Proof of Polignac's conjecture for gap equal to eight

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Abstract In this paper proof of the Polignac's Conjecture for gap equal to eight is going to be presented. It will be shown that consecutive primes with gap eight could be obtained through two stage sieve process, and that will be used to prove that infinitely many primes with gap eight exist. The proof represents an simple extension of the recently presented proof that infinitely many sexy prime exist. The major contribution of this paper is presentation of all elementary modules that are necessary for the proof of Polgnac's conjecture in general case.

1 Introduction

In number theory, Polignac's conjecture states: For any positive even number g, there are infinitely many prime gaps of size g. In other words: there are infinitely many cases of two consecutive prime numbers with the difference g [1].

In [2] it has been shown that exists infinitely many consecutive prime numbers that have gaps that are not bigger than 246. Recently, the Polignac's conjecture was proved for gaps of the size 2 and 4 [3], as well as gap equal to six [4]. The problem was addressed in generative space, which means that prime numbers were not analyzed directly, but rather their representatives that can be used to produce them. This paper represents a simple extension of the previous work [4]. Here, the gap equal to eight is going to be analyzed.

Basically, three groups of gaps bigger than 4 exist: the gaps of the size $6k$, the gaps of the size $6k+2$ and gaps of the size $6k+4$, $k \in N$. In the text that follows we mark the prime numbers in the form $6k - 1$ as *mps* primes and prime numbers in the form $6k + 1$ as *mpl* primes, $k \in N$. The gaps of the size $6k$ could be related to the prime pairs in both (*mps, mps*) and (*mpl, mpl*) form. The gaps in the form

$6k + 2$ can only be related to the pair of primes in (*mps, mpl*) form, while gaps in the form $6k + 4$ can only be related to the pair of primes in (*mpl, mps*) form. In other words there is not a single prime in *mpl* form that has consecutive prime that is $6k + 2$ apart, and there is not a single prime in *mps* form that has consecutive prime $6k + 4$ apart. It is trivial to show that by simple calculation. Here it will be shown that exist an infinite number of primes with gaps of the size 8 (we mark them as *g*8-primes). It will be shown that primes with gap of the size 8 could be generated by two stage recursive type sieve process. This process will be compared to other two stage recursion sieve process that leaves infinitely many numbers. Fact that sieve process that generate *g*8-primes leaves more numbers than the other sieve process will be used to prove that infinitely many consecutive primes with gap 8, exist.

Remark 1: *In this paper any infinite series in the form* $c_1 \cdot l \pm c_2$ *is going to be called a thread defined by number* c_1 *(in literature these forms are known as linear factors – however, it seems that the term thread is probably better choice in this context). Here* c_1 *and* c_2 *are numbers that belong to the set of natural numbers (*c_2 *can also be zero and usually is smaller than* c_1*) and* l *represents an infinite series of consecutive natural numbers in the form (1, 2, 3, …).*

Remark 2: *In the text that follows we will use fact that number of even/odd numbers is equal to the one half of the number of natural numbers (also the 1/3, 4/5 and other fractions of the number of natural numbers are used). This is not an usual way of discussing the number of numbers that are infinitely big, but it is quite suitable in this context.* ***It is important to notice that there is no context in which is correct to state that number of natural numbers is equal to the number of odd or even numbers (or that number of natural numbers is equal to the number of numbers divisible by 3, or 5, and so on). What can be said is that it is possible to generate the same number of unique labels for odd or even numbers using the same number of unique labels for natural numbers.*** *However, if we want to produce all even and odd labels at the same time,* *obviously, it is necessary to have two sets of natural numbers at the same time (it is necessary to clone the set of natural numbers), which means that the number of the natural numbers in that moment is two times bigger than the number of even or odd numbers. This analysis can also be made in quantum probabilistic context, but it is beyond the scope of this paper.*

2 Proof of Polignac's conjecture for gap equal to eight

As it was already explained in the introduction part, if two consecutive prime numbers have the gap of the size 8 it is clear that smaller of those numbers has to be in *mps* form while the bigger one has to be in *mpl* form. Here, we are going to present a two stage process that can be used for generation of the *g*8-primes.

In the first stage we are going to produce prime numbers by removing all composite numbers from the set of natural numbers.

