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Is complex number theory free from contradiction?

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Abstract With simple basic mathematics it is possible to demonstrate a conflicting result in complex number theory using Euler's identity, simple trigonometry and deMoivre's formula for n=2.

Keywords Basic complex number theory · Euler's identity & the DeMoivre rule · contradiction

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1 Introduction

Complex number theory is a widely applied theory of numbers. We mention e.g. Fourier analysis [1]. Despite the fact that it is considered a well established theory we will have a closer look at it. In the paper only one textbook reference is presented. It is unknown to the author if other modern research into the matter of conflicting result in complex numbers exists. In our paper we will use two basic principles. The reader is referred to [2]. The first principle is Euler's identity. This is

$$\forall_{t\in\mathbb{R}}\; e^{it} = \cos(t) + i\sin(t) \tag{1}$$

The second one is the power rule of DeMoivre. This is, $(x \in \mathbb{R})$

$$\forall_{n\in\mathbb{N}}\,(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx) \tag{2}$$

Here we will look at $n = 2$ and have as is usual, $i = \sqrt{-1}$.

2 Algebraic considerations

The equations under our attention will be

$$\begin{aligned} z_\epsilon &= e^{i(\gamma+\epsilon)} \\ z'_{\epsilon'} &= (1+\epsilon')e^{i\left(\frac{\chi+\eta\pi}{2}\right)} \end{aligned} \tag{3}$$

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with $0 < \epsilon$ and $\epsilon' = \sin(\epsilon)$, together with $\eta = \pm 1$ and γ and χ in $\mathbb{R}$. Under $0 < \epsilon \to 0$ we can look at

$$\lim_{0<\epsilon\to 0} \left(z_\epsilon - z'_{\epsilon'}\right) = 0 \tag{4}$$

This entails the solution: $\gamma_\eta = \frac{\chi+\eta\pi}{2}$. Let us concentrate in this paper on $\gamma_+ = \frac{\chi+\pi}{2}$. This means $\eta = 1$ and we may write:

$$e^{i(\gamma+\epsilon)} = (1+\epsilon')e^{i\left(\frac{\chi+\pi}{2}\right)} + |z_\epsilon - z'_{\epsilon'}|e^{i\varphi_\epsilon} \tag{5}$$

2.1 Real and Imaginary

If we compare right and left hand of (5) then the following two equations arise

$$\cos(\gamma+\epsilon) = -(1+\epsilon')\sin\left(\frac{\chi}{2}\right) + |z_\epsilon - z'_{\epsilon'}|\cos(\varphi_\epsilon) \tag{6}$$

$$\sin(\gamma+\epsilon) = (1+\epsilon')\cos\left(\frac{\chi}{2}\right) + |z_\epsilon - z'_{\epsilon'}|\sin(\varphi_\epsilon)$$

2.2 Substitutions & limits

In the follow up we employ $\gamma = \gamma_+ = \frac{\chi+\pi}{2}$ so that $\cos(\gamma) = \cos\left(\frac{\chi+\pi}{2}\right) = -\sin\left(\frac{\chi}{2}\right)$. For $\sin(\gamma)$ we would in the same way have that $\sin(\gamma) = \cos\left(\frac{\chi}{2}\right)$. From $\epsilon' = \sin(\epsilon)$, we can look at the replacement $\epsilon = 1/n$ and $n \to \infty$ and inspect the limit

$$L = \lim_{n\to\infty} \frac{\sin(1/n)}{|z_{1/n} - z'_{\sin(1/n)}|} = \tag{7}$$

$$\lim_{n\to\infty} \frac{\sin(1/n)}{\sqrt{1 + (1+\sin(1/n))^2 - 2(1+\sin(1/n))\sin[(1/n)+\pi/2]}}$$

With a numerical approach (appendix A) we find $L = \frac{1}{\sqrt{2}}$. See also appendix B. Moreover,

$$L' = \lim_{n\to\infty} \frac{\cos(1/n) - 1}{\sin(1/n)} = 0 \tag{8}$$

And finally,

$$\lim_{0<\epsilon\to 0} \varphi_\epsilon = \lim_{n\to\infty} \varphi_{1/n} = \varphi$$

2.2.1 Cosine Real

The first equation of (6), in n, is

$$-\sin\left(\frac{\chi}{2}\right)\left(\frac{\cos(1/n)-1}{\sin(1/n)}\right)\frac{\sin(1/n)}{|z_{1/n} - z'_{\sin(1/n)}|} - \cos\left(\frac{\chi}{2}\right)\frac{\sin(1/n)}{|z_{1/n} - z'_{\sin(1/n)}|} = \tag{9}$$

$$= -\sin\left(\frac{\chi}{2}\right)\frac{\sin(1/n)}{|z_{1/n} - z'_{\sin(1/n)}|} + \cos(\varphi_{1/n})$$

