A new inequality equivalent to the Riemann hypothesis

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Abstract: There have been published many research results on the Riemann hypothesis. The Robin inequality is one of the most important propositions equivalent to the Riemann hypothesis. At present, it is known that the Robin inequality holds for all odd and many even numbers, but unknown for all numbers yet.
In this paper, we first find a new inequality equivalent to the Riemann hypothesis on the basis of Robin theorem and Nicolas' result. Next, we introduce the error terms suitable to Mertens' formula and Chebyshev's function, and obtain their estimates. With such estimates and primorial numbers, we finally prove that the new inequality holds unconditionally for all numbers. The result of this paper shows that the Riemann hypothesis is true.
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I. Introduction

Let N be the set of the natural numbers. The function $\varphi(n) = n \cdot \prod_{p|n} (1 - p^{-1})$ is called Euler's function of $n \in N$ ([1]), where $\varphi(1) = 1$ and $p|n$ denotes p is the prime divisor of n.
The function $\sigma(n) = \sum_{d|n} d$ is called the sum of divisors function of $n \in N$ ([1]), where $\sigma(1) = 1$ and $d|n$ denotes d is the divisor of n ([1]).
G. Robin showed in his paper [2] (also see [3]).
[Proposition 1] If the Riemann hypothesis (RH [1]) is true, then

$$\frac{\sigma(n)}{n} < e^{\gamma} \cdot \log\log n \tag{0.1}$$

holds for any $n \geq 5041$, where $\gamma = 0.577\cdots$ is Euler's constant ([1]).
[Proposition 2] If the RH is false, then there exist constants $c > 0$ and $0 < \beta < 1/2$ such that

$$\frac{\sigma(n)}{n} \geq e^{\gamma} \cdot \log\log n + \frac{c \cdot \log\log n}{(\log n)^{\beta}} \tag{0.2}$$

holds for infinitely many $n \in N$.
From (0.1) and (0.2), one easily see that (0.1) is equivalent to the RH. So (0.1) is called the Robin criterion or the Robin inequality for the RH ([6-9]). Much papers have been attempted to the Robin inequality. At present, it is known that the Robin inequality holds for all odd ≥ 17 ([6]) and many even numbers ([7, 8, 9]), but unknown for all numbers yet. Now new idea is required to prove it in full generality ([7]).
There is another one than the Robin inequality.
Let $p_1 = 2, p_2 = 3, p_3 = 5, \cdots$ be the first consecutive primes. The number $\Im_m := (p_1 \cdot p_2 \cdots p_m)$ is called the primorial number of order m ([7, 10]).

J. L. Nicolas showed in his paper [10];
[Proposition 3] If the RH is true, then

$$c(n) := \left(\frac{n}{\varphi(n)} - e^{\gamma} \cdot \log\log n\right) \cdot \sqrt{\log n} \le c(\Im_{66}) \tag{0.3}$$

holds for any $n \ge 2$, where

$$c(\Im_{66}) = c(2 \cdot 3 \cdots 317) = 4.0628356921 \cdots .$$

Nicolas indicated in [10] that the inequality (0.3) is also sufficient condition for the RH. Thus (0.3) is one of the inequalities equivalent to the RH.
In this paper, we first find a new inequality equivalent to the Riemann hypothesis on the basis of the Robin theorem and Nicolas' result in [10]. Next, we introduce the error terms suitable to well- known Mertens' formula and Chebyshev's function, and obtain their estimates. With such estimates, we finally prove that the new inequality holds unconditionally for all numbers by the primorial numbers.
The result of this paper shows that the Riemann hypothesis is true.

II. Main results of paper

The main results of this paper are as follows;
[Theorem 1] The RH is true if and only if there exists a constant $c_0 \ge 1$ such that

$$\frac{n}{\varphi(n)} \le e^{\gamma} \cdot \log\log(n \cdot \exp(c_0 \cdot \sqrt{\log n})) \tag{0.4}$$

holds for any number $n \ge 2$.
Here (0.4) is a new inequality equivalent to the RH. However, it is of our interest whether (0.4) holds unconditionally in deed or not. In this connection, we introduce the function;

$$\Phi_0(n) := \frac{\exp(e^{-\gamma} \cdot n/\varphi(n)) - \log n}{\sqrt{\log n}}.$$

Then it is obvious that (0.4) is equivalent to $\Phi_0(n) \le c_0$ $(n \ge 2)$. Our aim is to determinate such constant $c_0 \ge 1$. We give;
[Theorem 2] We have $\Phi_0(n) \le 8$ for any $n \ge 2$.
(Remark) 1) In fact, $\Phi_0(n) \le 3$ holds for any $n \ge 2$. But it is difficult to guarantee it in the theoretical description. The number 8 is the most pertinent one to the question.
2) If the theorems 1, 2 hold, then it is not difficult to see that the Robin inequality holds for all numbers≥ 5041. (See the proof of the theorem 1 below)

III. Proof of the Theorem 1

If the RH is true, then by the proposition 3 (theorem 1.1 of [10]), there exists a constant $c_1 = 4.07 \ge 1$ such that

$$\frac{n}{\varphi(n)} \le e^{\gamma} \cdot \log\log n + \frac{c_1}{\sqrt{\log n}}$$

holds for any $n \ge 2$. Thus there exists a constant $c_0 \ge 13 \ge 1$ such that

$$e^{\gamma} \cdot \log\log n + \frac{c_1}{\sqrt{\log n}} \le$$

$$\le e^{\gamma} \cdot \log\log n + e^{\gamma} \cdot \log\left(1 + \frac{c_0}{\sqrt{\log n}}\right) =$$

$$= e^{\gamma} \cdot \log\log(n \cdot \exp(c_0 \cdot \sqrt{\log n})).$$

holds for any $n \geq 2$.
On the other hand, it is clear that $\sigma(n) \cdot \varphi(n) \leq n^2$ for any $n \geq 2$. If (0.4) holds for any $n \geq 2$, but the RH is false, then by the proposition 2 (Robin theorem 2), there exist constants $c > 0$ and $0 < \beta < 1/2$ such that