In the second stage, we are going to remove all twin primes and a quarter of sexy primes (or more precisely one half of *mps* sexy primes). In order to understand why is it so, we are going to consider 5 consecutive numbers $6k$-1, $6k$+1, $6k$+3, 6(k+1)-1, 6(k+1)+1. We can see that $6k$-1 and 6(k+1)+1 can create a *g*8-primes only in the case when $6k$+1 and 6(k+1)-1 are not primes. Here, we are going to analyze three situations. When $6k$+1 and 6(k+1)-1 are both primes, and in the caswe when only $6k$+1 is prime, removal of twin primes will remove *mps* primes that cannot have *g*8-prime pair. However, when $6k$+1 is not prime and 6(k+1)-1 is prime, it is also necessary to remove smaller number of sexy prime pair ($6k$-1, 6(k+1)-1) since it also cannot have a *g*8-prime pair. .

Also at the end of stage 2, we are going to remove all prime numbers in *mpl* form that are left, and all prime numbers in *mps* form that have an bigger odd *g*8-neighbor (odd number that has gap 8 with the prime of interest) that is a composite number. At the end, only the prime numbers in the *mps* form, that represent the smaller number of a *g*8-prime pair, are going to stay. Their number is a half of the number of *g*8-primes. It is going to be shown that that number is infinite.

STAGE 1

Prime numbers can be obtained in the following way:

First, we remove all even numbers (except 2) from the set of natural numbers. Then, it is necessary to remove the composite odd numbers from the rest of the numbers. In order to do that, the formula for the composite odd numbers is going to be analyzed. It is well known that odd numbers bigger than 1, here denoted by a, can be represented by the following formula

$$a = 2n + 1,$$

where $n \in N$. It is not difficult to prove that all composite odd numbers a_c can be represented by the following formula

$$a_c = 2(2ij + i + j) + 1 = 2((2j+1)i + j) + 1. \tag{1}$$

where $i, j \in N$. It is simple to conclude that all composite numbers could be represented by product $(i + 1)(j + 1)$, where $i, j \in N$. If it is checked how that formula looks like for the odd numbers, after simple calculation, equation (1) is obtained. This calculation is presented here. The form $2m + 1$, $m \in N$ will represent odd numbers that are composite. Then the following equation holds

$$2m + 1 = (i_1 + 1)(j_1 + 1) \quad ,$$

where $i_1, j_1 \in N$. Now, it is easy to see that the following equation holds

$$m = \frac{i_1 j_1 + i_1 + j_1}{2}.$$

In order to have $m \in N$, it is easy to check that i_1 and j_1 have to be in the forms

$$i_1 = 2i \text{ and } j_1 = 2j,$$

where $i, j \in N$. From that, it follows that m must be in the form

$$m = 2ij + i + j = (2i + 1)j + i. \tag{2}$$

When all numbers represented by m are removed from the set of odd natural numbers bigger than 1, only the numbers that represent odd prime numbers are going to stay. In other words, only odd numbers that cannot be represented by (1) will stay. This process is equivalent to the sieve of Sundaram [5].

Let us denote the numbers used for the generation of odd prime numbers with *m2* (here we ignore number 2). Those are the numbers that are left after the implementation of Sundaram sieve. The number of those numbers that are smaller than some natural number n, is equivalent to the number of prime numbers smaller than n. If we denote with $\pi(n)$ number of primes smaller than n, than the following equation holds

$$\pi(n) \approx \frac{n}{\ln(n)}.$$

From [6] we know that following holds

$$\pi(n) > \frac{n}{\ln(n)}, \quad n \geqslant 17. \tag{3}$$

STAGE 2

What was left after the first stage are prime numbers. With the exception of number 2, all other prime numbers are odd numbers. All odd primes can be expressed in the form $2n + 1$, $n \in N$. It is simple to understand that their bigger odd $g8$-neighbor must be in the form $2n + 9$, $n \in N$. Now, we should implement a second stage in which we are going to **remove**:

A. Number 2 (since 2 cannot make a $g8$-prime pair);

B. All twin primes and quarter of sexy primes– so number of numbers that is going to be left is number of primes minus the number of twin primes minus one quarter of sexy primes (number 2 is ignored, and that has no impact on the analysis that follows). It is not difficult to prove that number of numbers left, is infinite. In Appendix C is going to be explained how that can be done.

