V_{1-\alpha} = \{\overrightarrow{A_{1-\alpha}}, \overrightarrow{B_{1-\alpha}}, \overrightarrow{C_{1-\alpha}}\}$.

Figure 6. Last vector examples.



(a) Two last vectors can't end at the origin.



(b) Two last vectors end at the origin.

* For convenience, we omitted the vector arrows in Figures.

The last vectors are $\overrightarrow{C_\alpha} = e^{-\alpha ln2} e^{-i\beta ln2}$ and $\overrightarrow{C_{1-\alpha}} = e^{-(1-\alpha)ln2} e^{-i\beta ln2}$, respectively. The arguments of the two last vectors are same. If the two last vectors are not on the same line as in figure (a), it can't be $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) = 0$, so, the two last vectors should be on the same line, as in figure (b). ■

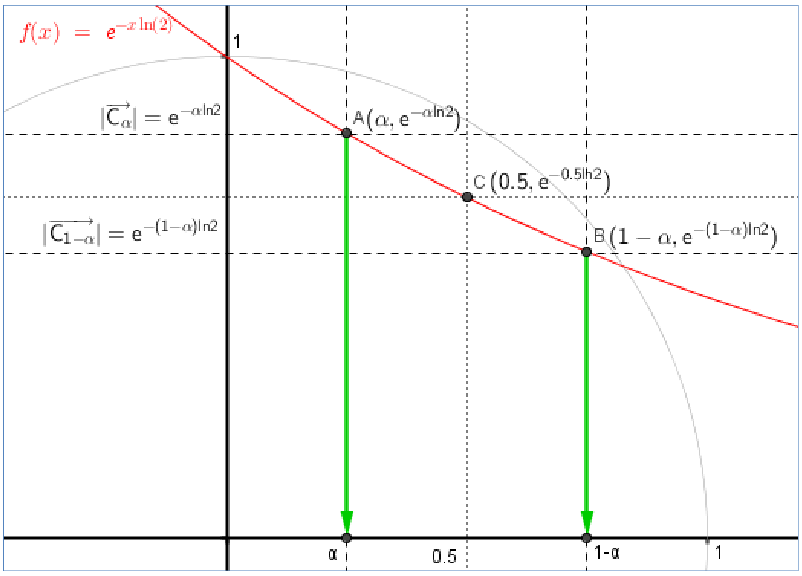
Definition 4.4. Same line restriction: The result of Lemma 4.3.

Lemma 4.5. If there exists α such that $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) = 0, 0 < \alpha < 0.5$, then, there are three possible trajectories of $\overrightarrow{A_\alpha} + \overrightarrow{B_\alpha}$, while α approaches to $1 - \alpha$, that will cause a contradiction, respectively. So, there can't exist α such that $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) = 0, 0 < \alpha < 0.5$, i.e., RH is true.

Proof. Following steps will lead to the proof.

Step 1: The magnitude of the two last vectors are $e^{-\alpha ln2}$ and $e^{-(1-\alpha)ln2}$, as in figure 7. The red graph $f(x) = e^{-xln2}$ represents the magnitude of the term of RZF for $n = 2$.

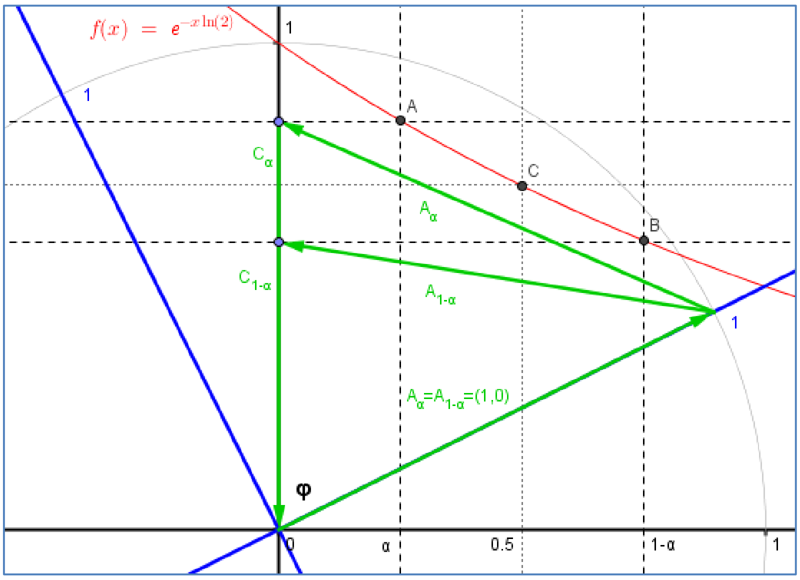
Figure 7. Last vector magnitude graph.



* The graph e^{-xln2} represents the magnitude of all vectors for $n = 2$.

Step 2: The argument of the two last vectors is $-i\beta ln2$, so, $\varphi = -i\beta ln2 - \pi$, as in figure 8. The new blue axis is introduced to reflect the argument of the last vectors. Also, the two TVSs $V_\alpha = \{\overrightarrow{A_\alpha}, \overrightarrow{B_\alpha}, \overrightarrow{C_\alpha}\}$ and $V_{1-\alpha} = \{\overrightarrow{A_{1-\alpha}}, \overrightarrow{B_{1-\alpha}}, \overrightarrow{C_{1-\alpha}}\}$ are shown.