$$e^{\gamma} \cdot \log\log n + \frac{c \cdot \log\log n}{(\log n)^{\beta}} \leq \frac{\sigma(n)}{n} \leq$$

$$\leq \frac{n}{\varphi(n)} \leq e^{\gamma} \cdot \log\log(n \cdot \exp(c_0 \cdot \sqrt{\log n}))$$

holds for infinitely many $n \in N$. Here, since $\log(1+t) \leq t \ \ (t > 0)$, we have

$$\log\log(n \cdot \exp(c_0 \cdot \sqrt{\log n})) \leq$$

$$\leq \log\log n + \frac{c_0}{\sqrt{\log n}}$$

and

$$1 \leq \frac{\mathrm{e}^{\gamma} \cdot c_0 \cdot c^{-1}}{(\log n)^{1/2-\beta} \cdot (\log\log n)} \to 0 \ \ (n \to \infty),$$

but it is a contradiction.

IV. Reduction to the primorial numbers

We will make ready to prove the Theorem 2 from the section IV to the section VII.
Assume that $n = q_1^{\lambda_1} \cdots q_m^{\lambda_m}$ is the prime factorization of $n \in N$. Here $q_1, \cdots, q_m$ are distinct primes, $\lambda_1, \cdots, \lambda_m$ are nonnegative integers ≥ 1 and $\omega(n) = m$ ([6]).
Then it is easy to see that $n \geq \Im_m$ and

$$\frac{n}{\varphi(n)} = \prod_{i=1}^{m}(1 - q_i^{-1})^{-1} \leq \prod_{i=1}^{m}(1 - p_i^{-1})^{-1} = \frac{\Im_m}{\varphi(\Im_m)}$$

and so $\Phi_0(n) \leq \Phi_0(\Im_m)$. This shows that the boundedness of the function $\Phi_0(n)$ for $n \in N$ ($n \neq 1$) is reduced to one for the primorial numbers.

V. Some symbols

Recall some concepts and introduce some notes. The formula

$$\sum_{p \leq t} p^{-1} = \log\log t + b + E(t)$$

is called Mertens' formula [6], where $t > 1$ is a real number, p is the prime number,

$$b = \gamma + \sum_{p} (\log(1 - 1/p) + 1/p) =$$

$$= 0.261497212847643 \cdots$$

is Mertens' constant ([6]). We will call $E(t)$ the error term of Mertens' formula.
By (3.18), (3.20) of [5], we could know

$$\frac{-1}{\log^2 t} < E(t) < \frac{1}{\log^2 t} \quad (t > 1). \tag{0.5}$$

And $\vartheta(t) = \sum_{p \le t} \log p$ is called the Chebyshev's function ([1]). By the prime number theorem ([1]), we could write as

$$\vartheta(t) = t \cdot (1 + \theta(t))$$

for any real $t > 1$. We will call $\theta(t)$ the error term of $\vartheta(t)$. By (3.15) and (3.16) of [5], we see

$$\frac{-1}{\log t} < \theta(t) < \frac{1}{\log t} \quad (t \ge 41). \tag{0.6}$$

Put $F_m := \Im_m / \varphi(\Im_m)$, then

$$\log(F_m) = -\sum_{i=1}^{m} (\log(1 - 1/p_i) + 1/p_i) + \sum_{i=1}^{m} 1/p_i =$$

$$= \log\log p_m + \gamma + E(p_m) + \varepsilon(p_m),$$

where

$$\varepsilon(p_m) := \sum_{p > p_m} (\log(1 - 1/p) + 1/p) = O(1/p_m).$$

From this

- $(e^{-\gamma} \cdot F_m) = \log p_m \cdot e_m,$

  $e_m := \exp(E(p_m) + \varepsilon(p_m)),$
- $\exp(e^{-\gamma} \cdot F_m) = p_m \cdot e'_m,$

  $e'_m := \exp(\log p_m \cdot (e_m - 1)).$

Furthermore, put

- $\log \Im_m = p_m \cdot \alpha_m, \quad \alpha_m := 1 + \theta(p_m),$
- $K_m := \sqrt{\log \Im_m}.$

VI. Some estimates

6.1. An estimate of e_m and e'_m

Assume that $p_m \ge e^{14}$ below. The discussions for $p_m \le e^{14}$ are supported by MATLAB.
By (3.30) of [5],

$$(e^{-\gamma} \cdot F_m) = \log p_m \cdot e_m = e^{-\gamma} \cdot \prod_{i=1}^{m} (1 - p_i^{-1})^{-1} <$$

$$< \log p_m + \frac{1}{\log p_m} \quad (p_m \ge 2)$$

and for $p_m \ge e^{14}$ we have respectively

$$e_m < 1.01, \quad e'_m < 1.08, \quad (e_m \cdot e'_m) < 1.1. \tag{0.7}$$

6.2. An estimate of $(e_m \cdot e'_m)$

If $e_m \le 1$ then $e'_m \le 1$ and $(e_m \cdot e'_m) \le 1$. And if $e_m > 1$ then, since $\varepsilon(p_m) < 0$,

$$0 < r := E(p_m) + \varepsilon(p_m) < \frac{1}{\log^2 p_m} \le 0.01$$

and

$$e_m = 1 + r + \sum_{n=2}^{\infty} \frac{r^n}{n!} \le 1 + r + \frac{r^2}{2 \cdot (1 - r)} \le$$

$$\leq 1 + r + 0.51 \cdot r^2,$$

$$e_m \cdot e'_m = \exp(r + (\log p_m) \cdot (e_m - 1)) \leq$$

$$\leq 1 + h + \frac{h^2}{2 \cdot (1-h)},$$

where

$$h := (1 + \log p_m) \cdot r + 0.51 \cdot \log p_m \cdot r^2 \leq 0.08.$$

Therefore we have

$$(e_m \cdot e'_m - 1) \leq (1 + \log p_m) \cdot (E(p_m) + \varepsilon(p_m)) +$$

$$+ 0.6 \cdot (1 + \log p_m)^2 \cdot (E(p_m) + \varepsilon(p_m))^2 \quad (e_m > 1). \tag{0.8}$$

