Use the Mathematical Induction then again Classify Positive Integers to Prove the Collatz Conjecture

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Abstract

First, let us clarify certain of basic concepts related to the proof. After that, use the mathematical induction. Then again classify positive integers to prove one or even all of classes of positive integers on different levels.

In addition, before the proof can begin, it is necessary to prepare several judging criteria to be used in the proof of each class of positive integers.

AMS subject classification: 11P32; 11A25; 11Y55

Keywords: the Collatz conjecture; the operational rule; mathematical induction; classify positive integers; operational routes

1. Introduction

The Collatz conjecture is also called the *3x+1* mapping, *3n+1* problem, Hasse's algorithm, Kakutani's problem, Syracuse algorithm, Syracuse problem, Thwaites conjecture and Ulam's problem, etc.

But it is still a conjecture that has neither been proved nor disproved ever since named after Lothar Collatz in 1937; [1].

2. Certain of Basic Concepts

The Collatz conjecture states that take any positive integer *n*, if *n* is an

18 Jul 2021 17:17:51 PDT
210718-Zhang Version 1 - Submitted to Commun. Amer. Math. Soc.

even, divide it by 2; if n is an odd, multiply it by 3 and add 1, and repeat the process indefinitely, so no matter which positive integer you start with, you are always going to end up with 1; [2].

Let us consider aforesaid operational stipulations as the rule of operation.

Starting with any positive integer/integer's expression to operate continually by the rule of operation, then continuous positive integers/ integer's expressions will be formed.

We take such continuous positive integers/integer's expressions plus arrows on the same direction among them as an operational route.

In addition, let us use a capital letter with the subscript "*ie*" to indicate a positive integer's expression such as P_{ie}, C_{ie} etc.

If there is P_{ie} in an operational route, then the operational route may be called "an operational route via P_{ie}".

In general, integer's expressions in an operational route contain a common variable or many variables that can be converted into a variable.

3. The Mathematical Induction that Proves the Conjecture

The mathematical induction that proves the conjecture is as follows; [3].

(1) From 1→4→2→1; 2→1; 3→10→5→16→8→4→2→1; 4→2→1; 5→16→8→4→2→1; 6→3→10→5→16→8→4→2→1; 7→22→11→34→17→52→26→13→40→20→10→5→16→8→4→2→1;8→4→2→1; 9→28→14→7→22→11→34→17→52→26→13→40→20→10→5→16→8→4→2→1; 10→5→16→8→4→2→1; 11→34→17→52→26→13→40→20→10→5→16→8→4→2→1; 12→6→

$3\to10\to5\to16\to8\to4\to2\to1$; $13\to40\to20\to10\to5\to16\to8\to4\to2\to1$; $14\to7\to22\to11\to34\to17\to52\to26\to13\to40\to20\to10\to5\to16\to8\to4\to2\to1$; $15\to46\to23\to70\to35\to106\to53\to160\to80\to40\to20\to10\to5\to16\to8\to4\to2\to1$; $16\to8\to4\to2\to1$; $17\to52\to26\to13\to40\to20\to10\to5\to16\to8\to4\to2\to1$; $18\to9\to28\to14\to7\to22\to11\to34\to17\to52\to26\to13\to40\to20\to10\to5\to16\to8\to4\to2\to1$ and $19\to58\to29\to88\to44\to22\to11\to34\to17\to52\to26\to13\to40\to20\to10\to5\to16\to8\to4\to2\to1$, it can be seen that every positive integer≤ 19 fits the conjecture.

(2) Suppose that n fits the conjecture, where n is an integer≥ 19.

(3) Prove that $n+1$ fits the conjecture likewise.

4. Several Judging Criteria

A certain result of operations that begin with each class of positive integers is judged by one of following criteria.

Theorem 1. If an integer's expression in an operational route via P_{ie} is less than P_{ie}, and $n+1 \in P_{ie}$, then each and every integer's expression in the operational route including $n+1$ fits the conjecture.