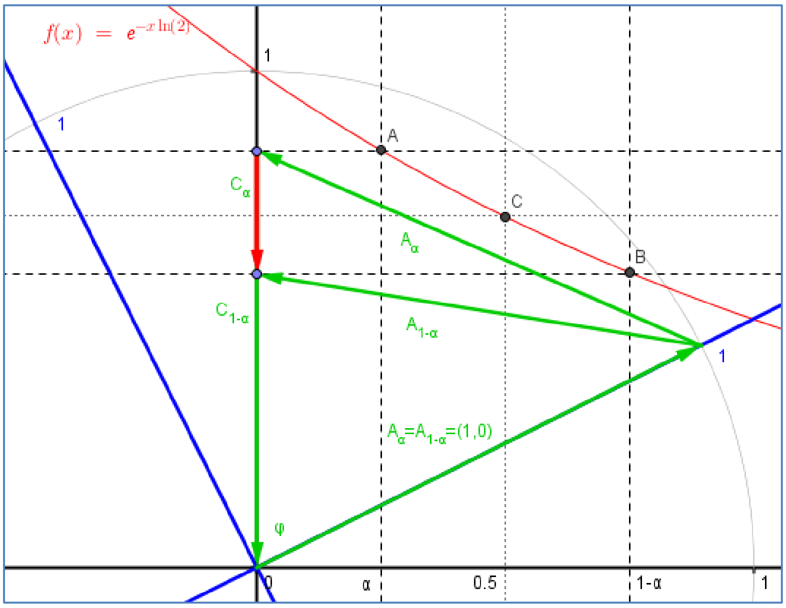
Figure 8. Last vector argument graph.



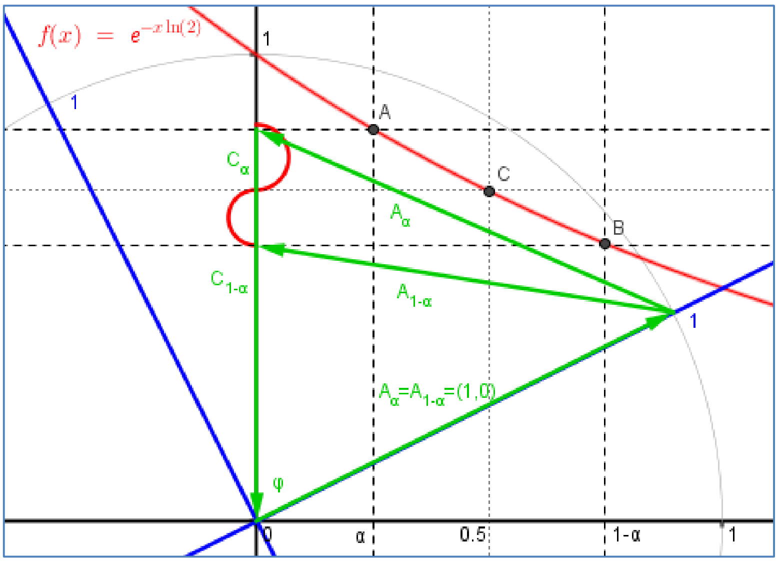
∗ The blue axis reflect the argument of the last vectors.

Step 3: The trajectory of $\overrightarrow{A_\alpha}, +\overrightarrow{B_\alpha}$ while α approaches to $1 - \alpha$ falls into one of the following three cases, as shown in Figure 9. Trajectories are marked as bold red line or curves.

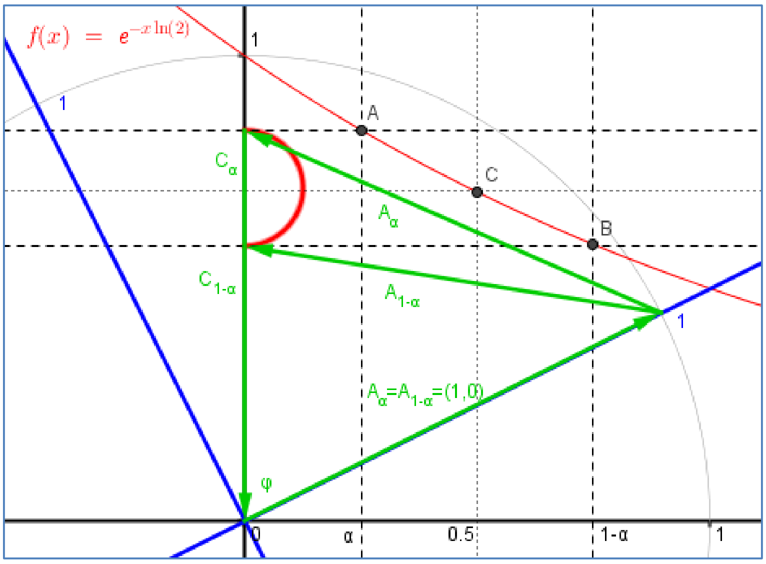
Figure 9. Three possible trajectories of $\overrightarrow{A_\alpha}, +\overrightarrow{B_\alpha}, \alpha \leq x \leq 1 - \alpha$.



(a) Line trajectory.



(b) Curve trajectory which crosses $\overrightarrow{C_\alpha}$.



(c) Curve trajectory which does not cross $\overrightarrow{C_\alpha}$.

Step 4: All three possible trajectories of $\overrightarrow{A_\alpha}, +\overrightarrow{B_\alpha}$ introduce a contradiction, respectively, so, RH is true.