C. The rest of *mpl* primes – it is trivial to see that it can be done by one thread that is defined by 3 – so in this step it is going to be removed, approximately (having in mind that the number of *mps* primes is a bit bigger than the number of *mpl* primes), one half of the numbers that are left after step

B;

D. All odd primes in the form $2m + 1$ such that $2m + 9$, $m \in N$ represents a composite number (all primes whose bigger odd *g*8-neighbor is composite number). If we make the same analysis, like in the Stage 1, it is simple to understand that *m* must be in the form

$$m = 2ij + i + j - 1 = (2i + 1)j + i - 4. \tag{4}$$

All numbers (in observational space) that are going to stay must be numbers in *mps* form and they represent a smaller primes of the *g*8-prime pairs in *mps* form. What has to be noticed is that thread in (4) that is defined by prime number 3 (for $i = 1$) is not going to remove any number from the numbers left.

Let us mark the number of *g*8-primes with π_{G8}. Also, let us define the number of numbers that is left after two consecutive implementations of Sundaram sieve as *pd8*. The numbers obtained after recursive implementation of two Sundaram sieves (where the second Sundaram sieve is implemented on prime numbers from which number 2, all twin primes and a quarter of sexy primes are removed) are going to be called *g*8-double primes. The second stage sieve that is identical to the first stage sieve can be obtained if the prime numbers left after the first stage and removal off 2, all twin primes and a quarter of sexy primes, are lined up next to each other and then the numbers are removed from the exactly same positions like in the first stage. In that case it is easy to understand that the following equation would holds ($n \in N$)

$$pd8(n) \approx \frac{\pi(n) - \pi_{G2}(n) - 0.25\pi_{G6}(n)}{\ln\left(\pi(n) - \pi_{G2}(n) - 0.25\pi_{G6}(n)\right)} \quad , \tag{5}$$

where *pd8*(*n*) represents the number of *g*8-double primes smaller than some natural number *n*. Since the *mps* *g*8-primes are obtained by implementation of the Sundaram sieve in the first stage and sieve that is similar to Sundaram sieve in the second stage, it can be intuitively concluded that numbers $\pi_{G8}/2$ and *pd8* should be comparable. However, in the case of generation of *mps* *g*8-primes the second stage sieve defined by (4) is not equivalent to the the first stage sieve since the second stage

“Sundaram” sieve is applied on an incomplete set, that is depleted by previously implemented Sundaram sieve. Here, it will be shown that number of *g*8-double primes *pd8* is smaller than the number $\pi_{G8}/2$. In order to understand why it is so, we are going to analyze (2) and (4) in more detail.

It is not difficult to be seen that *m* in (2) and (4) is represented by the threads that are defined by odd prime numbers. For details see Appendix A. Now we are going to compare stages 1 and 2 step by step, for a few initial steps (analysis can be easily extended to any number of steps). Starting point for the second stage is after removal of number 2, all twin primes and quarter of sexy primes.

Table 1 Comparison of the stages 1 and 2 – threads defined by a few smallest primes

Step	Stage 1	Step	Stage 2
1	Remove even numbers (except 2) amount of numbers left is 1/2	1	Remove numbers defined by thread defined by 3 (obtained for *i* = 1) amount of numbers left is 1/2
2	Remove numbers defined by thread defined by 3 (obtained for *i* = 1) amount of numbers left is 2/3 of the numbers that are left after previous step	2	Remove numbers defined by thread defined by 5 (obtained for *i* = 2) amount of numbers left is 3/4 of the numbers that are left after previous step
3	Remove numbers defined by thread defined by 5 (obtained for *i* = 2) amount of numbers left is 4/5 of the numbers that are left after previous step	3	Remove numbers defined by thread defined by 7 (obtained for *i* = 3) amount of numbers left is 5/6 of the numbers that are left after previous step
4	Remove numbers defined by thread defined by 7 (obtained for *i* = 3) amount of numbers left is 6/7 of the numbers that are left after previous step	4	Remove numbers defined by thread defined by 11 (obtained for *i* = 5) amount of numbers left is 9/10 of the numbers that are left after previous step
5	Remove numbers defined by thread defined by 11 (obtained for *i* = 5) amount of numbers left is 10/11 of the numbers that are left after previous step	5	Remove numbers defined by thread defined by 13 (obtained for *i* = 6) amount of numbers left is 11/12 of the numbers that are left after previous step

From the table, it can be noticed that threads defined by the same number in first and second stage will not remove the same percentage of numbers. The reason is obvious – consider for instance the thread defined by 3: in the first stage it will remove 1/3 of the numbers left, but in the second stage it will remove ½ of the numbers left, since the thread defined by 3 in stage 1 has already removed one third of the numbers (odd numbers divisible by 3 in observation space). So, only odd numbers (in observational space) that give residual 1 and -1 when they are divided by 3 are left, and there is approximately same number of numbers that give residual -1 and numbers that give residual 1, when the number is divided by 3 (see Appendix A). Same way of reasoning can be applied for all other threads defined by the same prime in different stages.

