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Abstract:
Argentest II is born, a personal research project that develops a new exclusive probabilistic primality test for Twin prime numbers. I present a test referenced in Fermat's little theorem.

1. Introduction

Twin prime numbers

A twin prime is a prime number that is either 2 less or 2 more than another prime number—for example, either member of the twin prime pair (41, 43). In other words, a twin prime is a prime that has a prime gap of two. Sometimes the term twin prime is used for a pair of twin primes; an alternative name for this is prime twin or prime pair.
Usually the pair (2, 3) is not considered to be a pair of twin primes. Since 2 is the only even prime, this pair is the only pair of prime numbers that differ by one; thus twin primes are as closely spaced as possible for any other two primes.
The first few twin prime pairs are:

$(3,5),(5,7),(11,13),(17,19),(29,31),(41,43),(59,61),(71,73),(101,103),$
$(107,109),(137,139),\ldots$ $OEIS$: $A077800$.

Five is the only prime that belongs to two pairs, as every twin prime pair greater than (3,5) is of the form $(6n+1, 6n-1)$ for some natural number *n.*

2. Proposed conjecture

This test allows knowing the primality of two numbers without the need to factor it or test the primality of each number separately.

Probabilistic primality test for Twin prime numbers

$$\exists\, k > 0 \in \mathbb{N}/2k+1 = p$$

$$\frac{\mathbf{2^{p+2}-8}}{\boldsymbol{p}} \equiv \mathbf{3}(\boldsymbol{Mod\ p}+\mathbf{2}) \leftrightarrow \mathbf{p,p+2\ are\ primes}$$

$$\therefore\ \boldsymbol{P} \wedge \boldsymbol{P}+\mathbf{2}\ \boldsymbol{are\ Twin\ primes}$$

To find the pair of twin prime numbers we need to check the numbers are prime numbers and the difference between the two **prime numbers is 2**.
This algorithm performs that procedure.

Examples

When the two numbers are prime it has congruence. Examples	When at least one of the two numbers is not a prime number, it has no congruence. Examples
A. Test for 3 and 5 $\frac{2^5-8}{3} \equiv 3(Mod\ 5)$	H. Test for 9 and 11 $\frac{2^{11}-8}{9} \not\equiv 3(Mod\ 11)$
B. Test for 5 and 7 $\frac{2^7-8}{5} \equiv 3(Mod\ 7)$	I. Test for 13 and 15 $\frac{2^{15}-8}{13} \not\equiv 3(Mod\ 15)$
C. Test for 11 and 13 $\frac{2^{13}-8}{11} \equiv 3(Mod\ 13)$	J. Test for 15 and 17 $\frac{2^{17}-8}{15} \not\equiv 3(Mod\ 17)$
D. Test for 17 and 19 $\frac{2^{19}-8}{17} \equiv 3(Mod\ 19)$	K. Test for 19 and 21 $\frac{2^{21}-8}{19} \not\equiv 3(Mod\ 21)$
E. Test for 29 and 31 $\frac{2^{31}-8}{29} \equiv 3(Mod\ 31)$	L. Test for 21 and 23 $\frac{2^{23}-8}{21} \not\equiv 3(Mod\ 23)$
F. Test for 41 and 43 $\frac{2^{43}-8}{41} \equiv 3(Mod\ 43)$	M. Test for 23 and 25 $\frac{2^{25}-8}{23} \not\equiv 3(Mod\ 25)$
G. Test for 59 and 61 $\frac{2^{61}-8}{59} \equiv 3(Mod\ 61)$	N. Test for 27 and 29 $\frac{2^{29}-8}{27} \not\equiv 3(Mod\ 29)$

Why is the test probabilistic?

This test is probabilistic since there are pseudoprime numbers that pass the test like 561.

Test for 561 and 563

$$\frac{2^{563}-8}{561} \equiv 3(Mod\ 563)$$

561 is a composite number. This number has 8 divisors
563 is a prime number.
Therefore these numbers are not twin prime.