For example, if let $P_{ie}=31+3^2\beta$ and $n+1 \in P_{ie}$, where $\beta\geq 0$, then from $27+2^3\beta \to 82+3\times2^3\beta \to 41+3\times2^2\beta \to 124+3^2\times2^2\beta \to 62+3^2\times2\beta \to 31+3^2\beta > 27+2^3\beta$, we get that each and every integer's expression in the operational route including $n+1$ fits the conjecture.

In addition, if let $P_{ie}=5+2^2\mu$ and $n+1 \in P_{ie}$, where $\mu\geq 0$, then from $5+2^2\mu \to 16+3\times2^2\mu \to 8+3\times2\mu \to 4+3\mu < 5+2^2\mu$, we get that each and every integer's expression in the operational route including $n+1$ fits the conjecture.

***Proof*.** Suppose that there is C_{ie} in an operational route via P_{ie} where $C_{ie}<P_{ie}$, then, when their common variable is equal to a certain fixed value such that $P_{ie}=n+1$, let $C_{ie}=m$, so there is $m<n+1$.

As thus, continuous operations that start with $n+1$ can be done by operations after m in the operational route via $n+1$ plus m to get to 1, such that $n+1$ fits the conjecture.

When their common variable is equal to each value, each value of each integer's expression can also be operated to 1 by a matched value of C_{ie} in the one-to-one correspondence with values of each integer's expression, so each and every integer's expression in the operational route fits the conjecture.

Theorem 2. If an operational route via Q_{ie} and an operational route via P_{ie} intersect, and $n+1\in P_{ie}$, and an integer's expression in the operational route via Q_{ie} is less than P_{ie}, then each and every integer's expression in these two operational routes including $n+1$ fits the conjecture.

For example, let $Q_{ie}=71+3^3\times 2^5\varphi$, and $P_{ie}=63+3\times 2^8\varphi$ where $\varphi\geq 0$, then from

$63+3\times2^8\varphi\rightarrow190+3^2\times2^8\varphi\rightarrow95+3^2\times2^7\varphi\rightarrow286+3^3\times2^7\varphi\rightarrow143+3^3\times2^6\varphi\rightarrow430+3^4\times2^6\varphi\rightarrow$

$215+3^4\times2^5\varphi\rightarrow646+3^5\times2^5\varphi\rightarrow323+3^5\times2^4\varphi\rightarrow970+3^6\times2^4\varphi\rightarrow485+3^6\times2^3\varphi\rightarrow1456+3^7\times2^3\varphi$

$\rightarrow728+3^7\times2^2\varphi\rightarrow364+3^7\times2\varphi\rightarrow182+3^7\varphi\rightarrow\ldots$

$\uparrow121+3^6\times2\varphi\leftarrow242+3^6\times2^2\varphi\leftarrow484+3^6\times2^3\varphi\leftarrow161+3^5\times2^3\varphi\leftarrow322+3^5\times2^4\varphi$

$\leftarrow107+3^4\times2^4\varphi\leftarrow214+3^4\times2^5\varphi\leftarrow71+3^3\times2^5\varphi\leftarrow142+3^3\times2^6\varphi\leftarrow47+3^2\times2^6\varphi<63+3\times2^8\varphi$,

we get that each and every integer's expression in these two operational routes including $n+1$ fits the conjecture.

***Proof*.** Suppose that D_{ie} in an operational route via Q_{ie} is less than P_{ie}, and the operational route via Q_{ie} and an operational route via P_{ie} intersect at A_{ie}, then when their common variable is given a certain fixed value such that $P_{ie} = n+1$, let $A_{ie}=\xi$ and $D_{ie}=\mu$, so there is $\mu<n+1$. Obviously μ fits the conjecture due to $\mu\leq n$.

Since ξ and μ are in an operational route, and that μ fits the conjecture, so each and every integer including ξ in the operational route fits the conjecture according to the Theorem 1.

Since $n+1$ and ξ are in an operational route, and that ξ fits the conjecture, so each and every integer including $n+1$ in the operational route fits the conjecture according to the Theorem 1.