Hence, because, $L' = 0$ and $L = \frac{1}{\sqrt{2}}$ in (7) and (8), with $n \to \infty$, we arrive at

$$\cos(\varphi) = \frac{1}{\sqrt{2}}\left(\sin\left(\frac{\chi}{2}\right) - \cos\left(\frac{\chi}{2}\right)\right) \tag{10}$$

2.2.2 Sine Imaginary

The second equation of (6), in n, is

$$\cos\left(\frac{\chi}{2}\right)\left(\frac{\cos(1/n)-1}{\sin(1/n)}\right)\frac{\sin(1/n)}{|z_{1/n}-z'_{\sin(1/n)}|}-\sin\left(\frac{\chi}{2}\right)\frac{\sin(1/n)}{|z_{1/n}-z'_{\sin(1/n)}|}= \tag{11}$$

$$=\cos\left(\frac{\chi}{2}\right)\frac{\sin(1/n)}{|z_{1/n}-z'_{\sin(1/n)}|}+\sin(\varphi_{1/n})$$

With $L'=0$ and $L=\frac{1}{\sqrt{2}}$ in (7) and (8) we arrive at

$$\sin(\varphi)=-\frac{1}{\sqrt{2}}\left(\sin\left(\frac{\chi}{2}\right)+\cos\left(\frac{\chi}{2}\right)\right) \tag{12}$$

2.3 The case $\chi/2=\pi/3$

Let us assume that $\chi=2\pi/3$. Then, $\sin(\chi/2)=\frac{\sqrt{3}}{2}\approx 0.866$ and $\cos(\chi/2)=1/2=0.500$. From (10) and (12) we get

$$\cos(\varphi)=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)\approx 0.259 \tag{13}$$

$$\sin(\varphi)=-\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)\approx -0.966$$

And here, in radians, $\varphi_1\approx\arccos(0.259)=1.3088$ and $\varphi_2\approx\arcsin(-0.966)=-1.3093$. The $\varphi=-\varphi_1=\varphi_2$ gives, in approximation, correct cos and sin in (13).

Let us nevertheless follow the path of the angular analysis. This gives

$$\cos(\varphi)+\sin(\varphi)=-\frac{1}{\sqrt{2}} \tag{14}$$

$$\cos(\varphi)-\sin(\varphi)=\frac{\sqrt{3}}{\sqrt{2}}$$

Let us then look at

$$\cos(\varphi)\sin(\varphi)=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)= \tag{15}$$

$$-\frac{1}{2}\left(\frac{3}{4}-\frac{1}{4}\right)=\left(-\frac{1}{2}\right)\times\frac{1}{2}=-\frac{1}{4}$$

Hence, also using $(a+b)(a-b)=a^2-b^2$, when $-\pi\leq\varphi\leq\pi\Leftrightarrow-2\pi\leq 2\varphi\leq 2\pi$

$$\sin(2\varphi)=2\cos(\varphi)\sin(\varphi)=-\frac{1}{2} \tag{16}$$

$$\cos(2\varphi)=\cos^2(\varphi)-\sin^2(\varphi)=-\frac{\sqrt{3}}{2}$$

Therefore, with $-2\pi\leq 2\varphi\leq 2\pi$ and both cos and sin negative in (16), we are allowed to set $2\varphi=\pi+\frac{\pi}{6}=\frac{7\pi}{6}$, with indeed *both* $\sin(2\varphi)=-\frac{1}{2}$ *and* $\cos(2\varphi)=-\frac{\sqrt{3}}{2}$. Hence, $\varphi=\frac{7\pi}{12}$ and the φ is in the interval $-\pi\leq\varphi\leq\pi$. But $\varphi=\frac{7\pi}{12}$ gives

$$\cos(\varphi)=\cos\left(\frac{7\pi}{12}\right)\approx -0.259 \tag{17}$$

$$\sin(\varphi)=\sin\left(\frac{7\pi}{12}\right)\approx 0.966$$

And this is in contradiction with (13) when we can restrict $-\pi \leq \varphi \leq \pi$.