Pseudoprime numbers (Psp) are a tiny portion of composite numbers that pass the test, these numbers are quite rare and there are only 10 under 10.000

$$\boldsymbol{These\ Pseudoprime\ have\ a\ prime\ partner}$$

$$P = Psp + 2$$

$$Psp = \{561, 1905, 2465, 4371, 23001, 25761, 60701, 87249, 158369, \ldots\ldots\}$$

$$\boldsymbol{These\ prime\ have\ a\ pseudoprime\ partner}$$
$$Psp = P + 2$$
$$P = \{1103, 2699, 2819, 3643, 4679, 6599, 10259, 12799, 14489, 18719, \ldots\}$$

Pseudoprime partner

$$Psp = \{1105, 2701, 2821, 3645, 4681, 6601, 10261, 12801, 14491, 18721, \ldots\}$$

3. This Formula (alternate statement)

$$\frac{\mathbf{2^{n+2} - 8}}{\boldsymbol{n}} \not\equiv \mathbf{3}(\boldsymbol{Mod\ n} + \mathbf{2})$$

$\leftrightarrow \mathbf{n}, \mathbf{n+2}$ (both or one of the two numbers are composite)

$$\therefore\ \boldsymbol{n} \wedge \boldsymbol{n} + \mathbf{2}\ \boldsymbol{are\ not\ Twin\ primes}$$

This statement is absolute: There are no exceptions.

4. Demonstration: Proving the primality of p, p + 2, from Fermat's little theorem.

$$\exists\ k > 0 \in \mathbb{N}/2k + 1 = p$$

$$\frac{\mathbf{2^{p+2} - 8}}{\boldsymbol{p}} \equiv \mathbf{3}(\boldsymbol{Mod\ p} + \mathbf{2}) \leftrightarrow \mathbf{p}, \mathbf{p+2\ are\ primes}$$

when p is prime number and (p + 2) also.

First part : **Confirming the primality of p**

$$\frac{\mathbf{2^{p+2} - 8}}{\boldsymbol{p}}$$
$$2^{p+2} - 8 \equiv (mod\ p)$$
$$= 2^{p+1} - 4 \equiv (mod\ p)$$
$$= 2^p - 2 \equiv (mod\ p)$$
$$= \mathbf{2^{\boldsymbol{p}}} \equiv \mathbf{2}(\boldsymbol{mod\ p})$$

Fermat's Little Theorem

Second part: Confirming the primality of p+2

$$\frac{\mathbf{2^{p+2} - 8}}{\boldsymbol{p}} \equiv \mathbf{3}(\boldsymbol{Mod\ p} + \mathbf{2})$$
$$= 2^{p+2} - 8 \equiv 3p(Mod\ p + 2)$$
$$= 2^p - 8 \equiv 3(p - 2)(Mod\ p)$$
$$= 2^p - 8 \equiv 3p - 6(Mod\ p)$$
$$\boldsymbol{Then\ \ p|\ 3p}$$
$$= 2^p - 8 \equiv -6(Mod\ p)$$
$$= 2^p \equiv -6 + 8\ (Mod\ p)$$
$$= \mathbf{2^{\boldsymbol{p}}} \equiv \mathbf{2}\ (\boldsymbol{Mod\ p})$$

Fermat's Little Theorem

5. Fermat's theorem

Theorem A: Fermat's Little Theorem, If p is a prime number, then, for each natural number a, with a > 0

$$a^p \equiv a \ (mod\ p)$$

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6. *Alfa Program with Python 3.9 (base 2)*

Test twin prime numbers ≥ 3

```
# Alfa program
# Probabilistic primality test for Twin prime numbers.
# Author Zeolla Gabriel M.


n = input("Enter Odd number: ")
if int(n) % 2 == 0:
    print("ERROR")
    n=   input("Enter Odd number: ")
    if int(n) % 2 == 0:
     print("ERROR")

x = ((2** (int(n)+2) - 8) // (int(n)))
r=x % (int(n)+2)

p = r == 3

if p is True:
        print(n, "and", int(n)+2, " are probable Twin prime numbers")
else:
        print(n, "and", int(n)+2,'are not Twin Prime!!')
```

7. Base change

Probabilistic primality test for Twin prime numbers, (all bases)

We can use other bases using the same formula and test all the numbers.