From Table 1 can be seen that in every step, except step 1, threads in the second stage will leave bigger percentage of numbers than the corresponding threads in the first stage. This could be easily understood from the analysis that follows:

- suppose that we have two natural numbers j, k such that $j - 1 \geq k$ ($j, k \in N$), then the following set of equations is trivially true

$$j + k - 1 \geqslant 2k$$

$$-j - k + 1 \leqslant -2k$$

$$jk - j - k + 1 \leqslant jk - 2k$$

$$(j-1)(k-1) \leqslant (j-2)k$$

$$\frac{k-1}{k} \leqslant \frac{j-2}{j-1}$$

The equality sign holds only in the case $j = k + 1$. In the set of prime numbers there is only one case when $j = k + 1$ and that is in the case of primes of 2 and 3. In all other cases $p(i) - p(i-1) > 1$, ($i > 1$, $i \in N$, $p(i)$ is i-th prime number). So, in all cases $i > 2$

$$\frac{p(i-1)-1}{p(i-1)}<\frac{p(i)-2}{p(i)-1} \quad .$$

From Table 1 (or last equation) we can see that bigger number of numbers is left in every step of stage 2 then in the stage 1 (except 1[st] step). From that, we can conclude that after every step bigger than 1, part of the numbers that is left in stage 2 is bigger than number of numbers left in the stage 1 (that is also noticeable if we consider amount of numbers left after removal of all numbers generated by threads that are defined by all prime numbers smaller than some natural number). From previous analysis we can safely conclude that the following equation holds

$$\pi_{G8}>\frac{\pi_{G8}}{2}>pd8=\lim_{n\to\infty} pd8(n) \quad .$$

Having in mind (3) and (5), we can say that for some n big enough the following inequality holds

$$pd8(n)>\frac{\pi(n)-\pi_{G2}(n)-0.25\pi_{G6}(n)}{\ln\left(\pi(n)-\pi_{G2}(n)-0.25\pi_{G6}(n)\right)} . \tag{6}$$

It can be realized that n that is big enough is $n\geq 317$, since 317 is the 17[th] prime left, when number 2, all twin primes and a quarter of the sexy primes are eliminated from the prime numbers set. It is not difficult to prove that the number of primes that are left when 2 and all twin primes and quarter of sexy primes are eliminated, is infinite. How it can be done is explained in Appendix C.

Having that in mind it it easy to show that following holds

$$\lim_{n\to\infty}\frac{\pi(n)-\pi_{G2}(n)-0.25\pi_{G6}(n)}{\ln\left(\pi(n)-\pi_{G2}(n)-0.25\pi_{G6}(n)\right)}=\infty .$$

Then, the following equation holds

$$pd8=\lim_{n\to\infty} pd8(n)=\infty .$$

Now, we can safely conclude that the number of $g8$-primes is infinite. That concludes the proof.

Here we will state the following conjecture: for n big enough, number of $g8$-primes is given by the following equation

$$\pi_{G8}(n) \sim 4C_2 \cdot \left(\frac{(\pi(n) - \pi_{G2}(n) - 0.25\pi_{G6}(n))}{\ln(\pi(n) - \pi_{G2}(n) - 0.25\pi_{G6}(n))} \right).$$

Constant C_2 s known as twin prime constant [7]. The equation can be expressed by using only *n*, but it would be cumbersome. Why it is reasonable to make such conjecture is explained in Appendix B. If we mark the number of primes smaller than some natural number n with $\pi(n) = f(n)$, where function $f(n)$ gives good estimation of the number of primes smaller than *n,* than $\pi_{G8}(n)$, for n big enough, is given by the following equation

$$\pi_{G8}(n) \sim 4C_2 \cdot \left(f\left(f(n) - \pi_{G2}(n) - 0.25\pi_{G6}(n) \right) \right).$$

Here we can see that constant C_2 has a misleading name. It is connected with repeated implementation of a sieve that produces prime numbers which is also, but not exclusively, connected to the twin primes. Where C_2 is coming from is explained in Appendix B.

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APPENDIX A.

Here it is going to be proved that m in (2) is represented by threads defined by odd prime numbers. Now, the form of (2) for some values of i will be checked.