① **Trajectory (a):** For all x in $\alpha < x < 1-\alpha$, the last vectors $\overrightarrow{C_x}$ will end at the origin which means that $\zeta(x+i\beta) = 0$ for all x, which contradicts.

② **Trajectory (b):** For some x in $\alpha < x < 1-\alpha$, the trajectory will cross $\overrightarrow{C_\alpha}$, and at that moment, $\zeta(x+i\beta) = 0$, which contradicts.

③ **Trajectory (c):** There always exist x_1, x_2 such that, $s_1 = x_1 + i\beta, s_2 = x_2 + i\beta, \alpha < x_1, x_2 < 1-\alpha, x_1 \neq x_2$, which satisfy $\zeta(s_1) = \zeta(s_2)$, as in Figure 10.

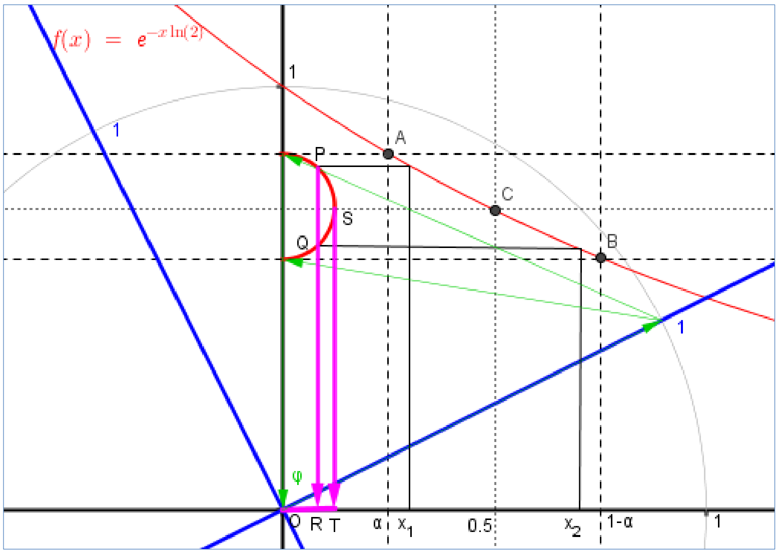
In Figure 10, while x moves from α to $1-\alpha$, $\zeta(x+i\beta)$ moves on the pink line segment $\overline{\mathrm{OT}}$ from left to right and exactly back from right to left. So, there always exists (s_1, s_2) pair that satisfy $\zeta(s_1) = \zeta(s_2)$ for all $x_1, x_2, \alpha < x_1, x_2 < 1-\alpha, x_1 \neq x_2$. The last two pink vectors, $\overrightarrow{\mathrm{PR}}$ of x_1 and $\overrightarrow{\mathrm{QR}}$ of x_2, show why $\zeta(x_1 + i\beta,) = \zeta(x_2 + i\beta)$ should have the same value.

The last vector $\overrightarrow{\mathrm{ST}}$ is any rightmost tangent vector to the trajectory (c).

According to the curve shape of trajectory (c), there can be more than two same values $\zeta(s_1) = \zeta(s_2) = \cdots = \zeta(s_n)$.

It does not matter whether the trajectory (c) is on the right side or left side.

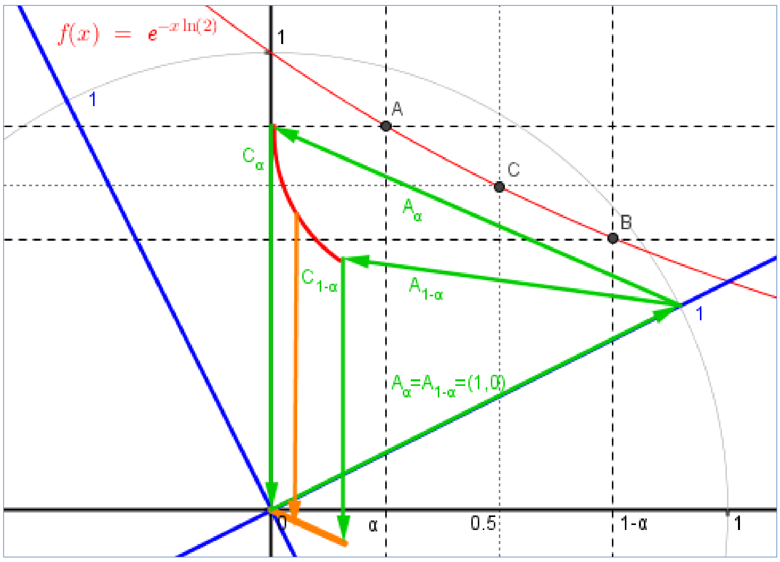
Figure 10. Trajectory (c) with $\zeta(s_1) = \zeta(s_2)$ example.



This lead to a contradiction, because $\zeta(s_1) = \zeta(s_2)$ for infinitely many $x_1 \neq x_2$, means that there exist reflection symmetry with respect to some x in $\alpha < x < 1 - \alpha$. ■

This result stems from the restriction that there exist α such that $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) = 0, 0 < \alpha < 0.5$. It leads to the restriction that the two last vectors must be on the same line as in Figure 6 (b). If the two last vectors are freed from the *'same line restriction'*, the contradictions of Lemma 4.5 are not necessary, as in Figure 11.