6.3. An estimate of V_m

Here

$$V_m := p_{m+1} \cdot (e'_{m+1} - \alpha_{m+1}) - p_m \cdot (e'_m - \alpha_m).$$

It is clear that

$$p_{m+1} \cdot \alpha_{m+1} - p_m \cdot \alpha_m = \log p_{m+1}.$$

Since

$$E(p_{m+1}) - E(p_m) = \frac{1}{p_{m+1}} - \log\left(\frac{\log p_{m+1}}{\log p_m}\right),$$

$$\varepsilon(p_{m+1}) - \varepsilon(p_m) = -\log\left(1 - \frac{1}{p_{m+1}}\right) - \frac{1}{p_{m+1}},$$

we have

$$\frac{e_{m+1}}{e_m} = \left(\frac{\log p_m}{\log p_{m+1}}\right) \cdot \left(1 + \frac{1}{p_{m+1} - 1}\right),$$

$$\frac{e'_{m+1}}{e'_m} = \frac{p_m}{p_{m+1}} \cdot \exp\left(\frac{\log p_m \cdot e_m}{p_{m+1} - 1}\right)$$

and

$$V_m = p_m \cdot e'_m \cdot \left(\frac{p_{m+1} \cdot e'_{m+1}}{p_m \cdot e'_m} - 1\right) - \log p_{m+1} =$$

$$= \log p_{m+1} \cdot (\mu \cdot e'_m - 1),$$

where

$$\mu := \frac{p_m}{\log p_{m+1}} \cdot \left(\exp\left(\frac{\log p_m \cdot e_m}{p_{m+1} - 1}\right) - 1\right).$$

Moreover

$$\mu \leq e_m + \frac{1}{2} \cdot \frac{\log p_m \cdot e_m^2}{p_m} \cdot \left(1 - \frac{\log p_m \cdot e_m}{p_m}\right)^{-1} \leq$$

$$\leq e_m + 0.52 \cdot \frac{\log p_m}{p_m}$$

and

$$\mu \cdot e'_m - 1 \leq (e_m \cdot e'_m - 1) + 0.6 \cdot \frac{\log p_m}{p_m}. \tag{0.9}$$

6.4. An estimate of W_m

Here

$$W_m := \left(U_m - (K_{m+1} - K_m)\right) \cdot \frac{1}{K_{m+1}}$$

and

$$U_m := \frac{\log p_{m+1}}{2 \cdot \sqrt{\log \Im_m}}.$$

Then

$$K_{m+1} - K_m = \sqrt{\log \Im_{m+1}} - \sqrt{\log \Im_m} \geq$$
$$\geq \frac{\log p_{m+1}}{2 \cdot \sqrt{\log \Im_{m+1}}}.$$

From this

$$U_m - (K_{m+1} - K_m) \leq$$
$$\leq \frac{\log p_{m+1}}{2} \cdot \left(\frac{1}{\sqrt{\log \Im_m}} - \frac{1}{\sqrt{\log \Im_{m+1}}} \right) \leq$$
$$\leq \frac{\log^2 p_{m+1}}{4 \cdot (\log \Im_m)^{3/2}}.$$

And it is known that $p_{k+1}^2 \leq 2 \cdot p_k^2$ for $p_k > 7$ by 247p of [4] and so

$$\log p_{m+1} \leq \Omega_0 \cdot \log p_m,$$

$$\Omega_0 := \left(1 + \frac{\log \sqrt{2}}{\log p_m} \right) \leq 1.025. \tag{0.10}$$

Since $\alpha_m \geq (1 - 1/14)$, $K_{m+1} \geq K_m$ and the function $(\log^3 t)/t$ is decreasing on the interval $(e^3, +\infty)$, we have

$$W_m \leq \frac{\log^3 p_m}{4 \cdot p_m \cdot \alpha_m^2} \cdot \frac{\Omega_0^2}{p_m \cdot \log p_m} \leq$$
$$\leq \frac{1.9 \cdot 10^{-4}}{p_m \cdot (\log p_m)^{1/2}} \quad (p_m \geq e^{14}). \tag{0.11}$$

6.5. An estimate of $S(p', \sigma)$

Here

$$S(p', \sigma) := \sum_{p \geq p'} \frac{1}{p \cdot (\log p)^\sigma} \quad (\sigma > 0).$$

Put

$$s(t) := \sum_{p \leq t} p^{-1} = \log \log t + b + E(t).$$

Then by Abel's identity ([1]), we have

$$S(p', \sigma) = \sum_{p > p' - 0} \frac{1}{p \cdot (\log p)^\sigma} = \int_{p'}^{+\infty} \frac{1}{(\log t)^\sigma} \cdot ds(t) =$$

$$= \int_{p'}^{+\infty} \frac{1}{(\log t)^\sigma} \cdot \left(\frac{dt}{t \cdot \log t} + dE(t) \right) =$$

$$= \int_{p'}^{+\infty} \frac{dt}{t \cdot (\log t)^{\sigma+1}} + \int_{p'}^{+\infty} \frac{dE(t)}{(\log t)^\sigma} =$$

$$\leq \frac{-1}{\sigma \cdot (\log t)^\sigma} \Big|_{p'}^{+\infty} + \frac{E(t)}{(\log t)^\sigma} \Big|_{p'}^{+\infty} + \int_{p'}^{+\infty} \frac{\sigma \cdot E(t)}{t \cdot (\log t)^{\sigma+1}} \cdot dt \leq$$

$$\leq \frac{1}{\sigma \cdot (\log p')^{\sigma}} + \frac{1}{(\log p')^{\sigma+2}} + \int_{p'}^{+\infty} \frac{\sigma}{t \cdot (\log t)^{\sigma+3}} \cdot dt =$$

$$= \frac{1}{\sigma \cdot (\log p')^{\sigma}} + \frac{1}{(\log p')^{\sigma+2}} + \frac{\sigma}{(\sigma+2) \cdot (\log p')^{\sigma+2}} =$$

$$= \frac{1}{\sigma \cdot (\log p')^{\sigma}} + \frac{2 \cdot \sigma + 2}{(\sigma+2) \cdot (\log p')^{\sigma+2}}$$