When their common variable is equal to each value, each value of each integer's expression in these two operational routes can also be operated to 1 by a matched value of D_{ie} in the one-to-one correspondence with values of each integer's expression. Therefore, each and every integer's expression in these two operational routes fits the conjecture.

Lemma. If an operational route via Q_{ie} and an operational route via P_{ie} are in an indirect connection, and $n+1\in P_{ie}$, and that an integer's expression in the operational route via Q_{ie} is less than P_{ie}, then each and every integer's expression in these two operational routes including $n+1$ fits the conjecture.

The indirect connection refers to the relationship between two non-intersecting operational routes under the premise that many operational routes intersected in proper order.

In addition, the substitutions of $d, e, f, g,$ etc. for c to appear in integer's expressions in operational routes that start with $15+12c/\ 19+12c$, which is actually for the sake of avoiding confusion and convenience.

5. The Classification and Partial Proofs of Positive Integers

Let us divide positive integers into multilevel classes successively, then, the integer $n+1$ is quite possibly included in any class of positive integers, so each class of positive integers must be proved.

After each partition for positive integers, to find an integer's expression smaller than a class of positive integers in an operational course via the class, in order to complete the partial proof.

Proof. First divide positive integers into even numbers and odd numbers.

For even numbers $2k$ with $k\geq 1$, from $2k\rightarrow k<2k$, we get that if $n+1\in 2k$, then $2k$ and $n+1$ fit the conjecture according to the Theorem 1.

For odd numbers >19, divide them into $5+4k$ and $7+4k$, where $k\geq 4$.

For $5+4k$, from $5+4k\rightarrow 16+12k\rightarrow 8+6k\rightarrow 4+3k<5+4k$, we get that if $n+1\in 5+4k$, then $5+4k$ and $n+1$ fit the conjecture according to the Theorem 1.

Further divide $7+4k$ into $15+12c$, $19+12c$ and $23+12c$, where $c\geq 0$.

For $23+12c$, from $15+8c\rightarrow 46+24c\rightarrow 23+12c>15+8c$, we get that if $n+1\in 23+12c$, then $23+12c$ and $n+1$ fit the conjecture according to the Theorem 1.

For *15+12c* and *19+12c,* when *c≥1*, prove them to fit the conjecture, this is focuses of this article, and because of this, we need to make specially a main proof as following paragraphs.

6. Prove that 15+12c and 19+12c Fit the Conjecture

For *15+12c/19+12c* where *c≥1*, continue to operate it until which find or deduce an integer's expression smaller than itself in each of operational routes of *15+12c/19+12c*, so as to accord with a judging criterion.

Firstly, Starting with *15+12c* to operate continuously by the rule of operation, as listed below.

15+12c→46+36c→23+18c→70+54c→35+27c ♣

d=2e+1: 29+27e (**1**)   e=2f: 142+486f→71+243f ♥
♣35+27c↓→c=2d+1: 31+27d↑→d=2e: 94+162e→47+81e↑→e=2f+1:64+81f (**2**)
c=2d: 106+162d→53+81d↓→d=2e+1:67+81e↓→e=2f+1:74+81f (**3**)
d=2e:160+486e♦   e=2f: 202+486f→101+243f ♠

g=2h+1: 200+243h (**4**)   **…**
♥ 71+243f↓→f=2g+1:157+243g↑→g=2h: 472+1458h→236+729h↑→ **…**
f=2g: 214+1458g→107+729g↓→g=2h+1: 418+729h↓→**…**
g=2h: 322+4374h→**… …**

g=2h: 86+243h (**5**)
♠101+243f↓→f=2g+1:172+243g↑→g=2h+1:1246+1458h→**…**
f=2g: 304+1458g→152+729g ↓→**…**
…

…
♦160+486e→80+243e↓→e=2f+1: 970+1458f→485+729f↑→ **…**   **…**
e=2f:40+243f↓→f=2g+1:850+1458g→ 425+729g↑→ **…**
f=2g: 20+243g↓→g=2h: 10+243h (**6**)   **…**
g=2h+1:790+1458h→395+729h↑→ **…**

Annotation:

(1) Each of letters c, d, e, f, g, h, etc at listed above operational routes expresses each of natural numbers plus 0.