Case i = 1: $m = 3j + 1$,

Case i = 2: $m = 5j + 2$,

Case i = 3: $m = 7j + 3$,

Case i = 4: $m = 9j + 4 = 3(3j + 1) + 1$,

Case i = 5: $m = 11j + 5$,

Case i = 6: $m = 13j + 6$,

Case i = 7: $m = 15y + 7 = 5(3j + 1) + 2$,

Case i = 8: $m = 17j + 8$,

It can be seen that m is represented by the threads that are defined by odd prime numbers. From examples (cases $i = 4$, $i = 7$), it can be seen that if $(2i + 1)$ represent a composite number, m that is represented by thread defined by that number also has a representation by the the thread defined by one of the prime factors of that composite number. That can be proved easily in the general case, by direct calculation, using representations similar to (2). Here, that is going to be analyzed. Assume that $2i + 1$ is a composite number, the following holds

$$2i + 1 = (2l + 1)(2s + 1)$$

where ($l, s \in N$). That leads to

$$i = 2ls + l + s.$$

The simple calculation leads to

$$m = (2l + 1)(2s + 1)j + 2ls + l + s = (2l + 1)(2s + 1)j + s(2l + 1) + l$$

or

$$m = (2l+1)((2s+1)j + s) + l$$

which means

$$m = (2l + 1)f + l,$$

and that represents the already exiting form of the representation of m for the factor $(2l + 1)$, where

$$f = (2s + 1)j + s.$$

In the same way this can be proved for (4), (5) and (7).

Note: It is not difficult to understand that after implementation of stage 1, the number of numbers in residual classes of some specific prime number are equal. In other words, after implementation of stage 1, for example, all numbers divisible by 3 (except 3, but it does not affect the analysis) are removed. However, the number of numbers in the forms $3k + 1$ and $3k + 2$ (alternatively, $3k – 1$) are equal. The reason is that the thread defined by any other prime number (bigger than 2) will remove the same number of numbers from the numbers in the form $3k + 1$ and from the numbers in the form $3k + 2$. It is simple to understand that, for instance, thread defined by number 5, is going to remove 1/5 of the numbers in form $3k + 1$ and 1/5 of the numbers in form $3k + 2$. This can be proved by elementary calculation. That will hold for all other primes and for all other residual classes.

APPENDIX B.

Here asymptotic density of numbers left, after implementation of the first and second Sundaram sieve is calculated.

After first k steps of the first Sundaram sieve, after removal of all composite even numbers, density of numbers left is given by the following equation

$$c_k = \frac{1}{2}\prod_{j=2}^{k+1}\left(1 - \frac{1}{p(j)}\right),$$

where $p(j)$ is j-th prime number.

In the case of second "Sundaram" sieve the density of numbers left after the first k-steps is given by the following equation

$$c2_k = \prod_{j=2}^{k+1}\left(1 - \frac{1}{p(j) - 1}\right) = \prod_{j=2}^{k+1}\left(\frac{p(j) - 2}{p(j) - 1}\right).$$

So, if implementation of first sieve will result in the number of prime numbers smaller than n which we denote as $\pi(n)$, than implementation of the second sieve on some set of size $\pi(n)$-$\pi_{G2}(n)$ should result in the number of numbers $gp(n)$ that are defined by the following equation (for some big enough n)

$$gp(n) = r_{S2S1}(n) \cdot \frac{\pi(n) - \pi_{G2}(n)}{\ln\left(\pi(n) - \pi_{G2}(n)\right)} \quad ,$$

where $r_{S2S1}(n)$ is defined by the following equation (k is the number of primes smaller or equal to $n1$ = $sqrt(n)$, where $sqrt$ marks square root function)

$$r_{S2S1}(n) = \frac{c2_k}{c_k} = \frac{\prod_{p>2,\, p\leq n1}\left(\frac{p-2}{p-1}\right)}{\prod_{p\leq n1}\left(\frac{p-1}{p}\right)} = 2\prod_{p>2,\, p\leq n1}\left(\frac{p-2}{p-1}\right)\left(\frac{p}{p-1}\right) \approx 2\mathrm{C}_2 .$$

For n that is not big, $gp(n)$ should be defined as

$$gp(n) = f_{COR}(n) \cdot 2C_2 \cdot \frac{\pi(n) - \pi_{G2}(n)}{\ln\left(\pi(n) - \pi_{G2}(n)\right)},$$

where $f_{COR}(n)$ represents correction factor that asymptotically tends toward 1 when n tends to infinity.