So

$$S(p', \sigma) \leq \frac{1}{\sigma \cdot (\log p')^{\sigma}} + \frac{2 \cdot \sigma + 2}{(\sigma+2) \cdot (\log p')^{\sigma+2}}$$

and by the same method,

$$\frac{1}{\sigma \cdot (\log p')^{\sigma}} - \frac{2 \cdot \sigma + 2}{(\sigma+2) \cdot (\log p')^{\sigma+2}} \leq S(p', \sigma).$$

If p' is the first prime $\geq e^{14}$, then $p' = 1202609$ and it is 93118-th prime.
And if $\sigma = 1/2$ and $p' = 1202609$, then

$$0.5328 \leq S(p', \ 1/2) \leq 0.5362. \tag{0.12}$$

VII. A Lemma for $(1 + \log p) \cdot E(p)$

Now for any $t > 1$ we introduce the following functions;

$$f(t) := t \cdot \log t \cdot E(t) - t \cdot \theta(t),$$

$$g(t) := \sqrt{t}, \quad d(t) := \frac{f(t)}{g(t)}. \tag{0.13}$$

Then the functions $f(t),\ g(t)$ and $d(t)$ are continuous and piece differentiable functions on the interval $(1, \ +\infty)$.

[Lemma] We have

$$(1 + \log p) \cdot E(p) \leq d(p) \cdot g'(p) + \frac{8}{\sqrt{p} \cdot (\log p)^{3/2}}$$

for any prime $p \geq e^{14}$, where the functions $d(p)$ is given in (0.13).
For the proof of the Lemma, we make some ready from the section 7.1 to the section 7.3.

7.1. A condition $(\bar{d})$

If the Lemma does not hold, then there exists a prime number $p \geq e^{14}$ such that

$$(1 + \log p) \cdot E(p) > d(p) \cdot g'(p) + \frac{8}{\sqrt{p} \cdot (\log p)^{3/2}}. \tag{0.14}$$

We fix one of such primes $p \geq e^{14}$ and take the functions $f(t),\ g(t)$ and $d(t)$ as in (0.13) on the interval $[p, \ p+1]$. Then the functions $f(t),\ g(t)$ and $d(t)$ are continuous on the interval $[p, \ p+1]$. And $f(t)$, $g(t)$ and $d(t)$ are n-th continuously differentiable on the interval $(p, \ p+1)$. Moreover, $f'(t)$ is the right continuous at $t = p$. Put

$$f'(p) := f'(p+0), \quad f'(p+1) := f'(p+1-0).$$

Then $f'(p) = (1 + \log p) \cdot E(p)$ and (0.14) is equivalent to

$$d'(p) \cdot g(p) > \frac{8}{\sqrt{p} \cdot (\log p)^{3/2}}.$$

Here put $H(t) := d'(t) \cdot g(t)\ \ (t \in (p,\ \ p+1))$, then $H'(t) < 0\ \ (t \in (p,\ \ p+1))$. In fact,

$$H'(t) = d''(t) \cdot g(t) + d'(t) \cdot g'(t) =$$

$$= \frac{1}{t} \cdot \left(\partial_0(t) - 1 - \frac{1}{\log t} \right),$$

where

$$\partial_0(t) := E(t) - t \cdot d(t) \cdot g''(t) - t \cdot d'(t) \cdot g'(t).$$

By (0.5), (0.6) for any $t \in (p,\ p+1)$ it is easy to see that

$$|E(t)| \leq \frac{1}{\log^2 t} \leq 0.01,$$

$$|t \cdot d(t) \cdot g''(t)| = |t \cdot d(t) \cdot g(t)| \cdot \left| \frac{g''(t)}{g(t)} \right| =$$

$$= |t \cdot f(t)| \cdot \left| \frac{g''(t)}{g(t)} \right| \leq \frac{1}{4} \cdot (|\log t \cdot E(t)| + |\theta(t)|) \leq$$

$$\leq \frac{1}{2 \cdot \log t} \leq 0.04,$$

$$|t \cdot d'(t) \cdot g'(t)| = |t \cdot d'(t) \cdot g(t)| \cdot \left| \frac{g'(t)}{g(t)} \right| =$$

$$= \frac{1}{2} \cdot |f'(t) - d(t) \cdot g'(t)| \leq$$

$$\leq \frac{1}{2} \cdot \left(|(1 + \log t) \cdot E(t)| + \frac{|f(t)|}{2 \cdot t} \right) \leq$$

$$\leq \frac{1}{2} \cdot \left(\frac{2}{\log t} + \frac{1}{\log^2 t} \right) \leq 0.08.$$

Thus we have

$$|\partial_0(t)| \leq 0.01 + 0.04 + 0.08 = 0.13.$$

Therefore $H'(t) < 0\ \ (t \in (p,\ \ p+1))$. So there exists a point t_0 such that $p < t_0 < p+1$ and

$$H(p+1) = H(p) + H(p+1) - H(p) = H(p) + H'(t_0) >$$

$$> \frac{8}{h(p)} + \frac{1}{t_0} \cdot \left(\partial_0(t_0) - 1 - \frac{1}{\log t_0} \right) \geq$$

$$\geq \frac{8}{h(p)} + \frac{1}{t_0} \cdot \left(-0.13 - 1 - \frac{1}{\log t_0} \right) \geq$$

$$\geq \frac{8}{h(p)} - \frac{1.21}{p} \geq$$

$$\geq \frac{1}{h(p)} \cdot \left(8 - 1.21 \cdot \frac{h(p)}{p} \right) \geq \frac{7.9422}{\sqrt{p} \cdot (\log p)^{3/2}},$$

where $h(t) := \sqrt{t} \cdot (\log t)^{3/2}$. Hence for any $t \in (p,\ \ p+1)$, we have

$$H(t) \geq H(p+1) > \frac{7.9422}{\sqrt{p} \cdot (\log p)^{3/2}} \geq \frac{7.9422}{\sqrt{t} \cdot (\log t)^{3/2}}. \tag{0.15}$$

We will call (0.15) the condition $(\bar{d})$. To prove the Lemma, it is sufficient to obtain a certain contradiction from (0.15).