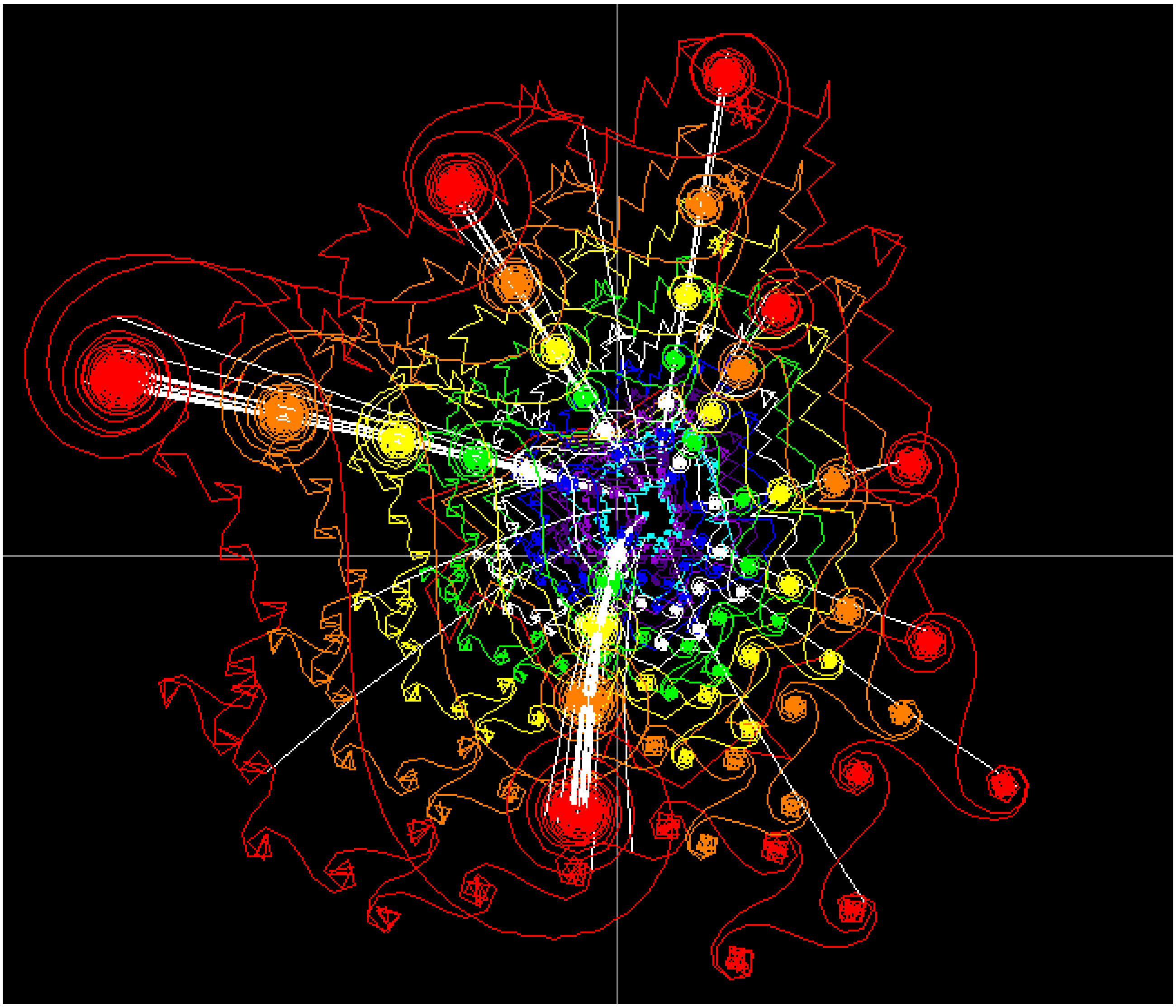
Figure 11. Example trajectory with no *'same line restriction'*.



Note that, we selected second term of (1.1) as the last vector, but, we can select any term as the last vector.

Figure 12 shows nine parallel vector trace graphs for β = 24499.249265478 and α = 0.3(outside red), 0.35, 0.4, 0.45, 0.5, 0.55, 0.6, 0.65, 0.7(inside aqua). In Figure 12, the white lines from outside to inside are the end point links for every mod(n, 100) = 0 terms. The parallel vector trace program source is provided in Appendix B.

Figure 12. Parallel vector trace example.



4.2. Proof by Cauchy Integral Theorem

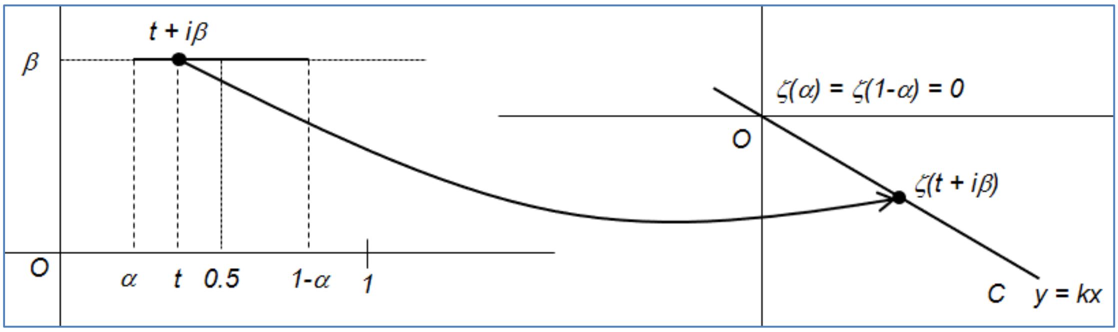
Cauchy integral theorem states that for any analytic complex function $h(z)$, the closed curve integral is always zero, as the following equation.

$$\oint_C h(z)dz = 0 \tag{4.6}$$

Lemma 4.6. If there exist α such that $\zeta(\alpha+i\beta) = \zeta(1-\alpha+i\beta) = 0, 0 < \alpha < 0.5$, then, the Cauchy integral on the close trajectory $\zeta(z)$ proves that $\alpha = 0.5$.