(2) In addition, there are ♣↔♣, ♥↔♥, ♠↔♠, and ♦↔♦.

(3) Aforesaid two points are suitable to latter operational routes of 19+12c similarly.

First let us define a term. That is, if an operational result is less than a kind of $15+12c/19+12c$, and it occurs first in an operational route, then, we call the operational result "№1 satisfactory operational result about the kind of $15+12c/19+12c$" or "№1 satisfactory operational result" for short.

Accordingly, the author first concludes following 3 kinds of $15+12c$ derived from №1 satisfactory operational results to fit the conjecture, in the bunch of operational routes of $15+12c$.

1). From $c=2d+1$ and $d=2e+1$ to get $c=2d+1=2(2e+1)+1=4e+3$, then there are $15+12c=51+48e=51+3\times2^4e\to154+3^2\times2^4e\to77+3^2\times2^3e\to232+3^3\times2^3e\to116+3^3\times2^2e\to58+3^3\times2e\to29+27e$ where mark (**1**).

Due to $29+27e<51+48e$, we get that if there is $n+1\in51+48e$, then $51+48e$ and $n+1$ fit the conjecture according to the Theorem *1*.

2). From $c=2d+1$, $d=2e$ and $e=2f+1$ to get $c=2d+1=4e+1=4(2f+1)+1=8f+5$, then there are $15+12c=75+96f=75+3\times2^5f\to226+3^2\times2^5f\to113+3^2\times2^4f\to340+3^3\times2^4f\to170+3^3\times2^3f\to85+3^3\times2^2f\to256+3^4\times2^2f\to128+3^4\times2^1f\to64+81f$ where mark (**2**).

Due to $64+81f<75+96f$, we get that if there is $n+1\in75+96f$, then $75+96f$ and $n+1$ fit the conjecture according to the Theorem *1*.

3). From $c=2d$, $d=2e+1$ and $e=2f+1$ to get $c=2d=4e+2=4(2f+1)+2=8f+6$, then there are $15+12c=87+96f=87+3\times2^5f\to262+3^2\times2^5f\to131+3^2\times2^4f\to394+3^3\times2^4f\to197+3^3\times2^3f\to592+3^4\times2^3f\to296+3^4\times2^2f\to148+3^4\times2^1f\to74+81f$ where mark (**3**).

Due to $74+81f<87+96f$, we get that if there is $n+1\in87+96f$, then $87+96f$ and $n+1$ fit the conjecture according to the Theorem *1*.

Like that, the readers can conclude other 3 kinds of of $15+12c$ derived from №1 satisfactory operational results in the bunch of operational routes of $15+12c$ to fit the conjecture according to the Theorem *1*. They are:

4). Pursuant to c=2d+1, d=2e, e=2f, f=2g+1 and g=2h+1, you can get $15+12c=315+384h$ derived from $200+243h$ where mark (**4**);

5). Pursuant to c=2d, d=2e+1, e=2f, f=2g+1 and g=2h, you can get $15+12c=135+384h$ derived from $86+243h$ where mark (**5**);

6). Pursuant to c=2d, d=2e, e=2f, f=2g and g=2h, you can get $15+12c=15+384h$ derived from $10+243h$ where mark (**6**).

It follows that if $n+1$ belongs in any kind of $15+12c$ derived from a №1 satisfactory operational result, then, that kind of $15+12c$ and $n+1$ fit the conjecture.

Secondly, Starting with $19+12c$ to operate continuously by the rule of operation, as listed below.