7.2. Estimates of the different derivatives

For any $t \in (p,\ p+1)$,

$$f'(t) = (1+\log t)\cdot E(t),$$

$$f''(t) = \frac{1}{t}\cdot\left(E(t) - 1 - \frac{1}{\log t}\right) < 0,$$

$$f'''(t) = \frac{1}{t^2}\cdot\left(1 + \frac{1}{\log^2 t} - E(t)\right) > 0,$$

and

$$g'(t) = \frac{1}{2\cdot\sqrt{t}} > 0,$$

$$g''(t) = \frac{-1}{4\cdot t\cdot\sqrt{t}} < 0,$$

$$g'''(t) = \frac{3}{8\cdot t^2\cdot\sqrt{t}} > 0,$$

and by the condition $(\bar{d})$

$$d'(t)\cdot g(t) = f'(t) - d(t)\cdot g'(t) > 0,$$

$$d''(t)\cdot g(t) = f''(t) - d(t)\cdot g''(t) - 2\cdot d'(t)\cdot g'(t) < 0,$$

$$d'''(t)\cdot g(t) = f'''(t) - d(t)\cdot g'''(t) -$$

$$-3\cdot\left(d''(t)\cdot g'(t) + d'(t)\cdot g''(t)\right) > 0.$$

On the other hand, we have

$$|E(t)| \le \frac{1}{\log^2 t}, \quad |\theta(t)| \le \frac{1}{\log t},$$

$$P_0(t) := |f(t)| \le \frac{2\cdot t}{\log t} \le 0.15\cdot t,$$

$$P_1(t) := |f'(t)| \le \frac{1}{\log t} + \frac{1}{\log^2 t} \le 0.08,$$

$$P_2(t) := |f''(t)| \le \frac{1}{t}\cdot\left(1 + \frac{1}{\log t} + \frac{1}{\log^2 t}\right) \le \frac{1.08}{t},$$

$$P_3(t) := |f'''(t)| \le \frac{1}{t^2}\cdot\left(1 + \frac{2}{\log^2 t}\right) \le \frac{1.02}{t^2},$$

$$Q_1(t) := \left|\frac{g'(t)}{g(t)}\right| = \frac{1}{2\cdot t} = \frac{0.5}{t},$$

$$Q_2(t) := \left|\frac{g''(t)}{g(t)}\right| \le \frac{1}{4\cdot t^2} = \frac{0.25}{t^2},$$

$$Q_3(t) := \left|\frac{g'''(t)}{g(t)}\right| \le \frac{3}{8\cdot t^3} \le \frac{0.38}{t^3},$$

From this for any $t \in (p,\ p+1)$ we have respectively

$$D_{01} := |d(t) \cdot g'(t)| = P_0(t) \cdot Q_1(t) \le 0.08,$$

$$D_{02} := |d(t) \cdot g''(t)| = P_0(t) \cdot Q_2(t) \le \frac{0.04}{t},$$

$$D_{03} := |d(t) \cdot g'''(t)| = P_0(t) \cdot Q_3(t) \le \frac{0.06}{t^2},$$

$$D_{10} := |d'(t) \cdot g(t)| \le P_1(t) + D_{01} \le 0.16,$$

$$D_{11} := |d'(t) \cdot g'(t)| = D_{10} \cdot Q_1(t) \le \frac{0.08}{t},$$

$$D_{12} := |d'(t) \cdot g''(t)| = D_{10} \cdot Q_2(t) \le \frac{0.04}{t^2},$$

$$D_{20} := |d''(t) \cdot g(t)| \le P_2(t) + D_{02} + 2 \cdot D_{11} \le \frac{1.3}{t},$$

$$D_{21} := |d''(t) \cdot g'(t)| = D_{20} \cdot Q_1(t) \le \frac{0.7}{t^2},$$

$$D_{30} := |d'''(t) \cdot g(t)| \le P_3(t) + D_{03} +$$

$$+3 \cdot (D_{12} + D_{21}) \le \frac{3.4}{t^2}.$$

Here $i,\ j$ in D_{ij} $(i,\ j = 0,\ 1,\ 2,\ 3)$ denote the orders of the derivatives of the functions $d(t),\ g(t)$ respectively. While, for any $t_1,\ t_2 \in (p,\ p+1)$

$$\rho_0 := \left|\frac{g(t_2)}{g(t_1)}\right| \le 1 + \left|\frac{g'(t_0)}{g(t_1)}\right| \le 1 + 0.5 \cdot 10^{-8},$$

where $t_1 < t_0 < t_2$ or $t_1 > t_0 > t_2$.
Thus by the condition $(\bar{d})$ for any $t_1,\ t_2 \in (p,\ p+1)$ we have

$$\Lambda_1 := \left|\frac{d''(t_1)}{d'(t_2)}\right| \le \left|\frac{D_{20}}{H(t_2)}\right| \cdot \rho_0 \le \frac{1.31}{7.9422} \cdot \frac{\sqrt{t_2} \cdot (\log t_2)^{3/2}}{t_1} \le 0.0079. \tag{0.16}$$

$$\Lambda_2 := \left|\frac{d'''(t_1)}{d'(t_2)}\right| \le \left|\frac{D_{30}}{H(t_2)}\right| \cdot \rho_0 \le \frac{3.41}{7.9422} \cdot \frac{\sqrt{t_2} \cdot (\log t_2)^{3/2}}{t_1^2} \le 1.8 \cdot 10^{-8}. \tag{0.17}$$