Proof. In our case, the three trajectory types in Figure 9 can be generalized as a line segment of $y = kx$, as in figure 13. So, the Cauchy integral theorem can be applied to that line segment, which is denoted as contour C in Figure 13. C starts from the origin and returns back to the origin. In the case of trajectory (a) in Figure 9, C always remains at the origin, regardless of the movement of $t, \alpha \le t \le 1 - \alpha$.

Figure 13. Cauchy integral domain and contour C.



Let $h(z) = 1$, which is entirely analytic, then,

$$\oint_C h(z)dz = \oint_C dz = 0 \tag{4.7}$$

$$z(t) = x(t) + iy(t) = t + ikt = (1 + ik)t \tag{4.8}$$

$$dz = (i + ik)dt, \alpha \le t \le 1 - \alpha \tag{4.9}$$

$$\oint_C dz = \int_\alpha^{1-\alpha} (1 + ik)dt = (1 + ik)[t]\ {}_\alpha^{1-\alpha} = (1 + ik)(1 - 2\alpha) = 0 \tag{4.10}$$

$$\alpha = 0.5. \tag{4.11}$$

The result (4.10) explicitly shows that RH is true. ■

5. Conclusion

In this thesis, we proved RH by analysing the vector properties of RZF and DEF. In a complex variable $s = \alpha + i\beta$, α only affects the magnitude of each vector and β affects only the argument of each vector, independently. We provided various vector trace examples of RZF and DEF from which the triangle vector geometry with last vector concept can be devised to induce contradictions. The parallel property of each vectors implies a very strict geometric restriction which lead to Lemma 4.5, which is a successful proof of RH. The contour trajectory of RZF that satisfies $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) = 0$ must be just a line segment that start from the origin and go back to origin. This simple restriction let us the second proof possible by using Cauchy integral theorem, as in Lemma 4.6.

References

[1] H.M. Edwards, Riemann's Zeta Function, Dover Publications, Inc., 1974.

[2] Aleksandar Ivic, The Riemann Zeta-Function, Theory and Applications, Dover Publications, Inc., 1985.

[3] https://en.wikipedia.org/wiki/Riemann_zeta_function

[4] https://www.youtube.com/watch?v=ZlYfEqdlhk0&list=PL32446FDD4DA932C9 Lectures on Euler-Riemann Zeta Function.