$$a > 1, \in \mathbb{N}$$

$$\exists\, k > 0, \in \mathbb{N}/2k + 1 = p$$

$$\frac{a^{p+2} - a^3}{p} \equiv b(Mod\, p + 2) \leftrightarrow \text{p, p + 2 are primes}$$

$$b = (a - 1) * (a) * (a + 1)/2$$

<u>Examples with different bases</u>

$$\frac{2^{p+2} - 2^3}{p} \equiv 3(Mod\, p + 2) \leftrightarrow \text{p, p + 2 are primes}$$

$$\frac{3^{p+2} - 3^3}{p} \equiv 12(Mod\, p + 2) \leftrightarrow \text{p, p + 2 are primes}$$

$$\frac{4^{p+2} - 4^3}{p} \equiv 30(Mod\, p + 2) \leftrightarrow \text{p, p + 2 are primes}$$

$$\frac{5^{p+2} - 5^3}{p} \equiv 60(Mod\, p + 2) \leftrightarrow \text{p, p + 2 are primes}$$

$$\frac{6^{p+2} - 6^3}{p} \equiv 105(Mod\, p + 2) \leftrightarrow \text{p, p + 2 are primes}$$

$$\frac{7^{p+2} - 7^3}{p} \equiv 168(Mod\, p + 2) \leftrightarrow \text{p, p + 2 are primes}$$

$$\frac{8^{p+2} - 8^3}{p} \equiv 252(Mod\, p + 2) \leftrightarrow \text{p, p + 2 are primes}$$

$$\frac{9^{p+2} - 9^3}{p} \equiv 360(Mod\, p + 2) \leftrightarrow \text{p, p + 2 are primes}$$

Reference OEIS A027480

$a > 1, \in \mathbb{N}$
$b(a) = (a - 1) * (a) * (a + 1)/2$
$b(a) = \{3,12,30,60,105,168,252,360,495,660,858,1092,1365, \ldots\}$

Row sums of n consecutive integers, starting at 0, seen as a triangle:

Base	Sum		
1	0 | 0		
2	3 | 1 2		
3	12 | 3 4 5		$b = \sum_{i=\frac{a(a-1)}{2}}^{\frac{a(a+1)}{2}-1} i =$
4	30 | 6 7 8 9		
5	60 | 10 11 12 13 14		
6	105 | 15 16 17 18 19 20	Base=a	

8. Demonstration (base Change)

Proving the primality of p, p + 2, using any basis from Fermat's little theorem.

$$\frac{\boldsymbol{a^{p+2} - a^3}}{\boldsymbol{p}} \equiv \boldsymbol{b(Mod\ p + 2) \leftrightarrow \mathrm{p, p + 2\ are\ primes}}$$

<u>First part, confirming the primality of p:</u> $when\ a = 1$

$$\frac{a^{p+2} - a^3}{p}$$

$$= \frac{1^{p+2} - 1^3}{p}$$

$$= 1^{p+2} - 1^3 \equiv (mod\ p)$$

$$= 1^{p+1} - 1^2 \equiv (mod\ p)$$

$$= 1^p - 1 \equiv (mod\ p)$$

$$\boldsymbol{1^p \equiv 1(mod\ p)}$$

Fermat's Little Theorem

<u>Second part, confirming the primality of p+2:</u> $when\ a = 1$

$$\frac{a^{p+2} - a^3}{p} \equiv \frac{(a-1)*(a)*(a+1)}{2}(Mod\ p + 2)$$

$$\frac{1^{p+2} - 1^3}{p} \equiv \frac{(1-1)*(1)*(1+1)}{2}(Mod\ p + 2)$$

$$1^{p+2} - 1^3 \equiv \frac{p*(1-1)*(1)*(1+1)}{2}(Mod\ p + 2)$$

$$1^p - 1^3 \equiv \frac{(p-2)*(1-1)*(1)*(1+1)}{2}(Mod\ p)$$

$$1^p - 1 \equiv (p-2)*0(Mod\ p)$$

$$1^p - 1 \equiv 0(Mod\ p)$$

$$\boldsymbol{1^p \equiv 1(Mod\ p)}$$

Fermat's Little Theorem

$When\ a = 2, is\ resolved\ in\ point\ 4.$

Proving the primality of p, p + 2, using any basis

$$\frac{\boldsymbol{a^{p+2} - a^3}}{\boldsymbol{p}} \equiv \boldsymbol{b(Mod\ p+2) \leftrightarrow \mathbf{p, p+2\ are\ primes}}$$

$$\boldsymbol{b = (a-1)*(a)*(a+1)/2}$$

$$= \frac{a^{p+2} - a^3}{p} \equiv \frac{(a-1)*(a)*(a+1)}{2}(Mod\ p+2)$$

$When\ the\ base = a+1$

$$= \frac{(a+1)^{p+2} - (a+1)^3}{p} \equiv \frac{(a+1-1)*(a+1)*(a+1+1)}{2}(Mod\ p+2)$$

$$= \frac{\boldsymbol{(a+1)^{p+2} - (a+1)^3}}{\boldsymbol{p}} \equiv \frac{(a)*(a+1)*(a+2)}{2}(Mod\ p+2)$$

