A Heuristic and Approximate Solution to the Oppermann's Conjecture

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Abstract: Here we have shown a heuristic and approximate solution to the unsolved oppermann's conjecture. Firstly we have generated a formula to calculate the approximate number of multiples of a prime number p_n less than or equal to a number k which are not the multiples of the prime numbers $p_1, p_2, p_3, \dots, p_{n-1}$ and $p_n < k$. Then we have generated another formula to calculate the number of prime numbers less than or equal to a number k if the prime numbers less than $\sqrt{n}$ are given where $\sqrt{n} \leq k \leq n$. By using these formulas and the main concept of these formulas we have presented our solution.

Keywords: Oppermann's conjecture, Legendre's conjecture, Andrica's conjecture, Brocard's conjecture, prime counting function, Bertrand's postulate.

1. Introduction

Oppermann's conjecture is an unsolved problem in mathematics on the distribution of prime numbers. It is closely related to but stronger than Legendre's conjecture, Andrica's conjecture, and Brocard's conjecture. It is named after Danish mathematician Ludvig Oppermann, who announced it in an unpublished lecture in March 1877.

Conjecture 1.1 (Oppermann's conjecture): For every integer $n > 1$, there is at least one prime number between $n(n-1)$ and n^2, and at least another prime between n^2 and $n(n+1)$.

Here in this paper we have shown a heuristic and approximate solution to the unsolved oppermann's conjecture. We have generated a formula to calculate the approximate number of multiples of a prime number p_n less than or equal to a number k which are not the multiples of the prime numbers $p_1, p_2, p_3, \dots, p_{n-1}$ and $p_n < k$. The formula is as follows,

$$\frac{k}{p_n} \prod_{r=1}^{n(p)-1} \left(1 - \frac{1}{p_r}\right)$$

Here $n(p)$ is the number of prime numbers. By using this formula we have generated another formula to calculate the number of primes less than or equal to a number k if the prime numbers less than or equal to $\sqrt{n}$ are given where $\sqrt{n} \leq k \leq n$. Suppose, the number of prime numbers less than or equal to $\sqrt{n}$ is $q > 1$ and the qth prime is p_q. So, the number of prime numbers less than or equal to k will be

$$\pi(k) = n(p) - 1 + k \prod_{r=1}^{n(p)<\sqrt{n}} \left(1 - \frac{1}{p_r}\right)$$

By using this formula we can show that approximately there exists at least one prime number between $n(n-1)$ and $n^2$and another prime number between n^2 and $n(n+1)$. By using the same method we can show that there exists at least one prime between n and $2n$ where $n > 1$ which is known as Bertrand's postulate. As we know that Bertrand's postulate is correct, so we can say that our heuristic and approximate solution to the oppermann's conjecture is logical and it does make sense.

2. Generating the formula to calculate the approximate number of multiples of a prime number p_n less than or equal to a number k which are not the multiples of the prime numbers $p_1, p_2, p_3, \dots, p_{n-1}$ and $p_n < k$ and the prime counting formula

Suppose, there are some prime numbers $p_1, p_2, p_3, \dots, p_n$ where $p_n < k$. Now we have to calculate the approximate number of multiples of a prime number p_n less than or equal to a number k which are not the multiples of the prime numbers $p_1, p_2, p_3, \dots, p_{n-1}$. Suppose, there are two prime numbers p_1 and p_2. Now we have to calculate the approximate number of multiples of p_2 less than or equal to k which are not the multiples of p_1. Now, the approximate number of multiples of p_1 and p_2 will be respectively

$$\frac{k}{p_1}, \frac{k}{p_2}$$

Now suppose, $A = \{\mathrm{x: x, multiples\ of\ p_1}\}$ and $B = \{\mathrm{x: x, multiples\ of\ p_2}\}$. So, the approximate number of multiples of p_2 less than or equal to k which are not the multiples of p_1 will be

$$n(B) - n(A \cap B) = \frac{K}{P_1} - \frac{K}{P_1 P_2} = \frac{K(P_1 - 1)}{P_1 P_2}$$

Again suppose, there are three prime numbers p_1, p_2 and p_3. Now, the approximate number of multiples of p_1, p_2 and p_3 will be respectively

$$\frac{k}{p_1}, \frac{k}{p_2}, \frac{k}{p_3}$$

Now suppose, $A = \{\mathrm{x: x, multiples\ of\ p_1}\}$, $B = \{\mathrm{x: x, multiples\ of\ p_2}\}$ and $C = \{\mathrm{x: x, multiples\ of\ p_3}\}$. So, the approximate number of multiples of p_3 less than or equal to k which are not the multiples of p_1 and p_2 will be

$$n(C) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$= \frac{k}{p_3} - \frac{K}{P_1 P_3} - \frac{K}{P_2 P_3} + \frac{K}{P_1 P_2 P_3} = K\left(\frac{P_1 P_2 - P_2 - P_1 + 1}{P_1 P_2 P_3}\right) = \frac{K(P_1 - 1)(P_2 - 1)}{P_1 P_2 P_3}$$