[5] https://en.wikipedia.org/wiki/Riemann_hypothesis

[6] https://en.wikipedia.org/wiki/Dirichlet_eta_function

[7] https://www.purebasic.com

[8] https://www.geogebra.org/m/UdjMfsKS

[9] https://en.wikipedia.org/wiki/Riemann_Xi_function

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Appendix A : Source Code For RZF or DEF Trace

```
;Code for zeta or eta vector trace visualization using PureBasic evaluation version.

;[1] graph window.
#Window1     = 0
#Image1      = 0
#ImgGadget = 0
#width              = 1370
#height             = 735

;[2] variables.
Define.d Dim x(10000000)       ;Re(z)
Define.d Dim y(10000000)       ;Im(z)
Define.d Dim t(10000000)       ;Arg(z), radian
Define.d Dim deg(10000000)    ;Arg(z), degree
Define.d Dim r(10000000)       ;r = |z|
Define.d a, b, r, t, delta, x0, y0, x1, y1, xsum, ysum, x2, y2, x3, y3, lnn, rr
Define.q i, m, n, thresh

;[3] sample zero values.
a = 1/2
;a = 0.501
;b = 14.134725141734693790
;b = 236.5242296658162058
;b = 5565.566217327
b = 24499.249265478
;b = 74908.108191005

;[4] font.
LoadFont (0, "Courier", 15) ;load Courier Font, Size 15.
LoadFont (1, "Arial", 24)     ;load Arial Font, Size 24.
OpenConsole()
If b>0
   header$ = "Riemann Zeta : s=" + a + "+" + b + "i"
   ;header$ = "Dirichlet Eta : s=" + a + "+" + b + "i"
Else
   header$ = "Riemann Zeta : s=" + a + "" + b + "i"
   ;header$ = "Dirichlet Eta : s=" + a + "" + b + "i"
EndIf
delta = 0.007    ;image zoom factor: small value for zoom in.
m = 7000 ;#vectors to plot.
pi.d = 3.1415926535

;[5] calculate vectors.
For n=1 To m Step 1
   lnn = Log(n)
   r(n) = Exp(-a*lnn)
   t(n) = -b*lnn
   deg(n) = Mod(Round(Degree(t(n)), #PB_Round_Down), 360)    ;#PB_Round_Up, #PB_Round_Nearest
   If deg(n)<0
      deg(n) = deg(n) + 360
   EndIf
   dg = deg(n) - deg(n-1)
```

```
  ;PrintN("n=" + n + " r(n)=" + r(n) + " θ=" + deg(n) + "° dθ=" + dg + "°" + " t(n)=" + t(n))   ;print values.
;for eta function, remove following 5 comments in If...Else...EndIf block.
;   If Mod(n, 2)=1
      x(n) = r(n)*Cos(t(n))
      y(n) = r(n)*Sin(t(n))
;   Else
;      x(n) = -r(n)*Cos(t(n))
;      y(n) = -r(n)*Sin(t(n))
;   EndIf
Next

;[6] graph origin.
x0 = #width/2
y0 = #height/2
xsum = 0
ysum = 0
If OpenWindow(#Window1, 0, 0, #width, #height, header$, #PB_Window_SystemMenu ) ;If 1
  If CreateImage(#Image1, #width, #height)                                  ;If 2
    ImageGadget(#ImgGadget, 0, 0, #width, #height, ImageID(#Image1))
    StartDrawing(ImageOutput(#Image1))
    Delay(2000)
    DrawingFont(FontID(1))   ;use the 'Courier' font
    c$ = "Riemann Zeta Function Vector Trace : s = " + a + " + " + b + "i"
    ;c$ = "Dirichlet Eta Function Vector Trace : s = " + a + " + " + b + "i"
    DrawText(200,200, c$, RGB(255, 255, 255))
    StopDrawing()
    ImageGadget(#ImgGadget, 0, 0, #width, #height, ImageID(#Image1))
    StartDrawing(ImageOutput(#Image1))
    Delay(1000)
    DrawingFont(FontID(1))
    c$ = "
"
    DrawText(150, 200, c$, RGB(0, 0, 0))   ;erase previous text.
    StopDrawing()
    ImageGadget(#ImgGadget, 0, 0, #width, #height, ImageID(#Image1))
    StartDrawing(ImageOutput(#Image1))
    ;axis.
    LineXY(0, y0, #width, y0, RGB(128,128,128))
    LineXY(x0, 0, x0, #height, RGB(128,128,128))
    x1 = Int(xsum/delta) + x0
    y1 = -Int(ysum/delta) + y0
    StopDrawing()

    ;[7]plot vectors.
    For i = 1 To m
      If Not(i>=startVector And i<=endVector)
        Gosub plotVector
      EndIf
    Next
    SetGadgetState(#ImgGadget, ImageID(#Image1))
    StartDrawing(ImageOutput(#Image1))
   EndIf   ;If 2
  Repeat
```

```
      Event = WaitWindowEvent()
   Until Event = #PB_Event_CloseWindow
EndIf ;If 1

;======================== plotVector Subroutine ================================
plotVector:
        xsum = xsum + x(i)
        ysum = ysum + y(i)
        xx = Int(xsum/delta)
        yy = Int(ysum/delta)
        x2 = xx + x0
        y2 = -yy + y0
        SetGadgetState(#ImgGadget, ImageID(#Image1))
        StartDrawing(ImageOutput(#Image1))
        ;vector colors.
        If Mod(i, 3) = 1
           color = RGB(255, 255, 255) ;white.
        ElseIf Mod(i, 3) = 2
           color = RGB(0, 255, 255)
        Else
           color = RGB(255, 0, 255)
        EndIf
        LineXY(x1, y1, x2, y2, color)
        x3 = Int(xsum*100)/100
        y3 = Int(ysum*100)/100
        rr = Sqr(xsum*xsum + ysum*ysum)
        c$ = "n = " + Str(i+jump) + " : (x, y) = (" + xsum + ", " + ysum + "), r = " + rr + ", θ = " + deg(i) + "°, dθ = "
+ Str(deg(i)-deg(i-1)) + "°                    "
        DrawText(20, 20, c$)
        PrintN(c$)
        If i=1 ;mark 0 and 1
           DrawText(x1, y1, "0")
           DrawText(x2, y2, "1")
        EndIf
        If i>=2 And i<=10    ;mark first 10 points
           DrawText(x2, y2, Str(i))
        EndIf
        Delay(1)    ;plot speed.
        StopDrawing()
        x1 = x2
        y1 = y2
Return
```