APPENDIX C.

Here, it is going to be shown that the number of primes that are not twin primes is infinite. One way to do it has been used in [4]. The idea was to use the fact that the sum of the reciprocals of primes is infinite and the sum of reciprocals of twin primes is finite [8]. Here we are going to present more general method that can be used in many different problems.

Using the same line of reasoning, it is not difficult to prove that the number of primes from which number 2, all twin primes and quarter of sexy primes are removed, is infinite. Since the proof is straight forward extension of the proof presented here, and since the equations in that case become much more complex, details of that proof are not presented here.

Here we are going to analyze the following table which is obtained by small modification of Table 1. Actually we are going to compare the sieve that is used in the second stage of generation of twin primes, with the recursive sieve that is almost equivalent to Sundaram sieve, with one small exception – in the first step we are not going to remove half of the numbers – rather we are going to start with the second step where we are going to remove one third of the numbers. What we clearly know is the fact that the newly proposed sieve will leave 2 times more numbers than the original standard sieve. So, we are going to call that sieve D-Sundaram sieve.

Table C Comparison of the two sieves – threads defined by a few smallest primes

Step	Stage 1	Step	Stage 2
1	Remove numbers defined by thread defined by 3 (obtained for $i = 1$) amount of numbers left is 2/3 of the numbers that are left after previous step	1	Remove numbers defined by thread defined by 3 (obtained for $i = 1$) amount of numbers left is 1/2
2	Remove numbers defined by thread defined by 5 (obtained for $i = 2$) amount of numbers left is 4/5 of the numbers that are left after previous step	2	Remove numbers defined by thread defined by 5 (obtained for $i = 2$) amount of numbers left is 3/4 of the numbers that are left after previous step
3	Remove numbers defined by thread defined by 7 (obtained for $i = 3$) amount of numbers left is 6/7 of the numbers that are left after previous step	3	Remove numbers defined by thread defined by 7 (obtained for $i = 3$) amount of numbers left is 5/6 of the numbers that are left after previous step
4	Remove numbers defined by thread defined by 11 (obtained for $i = 5$) amount of numbers left is 10/11 of the numbers that are left after previous step	4	Remove numbers defined by thread defined by 11 (obtained for $i = 5$) amount of numbers left is 9/10 of the numbers that are left after previous step
5	Remove numbers defined by thread defined by 13 (obtained for $i = 6$) amount of numbers left is 12/13 of the numbers that are left after previous step	5	Remove numbers defined by thread defined by 13 (obtained for $i = 6$) amount of numbers left is 11/12 of the numbers that are left after previous step

Using the same analysis like in the previous text after Table 1, we can conclude that number of numbers left by the sieve presented in the left column is higher than the number of the numbers left after implementation of the sieve in right column.

From [6] we know that following holds

$$\pi(n) < 1.26 \frac{n}{\ln(n)}, \quad n \geqslant 1. \tag{C1}$$

That means that number of numbers that are left after two recursive implementations of Sundaram sieve, or number of double primes smaller than some number *n,* denoted as *pd2*(n), is smaller than

$$pd2(n) < 1.26 \frac{\frac{n}{\ln(n)}}{\ln\left(\frac{n}{\ln(n)}\right)}. \tag{C2}$$

That means that we can write that the number of primes smaller than some *n* that is left after implementation of Sundaram and D-Sundaram sieve, which we mark as *pd2d*(n), is limited by the expression given in the following equation

$$pd2d(n) < 2 \cdot \frac{1.26 \frac{n}{\ln(n)}}{\ln\left(\frac{n}{\ln(n)}\right)}. \tag{C3}$$

Since we know that the following equation holds

$$pd2d = \lim_{n \to \infty} pd2d(n) > \frac{\pi_{G2}}{2}, \tag{C4}$$

we can conclude that the number of prime numbers left, when all twin primes are removed, is infinite since the following equation holds

$$\lim_{n \to \infty} \pi(n) - \pi_{G2}(n) > \lim_{n \to \infty} \frac{n}{\ln(n)} - 4 \frac{1.26 \frac{n}{\ln(n)}}{\ln\left(\frac{n}{\ln(n)}\right)} = \infty. \tag{C5}$$

That completes the proof.

It is clear that equation (C3), for limitation of *pd2d* can be made much tighter. However, that is beyond the scope of this paper.