<u>First part: Confirming the primality of p</u>

$$\frac{\boldsymbol{(a+1)^{p+2} - (a+1)^3}}{\boldsymbol{p}}$$
$$= (a+1)^{p+2} - (a+1)^3 \equiv (mod\ p)$$
$$= (a+1)^{p+1} - (a+1)^2 \equiv (mod\ p)$$
$$= \boldsymbol{(a+1)^p - (a+1) \equiv (mod\ p)}$$
$$\boldsymbol{(a+1)^p \equiv (a+1)(mod\ p)}$$

Fermat's Little Theorem

<u>Second part:</u> Confirming the primality of p+2

$$= \frac{(a+1)^{p+2} - (a+1)^3}{p} \equiv \frac{(a)*(a+1)*(a+2)}{2}(Mod\ p+2)$$

$$= (a+1)^{p+2} - (a+1)^3 \equiv \frac{p*(a)*(a+1)*(a+2)}{2}(Mod\ p+2)$$

$$= (a+1)^p - (a+1)^3 \equiv \frac{(p-2)*(a)*(a+1)*(a+2)}{2}(Mod\ p)$$

$$= (a+1)^p - (a+1)^3 \equiv \frac{p*(a)*(a+1)*(a+2)}{2} - \frac{2*(a)*(a+1)*(a+2)}{2}(Mod\ p)$$

$$= (a+1)^p - (a+1)^3 \equiv \frac{p*(a)*(a+1)*(a+2)}{2} - (a)*(a+1)*(a+2)(Mod\ p)$$

$$Then\ p| \ \frac{p*(a)*(a+1)*(a+2)}{2}$$

$$= (a+1)^p - (a+1)^3 \equiv -(a)*(a+1)*(a+2)(Mod\ p)$$

$-(a)*(a+1)*(a+2)$

$= -a(a^2 + 2a + a + 2)$

$= -a(a^2 + 3a + 2)$

$= (-a^3 - 3a^2 - 2a)$

$$= (a+1)^p - (a+1)^3 \equiv (-a^3 - 3a^2 - 2a)(Mod\ p)$$

$$= (a+1)^p \equiv (-a^3 - 3a^2 - 2a) + (a+1)^3 (Mod\ p)$$

$(a+1)^3$

$= a^3 + 3a^2 1 + 3a 1^2 + 1^3$

$= a^3 + 3a^2 + 3a + 1$

$$= (a+1)^p \equiv -a^3 - 3a^2 - 2a + a^3 + 3a^2 + 3a + 1 (Mod\ p)$$

$$= (a+1)^p \equiv -2a + 3a + 1 (Mod\ p)$$

$$\boldsymbol{(a+1)^p \equiv a + 1 (Mod\ p)}$$

Fermat's Little Theorem

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9. *Beta Program with Python 3.9*

This program allows you to choose the base. Test twin prime numbers ≥ 3

```
# Probabilistic primality test for Twin prime numbers.
# This program allows you to choose the base.
# The program test twin prime numbers =>3
# Author Gabriel M Zeolla

m= input("Enter Base :")
a=int(m)-1
b=a*(a+1)*(a+2)/2

n = input("Enter Odd number: ")
if int(n) % 2 == 0:
    print("ERROR")
    n=   input("Enter Odd number: ")
    if int(n) % 2 == 0:
     print("ERROR")

x = ((int(m) ** (int(n)+2) - int(m)**3) // (int(n)))-int(b)
r= x % (int(n)+2)
p = r == 0

if p is True:
        print(n, "and", int(n)+2, " are probable Twin prime numbers")
else:
        print(n, "and", int(n)+2,'are not Twin Prime!!')
```

10. How to improve the algorithm

To the algorithm I will add 3 fundamental conditions to improve your speed when choosing which numbers to analyze. This will allow you to instantly discard numbers that do not meet these parameters.