Note that when we select $2\varphi = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$ it is $-2\pi \leq 2\varphi \leq 2\pi$. Then we have $\cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2}$. This is in accordance with (16). So $2\varphi = -\frac{5\pi}{6}$ is correct. Hence, $\varphi = -\frac{5\pi}{12}$ is allowed. We observe $-\pi \leq \varphi \leq \pi$. It then follows

$$\cos(\varphi) = \cos\left(-\frac{5\pi}{12}\right) \approx 0.259 \tag{18}$$

$$\sin(\varphi) = \sin\left(-\frac{5\pi}{12}\right) \approx -0.966$$

This is the correct φ angle opposed to the one in (17).

3 Conclusion & discussion

There is no reason to reject the selection $\chi/2 = \pi/3$. The question now is:

Problem Statement: *What prevents the selection of the angle $\varphi = \varphi_1$ here, $\varphi_1 = \frac{7\pi}{12}$ and forces the use of the angle $\varphi = \varphi_2$, here $\varphi_2 = -\frac{5\pi}{12}$.*

If there is none beyond the required (16) and it is merely the avoiding of the situation indicated in (17), a contradiction is found.

In $-\pi \leq \varphi \leq \pi$ all possible values of both $\cos(\varphi)$ and $\sin(\varphi)$ are presented. This justifies the use of that interval. Further, a computer program to help the reader to check the computations is in the appendix A.

A point raised against all the previous is the multivaluedness of complex numbers, or the multiple ways to go to zero in the complex number field. But this is not a valid objection against the problem statement for, with multivaluedness broadly construed, we can write

$$z_\epsilon - z'_{\epsilon'} = \pm|z_\epsilon - z'_{\epsilon'}|e^{i\varphi_\epsilon} \tag{19}$$

In the previous we presented the analysis for $+$. Now for - we then see in case of $\chi/2 = \pi/3$ and looking at (10) and (12)

$$-\cos(\varphi) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \approx -0.259 \tag{20}$$

$$-\sin(\varphi) = -\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) \approx 0.966$$

Hence, in *this* case we would have in view of (17) that $\varphi = \frac{7\pi}{12}$ the correct one but $\varphi = -\frac{5\pi}{12}$ the incorrect one. Therefore, the ± 1 multivaluedness in (19) does not change a thing towards the problem given in the above. Furthermore, in view of

$$z_\epsilon - z'_{\epsilon'} = \pm i|z_\epsilon - z'_{\epsilon'}|e^{i\varphi_\epsilon} \tag{21}$$

a similar point, but with different φ can be raised as well. Let us for instance look at the $-i$ case of (21). This would lead us to for $\gamma = \gamma_+$

$$\begin{aligned} \cos(\gamma + \epsilon) + i\sin(\gamma + \epsilon) = \\ -(1 + \sin(\epsilon))\sin\left(\frac{\chi}{2}\right) + i(1 + \sin(\epsilon))\cos\left(\frac{\chi}{2}\right) + \\ -i|z_\epsilon - z'_{\epsilon'}|\cos(\varphi_\epsilon) + |z_\epsilon - z'_{\epsilon'}|\sin(\varphi_\epsilon) \end{aligned} \tag{22}$$

Hence, starting with a similar case as in (10) and (12) again

$$\sin(\varphi) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) > 0 \tag{23}$$

$$-\cos(\varphi) = -\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)$$

So along the lines (14)-(16), we have, $\sin(2\varphi) = 1/2$ and $\cos(2\varphi) = \sqrt{3}/2$. In $-2\pi \leq 2\varphi \leq 2\pi$ it is possible to write, $2\varphi = \pi/6$ and $2\varphi = -2\pi + (\pi/6) = -11\pi/6$. The latter because, $\cos[-2\pi+(\pi/6)] = \cos(-2\pi)\cos(\pi/6) = \cos(\pi/6)$ and $\sin[-2\pi+(\pi/6)] = \cos(-2\pi)\sin(\pi/6) = \sin(\pi/6)$. It is also easy to acknowledge that $\sin(\varphi) \approx -0.259 < 0$ when $2\varphi = -11\pi/6$, or $\varphi = -11\pi/12$ and, obviously, $-\pi \leq (-11\pi/12) \leq \pi$. But according to (23) we must have $\sin(\varphi) > 0$. Therefore in this case too we can have *two* possibilities for φ. One of those turns out to be invalid but still can be selected.

The conclusion is that multivaluedness, broadly construed, in complex numbers does *not* chase away the question in the problem statement above. Even if the reader comes with all kinds of different objections then:

- The interval $-2\pi \leq 2\varphi \leq 2\pi$ can be implemented. If not then why not. Note that restriction to suchlike interval also occurs in applications.
- The mathematics given is valid. If not, then where is the error.
- The two φ, i.e. φ_1 and φ_2 are associated to *one* single equation. It is unimportant that éither φ_1 ór φ_2 is non-contradictory. The "other" validly obtained contradictory φ is always there.