Appendix B: Source Code For Parallel RZF or DEF Trace

```
;Code for parallel zeta or eta vector trace visualization using PureBasic evaluation version.


#Window1     = 0
#Image1      = 0
#ImgGadget = 0
#width             = 1370
#height            = 735

Define.d Dim x(100000, 9)
Define.d Dim y(100000, 9)
Define.d Dim t(100000, 9)
Define.d Dim deg(100000, 9)
Define.d Dim r(100000, 9)
Define.d Dim a2(9)
Define.s Dim header$(9)

Define.d a, b, r, t, delta
Define.d Dim x0(9), Dim y0(9), Dim x1(9), Dim y1(9), Dim xsum(9), Dim ysum(9), Dim x2(9), Dim y2(9), Dim
x3(9), Dim y3(9)
Define.d lnn, rr
Define.q i, m, n, Dim color(9)

b = 14.134725141734693790
;b = 21.022039638771554993
;b = 69.546401711
;b = 124.256818554
;b = 236.5242296658162058
;b = 570.051114782
;b = 572.419984132
;b = 1201.810334857
;b = 2210.850941099
;b = 3156.300357947
;b = 5565.566217327
;b = 7776.955377123
;b = 9457.289938949
;b = 10000.065345417
;b = 10000.651847322
;b = 10000.918178956
;b = 12571.195309379
;b = 15536.816303095
;b = 24499.249265478
;b = 33945.406726423
;b = 48596.896626512
;b = 53243.675588739
;b = 74908.108191005

maxj = 9

a2(1) = 0.3
a2(2) = 0.35
a2(3) = 0.4
```

```
a2(4) = 0.45
a2(5) = 0.5
a2(6) = 0.55
a2(7) = 0.6
a2(8) = 0.65
a2(9) = 0.7

color(1) = RGB(255, 0, 0)
color(2) = RGB(255, 127, 0)
color(3) = RGB(255, 255, 0)
color(4) = RGB(0, 255, 0)
color(5) = RGB(255, 255, 255)
color(6) = RGB(0, 0, 255)
color(7) = RGB(75, 0, 130)
color(8) = RGB(148, 0, 211)
color(9) = RGB(0, 255, 255)

delta = 0.015 ;image size zoom factor
d = 0 ;delay
markN = 1 ;1=mark, 0=do not mark n on the image

m = 5500 ;# terms

LoadFont (0, "Courier", 15)                    ; Load Courier Font, Size 15
LoadFont (1, "Arial", 24)                      ; Load Arial Font, Size 24

OpenConsole()

For j=1 To maxj
   header$(j) = "Riemann Zeta : s" + j + "=" + a2(j) + "+" + b + "i"
Next

pi.d = 3.1415926535

For n=1 To m Step 1
   For j=1 To maxj

      lnn = Log(n)
      r(n, j) = Exp(-a2(j)*lnn)
      t(n, j) = -b*lnn
      deg(n, j) = Mod(Round(Degree(t(n, j)), #PB_Round_Down), 360) ;#PB_Round_Up, #PB_Round_Nearest
      If deg(n, j)<0
         deg(n, j) = deg(n, j) + 360
      EndIf
      dg = deg(n, j) - deg(n-1, j)
      ;PrintN("n=" + n + " r(n, j)=" + r(n, j) + " θ" + j + "=" + deg(n, j) + "° dθ" + j + "=" + dg + "°" + " t(n, j)=" + t(n,
j))
   ;   If Mod(n, 2)=1 ;For eta function, remove comments of If...Else...EndIf block.
         x(n, j) = r(n, j)*Cos(t(n, j))
         y(n, j) = r(n, j)*Sin(t(n, j))
   ;   Else
   ;      x(n, j) = -r(n, j)*Cos(t(n, j))
   ;      y(n, j) = -r(n, j)*Sin(t(n, j))
```

```
   ;    EndIf

   Next
Next

;origin
For j=1 To maxj
   x0(j) = #width/2
   y0(j) = #height/2
   xsum(j) = 0
   ysum(j) = 0
Next

j = 1

If OpenWindow(#Window1, 0, 0, #width, #height, header$(1) + "/" + header$(2), #PB_Window_SystemMenu ) ;If
1
   If CreateImage(#Image1, #width, #height)                                        ;If 2

      ImageGadget(#ImgGadget, 0, 0, #width, #height, ImageID(#Image1))
      StartDrawing(ImageOutput(#Image1))

      Delay(1000)

      DrawingFont(FontID(1)) ; Use the 'Courier' font

      c$ = "Riemann Zeta Function Vector Trace : s = " + a2(1) + " + " + b + "i"
      DrawText(200,200, c$, RGB(255, 255, 255))
      c$ = "Riemann Zeta Function Vector Trace : s = " + a2(maxj) + " + " + b + "i"
      DrawText(200,300, c$, RGB(255, 255, 255))

      StopDrawing()

      ImageGadget(#ImgGadget, 0, 0, #width, #height, ImageID(#Image1))
      StartDrawing(ImageOutput(#Image1))

      Delay(2000)

      DrawingFont(FontID(1))
      c$                                        =                                        "
"
      DrawText(150, 200, c$, RGB(0, 0, 0))
      DrawText(150, 300, c$, RGB(0, 0, 0))

      StopDrawing()

      ImageGadget(#ImgGadget, 0, 0, #width, #height, ImageID(#Image1))
      StartDrawing(ImageOutput(#Image1))


       LineXY(0, y0(j), #width, y0(j), RGB(128,128,128))
       LineXY(x0(j), 0, x0(j), #height, RGB(128,128,128))
```

```
    For j=1 To maxj
      x1(j) = Int(xsum(j)/delta) + x0(j)
      y1(j) = -Int(ysum(j)/delta) + y0(j)
    Next

    xo = 150
    yo = 150
    deg(0, 1) = 0
    deg(0, 2) = 0

    StopDrawing()

    lastMile = 0

    For i = 1 To m    ;For loop
      For j=1 To maxj
        Gosub plotVector
      Next


      If Mod(i, 100)=0
        For j=2 To maxj+0
          Delay(1)
          SetGadgetState(#ImgGadget, ImageID(#Image1))
          StartDrawing(ImageOutput(#Image1))
          LineXY(x1(j-1), y1(j-1), x1(j), y1(j), color(5))
          StopDrawing()
        Next
      EndIf

    Next ;For loop

   EndIf ;If 2

  Repeat
    Event = WaitWindowEvent()
  Until Event = #PB_Event_CloseWindow

EndIf ;If 1

;======================= plotVector Subroutine ==============================

plotVector:

  Delay(d)

      xsum(j) = xsum(j) + x(i, j)
      ysum(j) = ysum(j) + y(i, j)
      xx = Int(xsum(j)/delta)
      yy = Int(ysum(j)/delta)
      x2(j) = xx + x0(j)
      y2(j) = -yy + y0(j)
```

```
        SetGadgetState(#ImgGadget, ImageID(#Image1))
        StartDrawing(ImageOutput(#Image1))

        LineXY(x1(j), y1(j), x2(j), y2(j), color(j))

        x3(j) = Int(xsum(j)*100)/100
        y3(j) = Int(ysum(j)*100)/100

        rr = Sqr(xsum(j)*xsum(j) + ysum(j)*ysum(j))

        c$ = "n = " + Str(i+jump) + " : (x, y) = (" + xsum(j) + ", " + ysum(j) + "), r = " + rr + ", θ = " + deg(i, j) + "°,
dθ = " + Str(deg(i, j)-deg(i-1, j)) + "°                    "
        DrawText(20, 20, c$)
        ;PrintN(c$)

        If i=1 ;mark 0 and 1
          DrawText(x1(j), y1(j), "0")
          DrawText(x2(j), y2(j), "1")
        EndIf

        StopDrawing()

        x1(j) = x2(j)
        y1(j) = y2(j)
Return
```