7.3. Functions $\lambda_1(x)$ and $\lambda_2(x)$
For any $t \in [p,\ p+1]$, let

$$x := t - p.$$

Then $0 \le x \le 1$, when $p \le t \le p+1$. Put

$$\lambda_1(x) := \frac{\log(1+x)}{\log 2} - x,$$

$$\lambda_2(x) := \frac{\log(1 + 2 \cdot x)}{\log 3} - x,$$

$$\beta_1 := \frac{1}{4}, \quad \beta_2 := \frac{3}{4}.$$

Then the functions $\lambda_1(x)$ and $\lambda_2(x)$ are continuous on the interval $[0,\ 1]$ and differentiable on the interval $(0,\ 1)$. And

$$\lambda_1(0) = 0, \quad \lambda_1(1) = 0,$$

$$\lambda_2(0) = 0, \quad \lambda_2(1) = 0,$$

$$a_1 := \lambda_1(\beta_1) = \frac{\log(5/4)}{\log 2} - \frac{1}{4} =$$

$$= 0.07192809488736 \cdots \neq 0,$$

$$a_2 := \lambda_2(\beta_1) = \frac{\log(3/2)}{\log 3} - \frac{1}{4} =$$

$$= 0.11907024642854 \cdots \neq 0,$$

Also

$$\lambda_1'(x) := \frac{1}{\log 2} \cdot \frac{1}{(1+t)} - 1,$$

$$\lambda_1''(x) := \frac{1}{\log 2} \cdot \frac{(-1)}{(1+t)^2}$$

and

$$\lambda_2'(x) := \frac{1}{\log 3} \cdot \frac{2}{(1+2\cdot t)} - 1,$$

$$\lambda_2''(x) = \frac{1}{\log 3} \cdot \frac{(-4)}{(1+2\cdot t)^2}.$$

Put

$$G_1(x) := \frac{\lambda_1(x)}{a_1} - \frac{\lambda_2(x)}{a_2}$$

then

$$G_1(0) = 0, \quad G_1(1) = 0, \quad G_1(\beta_1) = 0,$$

$$|G_1(x)| \leq 0.1, \quad \forall x \in (0,\ 1)$$

$$G_1'(x) := \frac{\lambda_1'(x)}{a_1} - \frac{\lambda_2'(x)}{a_2},$$

$$|G_1'(x)| \leq 0.8, \quad \forall x \in (0,\ 1)$$

$$G_1''(x) := \frac{\lambda_1''(x)}{a_1} - \frac{\lambda_2''(x)}{a_2},$$

$$|G_1''(x)| \leq 10.6, \quad \forall x \in (0,\ 1)$$

On the one hand,

$$b_1 := \lambda_1(\beta_2) = \frac{\log(7/4)}{\log 2} - \frac{3}{4} =$$

$$= 0.05735492205760 \cdots \neq 0,$$

$$b_2 := \lambda_2(\beta_2) = \frac{\log(5/2)}{\log 3} - \frac{3}{4} =$$

$$= 0.08404376714647 \cdots \neq 0,$$

Also put

$$G_2(x) := \frac{\lambda_1(x)}{b_1} - \frac{\lambda_2(x)}{b_2},$$

then

$$G_2(0)=0,\quad G_2(1)=0,\quad G_2(\beta_2)=0,$$

$$|G_2(x)|\le 0.18,$$

$$-0.18<G_2(x)-G_1(x)<0,\quad \forall t\in(0,\ 1)$$

$$G_2'(x):=\frac{\lambda_1'(x)}{b_1}-\frac{\lambda_2'(x)}{b_2},$$

$$|G_2'(x)|\le 2.2,\quad \forall x\in(0,\ 1)$$

$$-1.4<G_2'(x)-G_1'(x)<0.4,$$

$$G_2''(x):=\frac{\lambda_1''(x)}{b_1}-\frac{\lambda_2''(x)}{b_2},$$

$$|G_2''(x)|\le 18.2,\quad \forall x\in(0,\ 1).$$

And for any $x\in(0,\ 1)$ we have

$$G_2'''(x)-G_1'''(x)<0.$$

This shows that the function $G_2''(x)-G_1''(x)$ is monotonic decreasing on the interval $(0,\ 1)$. So

$$G_2''(x)-G_1''(x)\ge \lim_{x\to 1-0}(G_2''(x)-G_1''(x))=$$

$$=0.14190388505204\cdots,\quad \forall x\in(0,1)$$

Thus for any $x\in(0,\ 1)$ we have

$$\delta_0:=\min_{0\le x\le 1}(G_2''(x)-G_1''(x))\ge 0.1419. \tag{0.18}$$

This value is really important for us below.

7.4. Proof of the Lemma

Let

$$F_1(x):=G_1(x)\cdot d'(x),$$

$$F_2(x):=G_2(x)\cdot d'(x)$$

then

$$F_1(0)=F_1(1)=F_1(\beta_1)=0.$$

Hence there exists a point η_1 such that $0<\eta_1<1$ and $F_1''(\eta_1)=0$, that is,

$$F_1''(\eta_1)=G_1''(\eta_1)\cdot d'(\eta_1)+2\cdot G_1'(\eta_1)\cdot d''(\eta_1)+G(\eta_1)\cdot d'''(\eta_1)=0.$$

Similarly, since

$$F_2(0)=F_2(1)=F_2(\beta_2)=0,$$

there exists a point η_2 such that $0<\eta_2<1$ and $F_1''(\eta_2)=0$, that is,

$$F_2''(\eta_2)=G_2''(\eta_2)\cdot d'(\eta_2)+2\cdot G_2'(\eta_2)\cdot d''(\eta_2)+G(\eta_2)\cdot d'''(\eta_2)=0.$$

From this $F_2''(\eta_2)-F_1''(\eta_1)=0$, that is,

$$G_2''(\eta_2)\cdot d'(\eta_2)-G_1''(\eta_1)\cdot d'(\eta_1)=$$

$$=(2\cdot G_1'(\eta_1)\cdot d''(\eta_1)-2\cdot G_2'(\eta_2)\cdot d''(\eta_2))+$$

$$+(G_1(\eta_1)\cdot d'''(\eta_1) - G_2(\eta_2)\cdot d'''(\eta_2)).$$

Here

$$G_2''(\eta_2)\cdot d'(\eta_2) - G_1''(\eta_1)\cdot d'(\eta_1) =$$
$$= (G_2''(\eta_2) - G_1''(\eta_1))\cdot d'(\eta_2)) + G_1''(\eta_1)\cdot(d'(\eta_2) - d'(\eta_1))$$

and, since $d'(t) > 0$ for any $t \in (x_1,\ x_2)$ by the condition $(\bar{d})$, we have

$$(G_2''(\eta_2) - G_2''(\eta_2))\cdot d'(\eta_2) \geq \delta_0 \cdot d'(\eta_2),$$

$$G_1''(\eta_1))\cdot(d'(\eta_2) - d'(\eta_1)) =$$
$$= G_1''(\eta_1))\cdot d''(\eta_0)\cdot(\eta_2 - \eta_1),$$

where $\eta_1 < \eta_0 < \eta_2$ or $\eta_2 < \eta_0 < \eta_1$.