Multivaluedness broadly construed is not the solution to get rid of the problem. Mathematics is the science of reason and reasoning. Readers critical to the existence of the problem as well as the author cannot escape from that. The claim is that possibilities discussed here are sufficient cause for the problem statement given.

There is indeed nothing that forces us to select the wrong φ. However, there could also be nothing that forces us to select the correct φ. The philosopher of language, Ludwig Wittgenstein, already warned against using "prose" as a forcing form of mathematical reasoning. At least, that is how the present author understands [3]. If this is a correct way of looking at it, then it makes sense to reject all arguments to "throw away" the wrong φ and only to keep the preferred correct one. Such reasoning implies something like an Axiom of avoiding Absurdity. The latter is not present in the Zermelo Frenkel system of foundational mathematical axioms. Moreover, all computations then should be serviced with an absurdity checker routine.

But then again, what about the concept of the possible basic absurdity in the nature of all things. We can think of something like Wittgenstein's language limit [4]. The existence of such a barrier is erased from our knowledge just because mathematics has no room for it *and* is in lack of reasons to have no room. Do finally also note that we accept absurdities in quantum mechanics. Tunneling through a potential energy barrier is even not considered absurd anymore. It is the conerstone of nuclear alpha decay. So some absurdities are liked. Others are not. But then what if there is more absurdity than quantum mechanics and we just failed to notice that?

A genuine contradiction can perhaps be related to [5].

Declarations

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Appendix A Simple computer program in R to support the algebra argument of the main text.

```
# numerical limit proofs
y<-array(0,1000)
for (n in 1:1000){
  f<-sin(1/n)
  g<-1+((1+sin(1/n))^2)
  g<-g-(2*(1+sin(1/n))*sin((1/n)+(pi/2)))
  h<-f/sqrt(g)
  y[n]<-h
  print(c(n,h))
}
plot(y,type='l')
for (n in 1:1000){
  f<-cos(1/n)-1
  g<-sin(1/n)
  h<-f/g
  y[n]<-h
  print(c(n,h))
}
plot(y,type='l')
#
chi2<-pi/3
print(paste0("sin(chi2)=",sin(chi2)))
print(paste0("cos(chi2)=",cos(chi2)))
print("****")
fcos<-(sin(chi2)-cos(chi2))/sqrt(2)
fsin<-(-sin(chi2)-cos(chi2))/sqrt(2)
print(paste0("cos(phi)=",fcos))
print(paste0("sin(phi)=",fsin))
fd<-2*fcos*fsin
print(paste0("sin(2*phi)=",fd))
c2<-(fcos^2)-(fsin^2)
print(paste0("cos(2phi)=",c2))
#therefore
phiPos<-pi+(pi/6)
print(paste0("sin(2*phiPos)=sin(2*phi) is ",abs(fd-sin(phiPos))<1e-10))
print(paste0("cos(2*phiPos)=cos(2*phi) is ",abs(c2-cos(phiPos))<1e-10))
# ... 7pi/12
phiPos<-phiPos/2
print(paste0("phiPos/2=7*pi/12=",phiPos))
print(paste0("cos(phiPos)=",cos(phiPos)))
print(paste0("sin(phiPos)=",sin(phiPos)))
print(paste0("fcos=cos(phi) is ",abs(fcos-cos(phiPos))<1e-10))
print(paste0("fsin=sin(phi) is ",abs(fsin-sin(phiPos))<1e-10))
```

Appendix B Proof of limit $L = 1/\sqrt{2}$ (7) in maintext. L'Hopital with ϵ. Limit presentation with $\epsilon = 1/n$.

$$L^2 = \lim_{n\to\infty} \frac{\sin^2(1/n)}{1 + (1+\sin(1/n))^2 - 2(1+\sin(1/n))\sin[(1/n)+\pi/2]} =$$

$$\lim_{n\to\infty} \frac{\sin(2/n)}{2\cos(1/n) + \sin(2/n) - 2\cos[(1/n)+\pi/2] - 2\sin[(2/n)+\pi/2]} =$$

$$\lim_{n\to\infty} \frac{2\cos(2/n)}{-2\sin(1/n) + 2\cos(2/n) + 2\sin[(1/n)+\pi/2] - 4\cos[(2/n)+\pi/2]} = \frac{1}{2}$$