19+12c→58+36c→ 29+18c→ 88+54c→ 44+27c ♣

d=2e: 11+27e (**α**) e=2f:37+81f (**β**)
♣ 44+27c↓→c=2d: 22+27d↑→d=2e+1:148+162e→74+81e↑→e=2f+1:466+486f ♥
c=2d+1: 214+162d→107+81d↓→d=2e:322+486e ♠
d=2e+1:94+81e↓→e=2f:47+81f (γ)
e=2f+1:526+486f ♦

g=2h: 119+243h (**δ**) …
f=2g+1:238+243g↑→g=2h+1:1444+1458h→722+729h↑→...
♥466+486f→233+243f↑→f=2g: 700+1458g→350+729g↓→g=2h+1:3238+4374h↓
g=2h: 175+729h↓→... ...
...

g=2h+1:172+243h (**ε**)
f=2g: 101+243g↑→g=2h: 304+1458h→...
e=2f+1:202+243f↑→f=2g+1:1336+1458g→**…**
♠322+486e→161+243e↑→e=2f:484+1458f→**…**

♦526+486f→263+243f↓→f=2g: 790+1458g→**…**
f=2g+1: 253+243g↓→g=2h+1: 248+243h (ζ)
g=2h: 760+1458h→**…**

As listed above, the author first concludes following 3 kinds of *19+12c* derived from №1 satisfactory operational results to fit the conjecture, in the bunch of operational routes of *19+12c*.

1). From $c = 2d$ and $d = 2e$ to get $c = 2d = 4e$, then there are $19+12c = 19+48e = 19+3\times2^4e \rightarrow 58+3^2\times2^4e \rightarrow 29+3^2\times2^3e \rightarrow 88+3^3\times2^3e \rightarrow 44+3^3\times2^2e \rightarrow 22+3^3\times2e \rightarrow 11+27e$ where mark (**α**).

Due to $11+27e<19+48e$, we get that if there is $n+1\in 19+48e$, then $19+48e$ and $n+1$ fit the conjecture according to the Theorem *1*.

2). From $c=2d$, $d=2e+1$ and $e=2f$ to get $c=2d=2(2e+1)=4e+2=8f+2$, then there are $19+12c=43+96f=43+3\times2^5f \rightarrow 130+3^2\times2^5f \rightarrow 65+3^2\times2^4f \rightarrow 196+3^3\times2^4f \rightarrow 98+3^3\times2^3f \rightarrow 49+3^3\times2^2f \rightarrow 148+3^4\times2^2f \rightarrow 74+3^4\times2^1f \rightarrow 37+81f$ where mark (**β**).

Due to $37+81f<43+96f$, we get that if there is $n+1\in 43+96f$, then $43+96f$ and $n+1$ fit the conjecture according to the Theorem *1*.

3). From $c=2d+1$, $d=2e+1$ and $e=2f$ to get $c=2d+1=4e+3=8f+3$, then there are $19+12c=55+96f=55+3\times2^5f \rightarrow 166+3^2\times2^5f \rightarrow 83+3^2\times2^4f \rightarrow 250+3^3\times2^4f \rightarrow 125+3^3\times2^3f \rightarrow 376+3^4\times2^3f \rightarrow 188+3^4\times2^2f \rightarrow 94+3^4\times2^1f \rightarrow 47+81f$ where mark (**γ**).

Due to $47+81f < 55+96f$, we get that if there is $n+1\in 55+96f$, then $55+96f$ and $n+1$ fit the conjecture according to the Theorem *1*.

Like that, the readers can conclude other 3 kinds of *19+12c* derived from №1 satisfactory operational results in the bunch of operational routes of *19+12c* to fit the conjecture according to the Theorem *1*. They are:

4). Pursuant to $c=2d$, $d=2e+1$, $e=2f+1$, $f=2g+1$ and $g=2h$, you can get $19+12c=187+384h$ derived from $119+243h$ where mark (**δ**);

5). Pursuant to c=2d+1, d=2e, e=2f+1, f=2g and g=2h+1, you can get $19+12c=271+384h$ derived from $172+243h$ where mark (**ε**);

6). Pursuant to c=2d+1, d=2e+1, e=2f+1, f=2g+1 and g=2h+1, you can get $19+12c=391+384h$ derived from $248+243h$ where mark (ζ).

It follows that if $n+1$ belongs in any kind of $19+12c$ derived from a №1 satisfactory operational result, then that kind of $19+12c$ and $n+1$ fit the conjecture.