Appendix C: Vector Trace Videos

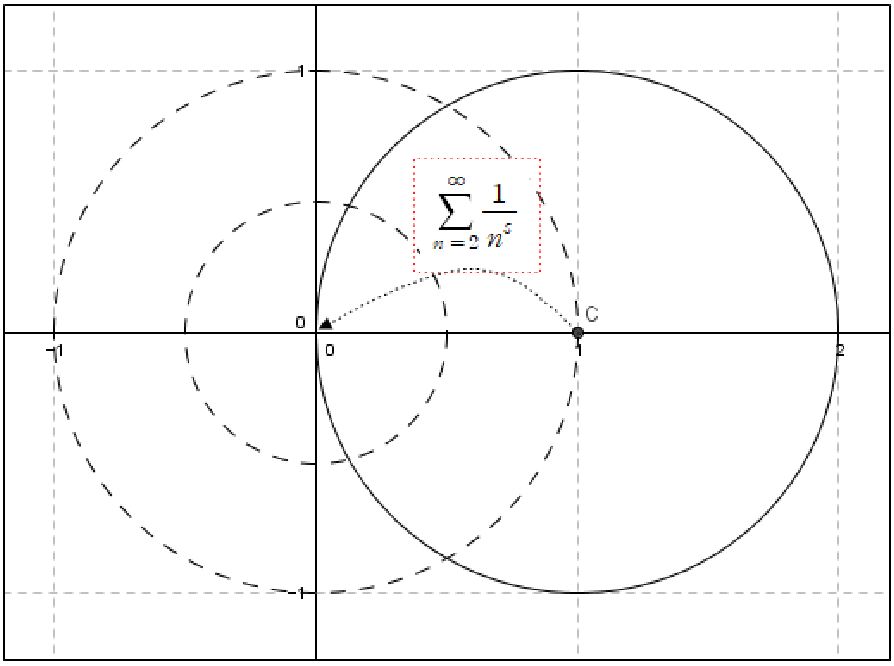
seq	type	s	link
1		0.4 + 10000.065i	https://www.youtube.com/watch?v=cIZIImNScll
2		0.6 + 10000.065i	https://www.youtube.com/watch?v=CHjCcqthuTc
3	DEF	0.5 + 74908.108i	https://www.youtube.com/watch?v=dE2fnWLzqxw
4		0.5 + 10000.065i	https://www.youtube.com/watch?v=3c2riWMV78I
5		0.5 + 14.135i	https://www.youtube.com/watch?v=5XPmdAfBphw
6		0.4 + 10000.065i	https://www.youtube.com/watch?v=54FGRm4mb_c
7	RZF	0.6 + 10000.065i	https://www.youtube.com/watch?v=pOeANPrMIRI
8		0.5 + 74908.108i	https://www.youtube.com/watch?v=W09mzoCTHEI

Appendix D: Other Possible Proofs

D.1. Possible Proof 1: By Lattice Hitting

RZF can be rewritten as $\zeta(s) = 1 + \sum_{n=2}^{\infty} 1/n^s$ and we can consider zero of ζ(s) as where $\sum_{n=2}^{\infty} 1/n^s$ hit the origin starting from (1, 0), as in the following figure.

Relativistic View of Zero of ζ(s)



Considering that the origin is also a lattice point, only some circle with radius $\sqrt{n}$ or $1/\sqrt{n}$ can hit the origin. To keep the radius to be of $1/\sqrt{n}$ pattern, α should be 1/2.

D.2. Possible Proof 2: By x-Axis Property

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$= A(\alpha)e^{iB(\beta)}$$

$$= A(\alpha)\{cosB(\beta) + isinB(\beta)\}$$

$$= u + iv$$

Eventually, RZF falls into just the two sine and cosine functions, but with a variable amplitude $A(\alpha)$ and a variable argument $B(\beta)$. So, the zeros of RZF must be on the x-axis, because the zeros must satisfy $cosB(\beta) = 0, sinB(\beta) = 0$, simultaneously.

The x-axis on the complex plane is just the critical line, where zeros are found. That is to say, zeros of sinusoidal functions are found only on *x*-axis, so, the crtical line should be the x-axis.