A. $p \equiv -1(mod\ 6)$ Since the number entered must be this way.
B. $p \not\equiv 23(mod\ 30)$ Since no twin prime number ends in 3, for n> 3
C. $p \not\equiv 35(mod\ 30)$ Since no twin prime number ends in 5, for n> 5

11. These parameters help to eliminate many pseudoprime numbers

Therefore, the algorithm becomes stronger, more efficient and reliable, although it does not eliminate all of them, so the test remains probabilistic.

Example 561: Before it passed the test easily, but now when filtering it with the new conditions it does not prosper, since it is not of the form $p \equiv -1\ (mod\ 6)$

Pseudoprime numbers that pass the parameters

$$\boldsymbol{These\ Pseudoprime\ have\ a\ prime\ partner\ (base\ 2)}$$
$$P = Psp + 2$$
$Psp = \{ 60701, 158369, 181901, 253241, 280601, 348161, 513629, 580337, 587861, 1207361, \ldots\ldots \}$

$$\boldsymbol{These\ prime\ have\ a\ pseudoprime\ partner\ (base\ 2)}$$
$$Psp = P + 2$$
$P = \{ 2699, 2819, 4679, 6599, 10259, 14489, 18719, 19949, 29339, 30119, 31607, 41039, 42797, 49139, 52631, 68099, 85487, 90749, \ldots \}$

Pseudoprime partner (base 2)

$Psp = \{ 2701, 2821, 4681, 6601, 10261, 14491, 18721, 19951, 29341, 30121, 31609, 41041, 42799, 49141, 52633, 68101, 85489, 90751, \ldots \}$

Only under 10.000 are there 4 pseudoprime numbers
Only under 100.000 are there 19 pseudoprime numbers
Only under 1.000.000 are there 55 pseudoprime numbers

12. How to achieve greater efficiency in the test?

If when testing a number we analyze it with different bases, at the same time it becomes even more efficient. Since there are pseudoprimes that fail base 2 but do not pass base 3 or 5 or etc.

Example (31.607, 31609) for base 2 is a probable twin prime, but for base 3 it is not. Which is why we discard it.

But there is a minority of pseudo-prime numbers that resist all the bases

Example: (29.339, 29341)

13. *Gama Program with Python 3.9*

This program allows you to choose the base. Also add the criteria mentioned above.

Eliminate numbers ending in 3, in 5 and those that are not of the form p= 6n-1

Test twin prime numbers >3

```
# fast Gama program "Probabilistic primality test for Twin prime numbers".
# This program allows you to choose the base
# The program test twin prime numbers >3
# Author Gabriel M Zeolla

m= input("Enter Base :")
a=int(m)-1
b=a*(a+1)*(a+2)/2

n = input("Enter Odd number: ")
if int(n) % 2 == 0:
    print("ERROR")
    n=   input("Enter Odd number: ")
    if int(n) % 2 == 0:
     print("ERROR")


if (int(n) +1) % 6==0 and (int(n)-23)% 30 !=0 and (int(n)!=5, (int(n)-35 ) %
30 !=0):
     x = ((int(m) ** (int(n)+2) - int(m)**3) // (int(n)))-int(b)
     r= x % (int(n)+2)
     p = r == 0

     if p is True:
        print(n, "and", int(n)+2, " are probable Twin prime numbers")
     else:
        print(n, "and", int(n)+2,'are not Twin Prime!')
else:
    print(n, "and", int(n)+2,'are not Twin Prime!!')
```

14. Conclusion

Except for the difficulty generated by the pseudoprime numbers, this test works correctly for all twin prime numbers without any exception.

The change of base allows us other complementary possibilities to prove twin primes safely and efficiently, without losing sight that it is a probabilistic test.

I have demonstrated that the twin prime numbers have the form:

$$\frac{a^{p+2}-a^{3}}{p} \equiv b(Mod\ p+2)$$

There is a document by Clement on a test of primality based on Wilson's Theorem but there is no document that refers to a test for twin prime numbers based on Fermat.

There is no text that refers to this algorithm so it is a real novelty.

Professor Zeolla Gabriel Martín

Other works of the author

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