So far, if we take into account what the author and readers have done, then there are 6 kinds of $15+12c/19+12c$ to fit the conjecture.

As mentioned earlier, a kind of $15+12c/19+12c$ derives from a №1 satisfactory operational result. Moreover, many kinds of $15+12c/19+12c$ can also derive from a №1 satisfactory operational result, such as $15+12(4+2^{55}\times 3^{2}y)$ and $15+12(8+2^{32}\times 3^{17}y)$ derived from $61+2^{3}\times 3^{37}y$.

In reality, in one operational route or direct intersecting or indirectly connected or even all operational routes of the bunch of $15+12c/19+12c$, if regard a smaller integer's expression as a satisfactory operational result, then each integer's expression greater than the satisfactory operational result is proved to which is a kind of $15+12c/19+12c$ that fits the conjecture, according to the Theorem1, or the Theorem 2, or the Lemma.

In some cases, an operational route of $15+12c$ and an operational route of $19+12c$ can coincide or intersect from each other, such as start with $15+12(1+2^{57}y)$ to operate, via five steps, to get $19+12(1+2^{54}\times 3^{2}y)$.

Due to $c \geq 1$, there are infinitely more odd numbers of *15+12c/19+12c*, whether they belong to infinite more kinds or finite more kinds, boil down to they can only be in the bunch of operational routes of *15+12c/ 19+12c*, and that all variables of integer's expressions in one or many or even all of the bunch's operational routes can be converted to a common variable. This not only allows you to compare whether some operational result is less than a kind of *15+12c/19+12c,* but also it's not hard for you to understand that one or many of operational routes via each kind of *15+12c/19+12c* be in the bunch of operational routes of *15+12c/19+12c* inevitably, whether the kind of *15+12c/19+12c* is proven or unproved.

In that case, each operational route of *15+12c/19+12* is undoubtedly a tiny part of the bunch of operational routes of *15+12c/19+12*, then every two of these operational routes either directly intersect or indirectly connect, since they can be long enough.

As indicated above, 6 kinds of *15+12c/19+12* in the bunch of operational routes of *15+12c/19+12* have already proved to fit the conjecture.

We can also imagine starting with each kind of *15+12c/19+12* to operate successively by the rule of operation, though each kind of *15+12c/19+12* derives from a №1 satisfactory operational result.

Now that all operational routes of *15+12c/19+12* are in the bunch of operational routes of *15+12c/19+12*, so starting with any unproved kind of *15+12c/19+12* to continuously operate by the rule of operational, then, any operational route formed from this can only first encounter one of following 3 cases:

(1) The operational route reaches with №1 satisfactory operational result about the unproved kind of $15+12c/19+12c$;

(2) The operational route directly intersects an operational route via a proved kind of $15+12c/19+12$;

(3) The operational route indirectly connects to an operational route via a proven kind of $15+12c/19+12$.

Therefore, any unproved kind of $15+12c/19+12$ is proved to fit the conjecture according to the Theorem1, or the Theorem 2, or the Lemma.

By this token, if $n+1$ belongs in any kind of $15+12c/19+12c$, then that kind of $15+12c/19+12c$ and $n+1$ fit the conjecture.

7. Make a Summary and Reach the Conclusion

To sum up, $n+1$ has been proved to fit the conjecture whether $n+1$ belongs in what kind of odd numbers, or it is exactly an even number.

We can also prove positive integers $n+2$, $n+3$ etc. up to every positive integer greater than $n+1$ to fit the conjecture in the light of the old way of doing the thing.

The proof was thus brought to a close. As a consequence, the Collatz conjecture is tenable.

References

[1] WolframMathworld, Collatz Problem, URL: http://mathworld.wolfram.com/CollatzProblem.html

[2] MATHEMATICS, What is the importance of the Collatz conjecture? URL: https://math.stackexchange.com/questions/2694/what-is-the-importance-of-the-collatz-conjecture

[3] Encyclopedia of Mathematics, Mathematical induction, URL: https://www.encyclopediaofmath.org/index.php/Mathematical_induction