Finally, by (0.16), (0.17) and (0.18), we have

$$1 \leq \left|\frac{G_1''(\eta_1)}{\delta_0}\right|\cdot\left|\frac{d''(\eta_0)}{d'(\eta_2)}\right| + 2\cdot\left|\frac{G_1'(\eta_1)}{\delta_0}\right|\cdot\left|\frac{d''(\eta_1)}{d'(\eta_2)}\right| +$$

$$+2\cdot\left|\frac{G_2'(\eta_2)}{\delta_0}\right|\cdot\left|\frac{d''(\eta_2)}{d'(\eta_2)}\right| + \left|\frac{G_1(\eta_1)}{\delta_0}\right|\cdot\left|\frac{d'''(\eta_1)}{d'(\eta_2)}\right| + \left|\frac{G_2(\eta_2)}{\delta_0}\right|\cdot\left|\frac{d'''(\eta_2)}{d'(\eta_2)}\right| \leq$$

$$\leq \frac{10.6}{0.1419}\cdot\Lambda_1 + \frac{2\times 0.8}{0.1419}\cdot\Lambda_1 +$$

$$+\frac{2\times 2.2}{0.1419}\cdot\Lambda_1 + \frac{0.1}{0.1419}\cdot\Lambda_2 + \frac{0.18}{0.1419}\cdot\Lambda_2 \leq$$

$$\leq 0.6 + 0.09 + 0.25 + 1.3\cdot 10^{-8} + 1.5\cdot 10^{-8} \leq 0.95,$$

but it is a contradiction. This shows that the condition $(\bar{d})$ is not valid.
So the proof of the Lemma is completed.

VIII. Proof of the Theorem 2

Put $C_m := \Phi_0(\Im_m)$. Then first, if $2 \leq p_m \leq e^{14}$, then we could verify $C_m \leq 3$ by MATLAB (see the table 1 and the table 2). Next, if $p_m \geq e^{14}$, then put $A_m := 8 - 9.3\cdot S(p_m,\ 1/2)$ and we will prove $C_m \leq A_m$ by the mathematical induction with respect to m.
If $m = 93118$ then we have

$$C_{93118} = 2.035\cdots \leq 8 - 9.3\cdot S(p_m,\ 1/2).$$

Now assume that $p_m \geq e^{14}$ and $C_m \leq A_m$. Then by the section 6.3, we have

$$C_{m+1} = \frac{1}{K_{m+1}}\cdot(p_m\cdot(e_m' - \alpha_m) + V_m) =$$

$$= C_m\cdot\frac{K_m}{K_{m+1}} + \frac{V_m}{K_{m+1}} \leq$$

$$\leq A_m\cdot\frac{K_m}{K_{m+1}} + \frac{1}{K_{m+1}}\cdot\log p_{m+1}\cdot(\mu\cdot e_m' - 1) \leq A_m + B_m,$$

where

$$B_m := \frac{1}{K_{m+1}}\cdot(\log p_{m+1}\cdot(\mu\cdot e_m' - 1) -$$
$$-A_m\cdot(K_{m+1} - K_m)).$$

By the assumption $C_m \leq A_m$, we get

$$e'_m \leq \alpha_m + A_m \cdot \frac{K_m}{p_m} = \alpha_m \cdot \left(1 + \frac{A_m}{\sqrt{p_m \cdot \alpha_m}}\right)$$

and by taking logarithm of both sides

$$\log e'_m = (\log p_m) \cdot (e_m - 1) \leq \theta(p_m) + \frac{A_m}{\sqrt{p_m \cdot \alpha_m}}.$$

From this

$$e_m \leq 1 + \frac{1}{\log p_m} \cdot \left(\theta(p_m) + \frac{A_m}{\sqrt{p_m \cdot \alpha_m}}\right),$$

$$E(p_m) + \varepsilon(p_m) \leq \frac{1}{\log p_m} \cdot \left(\theta(p_m) + \frac{A_m}{\sqrt{p_m \cdot \alpha_m}}\right)$$

and the both sides multiply by

$$\frac{p_m \cdot \log p_m}{\sqrt{p_m}}$$

then

$$d(p_m) := \frac{1}{\sqrt{p_m}} \cdot (p_m \cdot \log p_m \cdot E(p_m) - p_m \cdot \theta(p_m)) \leq$$

$$\leq \frac{A_m}{\sqrt{\alpha_m}} - \sqrt{p_m} \cdot \log p_m \cdot \varepsilon(p_m).$$

Thus by the Lemma,

$$(1 + \log p_m) \cdot (E(p_m) + \varepsilon(p_m)) \leq$$

$$\leq \frac{A_m}{2 \cdot \sqrt{p_m \cdot \alpha_m}} + \frac{8}{\sqrt{p_m} \cdot (\log p_m)^{3/2}},$$

because $\varepsilon(p_m) < 0$ and

$$\frac{\log p_m}{2} \leq (1 + \log p_m).$$

Since

$$0 < A_m = 8 - 9.3 \cdot S(p_m,\ 1/2) \leq 3.1$$

and $(1 - 1/14) \leq \alpha_m \leq (1 + 1/14)$, if $e_1 > 1$, then

$$(1 + \log p)^2 \cdot (E(p_m) + \varepsilon(p_m))^2 \leq \frac{3.2}{p_m}.$$

By the Lemma, the sections 6.3 and 6.4, we put

$$B_m \cdot K_{m+1} \leq T_1 + T_2 + T_3,$$

where

$$T_1 := \log p_{m+1} \cdot (1 + \log p_m) \cdot (E(p_m) + \varepsilon(p_m)) -$$

$$-A_m \cdot (K_{m+1} - K_m),$$

$$T_2 := 0.6 \cdot \log p_{m+1} \cdot \frac{\log p_m}{p_m},$$

$$T_3 := 0.6 \cdot \log p_{m+1} \cdot (1 + \log p_m)^2 \cdot (E(p_m) + \varepsilon(p_m))^2.$$

