Using the partial sums of the Alternating Harmonic Series to prove the Harmonic Series diverges

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Abstract: In this article we shall use the partial sums of the alternating harmonic series to (a) prove the harmonic series diverges, and (b) show that every harmonic number greater than 1 is the sum of partial sums of the alternating harmonic series.

There are many ways to prove the harmonic series diverges. We shall give a novel proof using the partial sums of the alternating harmonic series. Then we shall show that every harmonic number greater than 1 is the sum of partial sums of alternating harmonic series.

The partial sums of the harmonic series are the (finite) harmonic numbers:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad for\ n = 1, 2, 3, \dots .$$

Related to the harmonic numbers are the partial sums of the alternating harmonic series:

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n}, \quad for\ n = 1, 2, 3, \dots .$$

So we establish a connection between both partial sums.

Lemma:

$$\sum_{k=1}^{2^n} \frac{1}{k} = \sum_{k=1}^{2^n} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{2^{n-1}} \frac{1}{k}, \quad for\ n = 1, 2, 3, \dots .$$

Proof:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n - 1} + \frac{1}{2^n} - \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2^n - 1} - \frac{1}{2^n}\right)$$

$$= (1 - 1) + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{2^n - 1} - \frac{1}{2^n - 1}\right) + \left(\frac{1}{2^n} + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{n-1}}$$

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As an aside plugging in $n = 1, 2, 3, \dots$ gives us the following list of harmonic numbers:

$$H_2 = 1 + \frac{1}{2} = \left(1 - \frac{1}{2}\right) + 1$$

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + \left(1 + \frac{1}{2}\right)$$

$$H_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}\right) + \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$$

$$\vdots$$

As shown below this listing of the subsequence $\{H_{2^k}\}_{k=1}^{\infty}$ contains all that is required to prove the harmonic numbers diverge.

Theorem 1: The sequence $\{H_n\}_{n=1}^{\infty}$ is divergent.

Proof: As the identity holds for $n = 1, 2, 3, \ldots$ we list the subsequence $\{H_{2^k}\}_{k=1}^{\infty}$ as follows:

$$H_2 = 1 + \frac{1}{2} = \left(1 - \frac{1}{2}\right) + 1$$

$$\begin{aligned} H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} &= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + \left(1 + \frac{1}{2}\right) \\ &= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + H_2 \\ &= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + \left(1 - \frac{1}{2}\right) + 1 \end{aligned}$$

$$\begin{aligned} H_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} &= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}\right) + \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \\ &= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}\right) + H_4 \\ &= \left(1 - \frac{1}{2} + \cdots + \frac{1}{7} - \frac{1}{8}\right) + \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + \left(1 - \frac{1}{2}\right) + 1 \end{aligned}$$

$$\vdots$$

Therefore, as each consecutive harmonic number has an additional partial sum on the r.h.s. the subsequence $\{H_{2^k}\}$ is unbounded. Hence, the sequence $\{H_n\}$ is divergent.

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The proof above raises the question of whether *every* harmonic number greater than 1 is the sum of partial sums of the alternating harmonic series?

Lemma A:

$$\sum_{k=1}^{2n} \frac{1}{k} = \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{n} \frac{1}{k}, \quad for\ n = 1, 2, 3, \ldots . \qquad (A)$$

Proof:

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}-\left(1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{2n-1}-\frac{1}{2n}\right)$$

$$=(1-1)+\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n-1}\right)+\left(\frac{1}{2n}+\frac{1}{2n}\right)$$

$$=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

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Lemma B:

$$\sum_{k=1}^{2n+1}\frac{1}{k}=\sum_{k=1}^{2n+1}\frac{(-1)^{k-1}}{k}+\sum_{k=1}^{n}\frac{1}{k}, \quad for\ n=1,2,3,\dots . \qquad (B)$$

Proof:

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}+\frac{1}{2n+1}-\left(1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{2n}+\frac{1}{2n+1}\right)$$

$$=(1-1)+\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{2n}+\frac{1}{2n}\right)+\left(\frac{1}{2n+1}-\frac{1}{2n+1}\right)$$

$$=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

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Theorem 2: Every harmonic number greater than 1 is the sum of partial sums of the alternating harmonic series.

Proof: Similar to the previous proof using A and B allows us to consecutively list the harmonic numbers as follows:

$$H_1 = 1 = \mathbf{1}$$

$(A): n=1$ $$H_2 = 1+\frac{1}{2} = \left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2}}\right)+\mathbf{1}$$

$(B): n=1$ $$H_3 = 1+\frac{1}{2}+\frac{1}{3} = \left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2}}+\frac{\mathbf{1}}{\mathbf{3}}\right)+\mathbf{1}$$

$(A): n=2$ $$H_4 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2}}+\frac{\mathbf{1}}{\mathbf{3}}-\frac{\mathbf{1}}{\mathbf{4}}\right)+\left(1+\frac{1}{2}\right)$$

$$= \left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2}}+\frac{\mathbf{1}}{\mathbf{3}}-\frac{\mathbf{1}}{\mathbf{4}}\right)+\left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2}}\right)+\mathbf{1}$$

$(B): n=2$ $$H_5 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} = \left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2}}+\frac{\mathbf{1}}{\mathbf{3}}-\frac{\mathbf{1}}{\mathbf{4}}+\frac{\mathbf{1}}{\mathbf{5}}\right)+\left(1+\frac{1}{2}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}}\right) + \mathbf{1}$$

$$H_6 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}}\right) + \left(1 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}}\right) + \mathbf{1}$$

$$H_7 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}} + \frac{\mathbf{1}}{\mathbf{7}}\right) + \left(1 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}} + \frac{\mathbf{1}}{\mathbf{7}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}}\right) + \mathbf{1}$$

$$H_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}} + \frac{\mathbf{1}}{\mathbf{7}} - \frac{\mathbf{1}}{\mathbf{8}}\right) + \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \cdots + \frac{\mathbf{1}}{\mathbf{7}} - \frac{\mathbf{1}}{\mathbf{8}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}}\right) + \mathbf{1}$$

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