By $K_{m+1} \geq K_m$, the section 6.4 and (0.10) we have

$$\frac{T_1}{K_{m+1}} \leq \frac{A_m}{K_{m+1}} \cdot (U_m - (K_{m+1} - K_m)) +$$

$$+ \frac{1}{K_{m+1}} \cdot \frac{8 \cdot \log p_{m+1}}{\sqrt{p_m} \cdot (\log p_m)^{3/2}} \leq$$

$$\leq A_m \cdot W_m + \frac{1}{\sqrt{\alpha_m}} \cdot \frac{8 \cdot \Omega_0}{p_m \cdot (\log p_m)^{1/2}} \leq$$

$$\leq \frac{0.0007}{p_m \cdot (\log p_m)^{1/2}} + \frac{8.51}{p_m \cdot (\log p_m)^{1/2}} \leq \frac{8.6}{p_m \cdot (\log p_m)^{1/2}}$$

and

$$\frac{T_2}{K_{m+1}} \leq \frac{0.6 \cdot \log^3 p_m}{\sqrt{p_m \cdot \alpha_m}} \cdot \frac{\Omega_0}{p_m \cdot \log p_m} \leq$$

$$\leq \frac{0.5}{p_m \cdot (\log p_m)^{1/2}},$$

$$\frac{T_3}{K_{m+1}} \leq 0.6 \cdot 3.2 \cdot \frac{\log^2 p_m}{\sqrt{p_m \cdot \alpha_m}} \cdot \frac{\Omega_0}{p_m \cdot \log p_m} \leq$$

$$\leq \frac{0.1}{p_m \cdot (\log p_m)^{1/2}},$$

Thus if $e_m > 1$ then

$$B_m < \frac{9.3}{p_m \cdot (\log p_m)^{1/2}}.$$

Next, if $e_m \leq 1$ then by the section 6.3 we obtain

$$B_m \leq 0.6 \cdot \frac{\log^2 p_{m+1}}{p_m \cdot K_m} \leq \frac{0.5}{p_m \cdot (\log p_m)^{1/2}}.$$

Finally, we have $C_{m+1} \leq A_{m+1}$ and so the proof of the Theorem 2 is finished.

IX. Algorithm and Tables for Sequence $\{C_m\}$

The table 1 shows the values of C_m to $\omega(n) = m$ for $n \in N$. There are only values for $1 \leq m \leq 10$ here. But it is not difficult to verify $C_m \leq 3$ for $31 \leq p_m \leq e^{14}$. The table 2 shows the values of C_m for $93109 \leq m \leq 93118$. Of course, all the values in the table 1 and the table 2 are approximate with order 10^{-3}.

The algorithm for C_m to $\omega(n) = m$ by MATLAB is as follows:
Function RH-PN-Index, clc, gamma=0.57721566490153286060; format long,
$P = [2, 3, 5, 7, \cdots, 1202609]$; M=length(P);
for $m = 1 : M$; $p = P(1 : m)$; $q = 1 - 1./p$; $F = prod(1./q)$; $V = sum(\log(p))$; $V1 = (V)^{1/2}$;
m, $p(m)$, $C_m = (\exp(\exp(-gamma) * F) - V)/V1$, end.

Table 1

m	p_m	C_m
1	2	2.85947164195016
2	3	2.68745829155593
3	5	2.60801514536984
4	7	2.73115431266735
5	11	2.57452833561573
6	13	2.60523306367574
7	17	2.56004537210806
8	19	2.63431939241882
9	23	2.67311558160837
10	29	2.60637352799328

Table 2

m	p_m	C_m
93109	1202477	2.03539811396126
93110	1202483	2.03540315703560
93111	1202497	2.03540820013863
93112	1202501	2.03541335720468
93113	1202507	2.03541860543873
93114	1202549	2.03542353469470
93115	1202561	2.03542848676219
93116	1202569	2.03543350721711
93117	1202603	2.03543829985003
93118	1202609	2.03544318364830

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References

[1] Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, Heidelberg Berlin, 1976.

[2] G. Robin, Grandes valeurs de la fonction Somme des diviseurs et hypothese de Riemann, Journal of Math. Pures et appl. 63, 187-213, 1984.

[3] J. C. Lagarias, An elementary problem equivalent to the Riemann hypothesis, Amer. Math. Monthly 109, 534-543, 2002

[4] J. Sandor, D. S. Mitrinovic, B. Crstici, Handbook of Number theory 1, Springer, 2006.

[5] J. B. Rosser, L. Schoenfeld, Approximate formulars for some functions of prime numbers, Illinois J. Math. 6, 64-94, 1962

[6] Young Ju Choie, Nicolas Lichiardopol, Pieter Moree, Patrick Sole, On Robin's criterion for the Riemann hypothesis, Journal de Theorie des Nombers de Bordeaux 19, 357-372, 2007

[7] P. Sole, M. Planat, Robin inequality for 7-free integers, arXiv: 1012.0671v1 [math.NT] 3 Dec 2010.

[8] Ick Sun Eum, Ja Kyung Koo, The Riemann hypothesis and an upper bound of the divisor function for odd integers, J. Math. Anal. Appl. 421, 917-924, 2015

[9] A. Hertlein, Robin inequality for new families of integers, arXiv: 1612.0581v1 [math.NT] 15 Dec 2016

[10] J. L. Nicolas, Small values of the Euler function and the Riemann hypothesis, Acta arithmetica, 155.3, 311-321, 2012

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