Again suppose, there are four prime numbers p_1, p_2, p_3 and p_4. Now, the approximate number of multiples of p_1, p_2, p_3 and p_4 will be respectively

$$\frac{k}{p_1}, \frac{k}{p_2}, \frac{k}{p_3}, \frac{k}{p_4}$$

Now suppose, $A = \{\mathrm{x: x, multiples\ of\ p_1}\}$, $B = \{\mathrm{x: x, multiples\ of\ p_2}\}$, $C = \{\mathrm{x: x, multiples\ of\ p_3}\}$ and $D = \{\mathrm{x: x, multiples\ of\ p_4}\}$. So, the approximate number of multiples of p_4 less than or equal to k which are not the multiples of p_1 , p_2 and p_3 will be

$$n(D) - n(A \cap D) - n(B \cap D) - n(C \cap D) + n(A \cap B \cap D) + n(A \cap C \cap D) + n(B \cap C \cap D) - n(A \cap B \cap C \cap D)$$

$$= \frac{k}{p_4} - \frac{K}{P_1 P_4} - \frac{K}{P_2 P_4} - \frac{K}{P_3 P_4} + \frac{K}{P_1 P_2 P_4} + \frac{K}{P_1 P_3 P_4} + \frac{K}{P_2 P_3 P_4} - \frac{K}{P_1 P_2 P_3 P_4}$$

$$= K\left(\frac{P_1 P_2 P_3 - P_2 P_3 - P_1 P_3 - P_1 P_2 + P_3 + P_2 + P_1 - 1}{P_1 P_2 P_3 P_4}\right)$$

$$= K\left\{\frac{P_2P_3(P_2-1)-P_3(P_2-1)-P_2(P_2-1)+(P_2-1)}{P_1P_2P_3P_4}\right\}$$

$$= K\left\{\frac{(P_2-1)(P_2P_3-P_3-P_2+1)}{P_1P_2P_3P_4}\right\}$$

$$= \frac{K(P_1-1)(P_2-1)(P_3-1)}{P_1P_2P_3P_4}$$

We can see, here we get a beautiful pattern. Now, suppose the number of prime numbers is n and prime numbers are $p_1, p_2, p_3, \dots, p_n$ where $p_n < k$. According to the pattern, the approximate number of multiples of p_n less than or equal to k which are not the multiples of $p_1, p_2, p_3, \dots, p_{n-1}$ will be

$$\frac{K(P_1-1)(P_2-1)(P_3-1)\dots(p_{n-1}-1)}{P_1P_2P_3P_4\dots p_n}$$

$$= \frac{k}{p_n}\left(1-\frac{1}{P_1}\right)\left(1-\frac{1}{P_2}\right)\left(1-\frac{1}{P_3}\right)\dots\left(1-\frac{1}{P_{n-1}}\right)$$

Definition 1.1: We call $n(p)$ is the number of prime numbers.
So, we can write the formula to calculate the approximate number of multiples of a prime number p_n less than or equal to a number k which are not the multiples of the prime numbers $p_1, p_2, p_3, \dots, p_{n-1}$ and $p_n < k$,

$$\frac{k}{p_n}\prod_{r=1}^{n(p)-1}\left(1-\frac{1}{p_r}\right)$$

Now, we have to generate a formula to calculate the number of prime numbers less than or equal to a number k if the prime numbers less than or equal to $\sqrt{n}$ are given where $\sqrt{n} \le k \le n$. Suppose, the number of prime numbers less than or equal to $\sqrt{n}$ is q and the prime numbers are $p_1, p_2, p_3, \dots, p_q$. According to the trial division method, if we want to deduce weather a number n is prime, we have to just test for the prime factors up to $\sqrt{n}$. So, if we want to deduce weather a number k is prime where $\sqrt{n} \le k \le n$, it is enough to test for the prime factors up to $\sqrt{n}$. That means all the composite numbers from 2 to k will be the multiples of the prime numbers less than or equal to $\sqrt{n}$. Again, if k is an integer, then we can write,

$$k = 1 + \text{number of prime numbers } \le k + \text{number of composite numbers } \le k$$

As we know, 1 is not a prime number and also not a composite number. Now we can write,
The number of prime numbers less than or equal to k will be

$$\begin{aligned}\pi(k) = k - 1 &- \text{number of composite multiples of } p_1 \\ &- \text{number of composite multiples of } \mathrm{p_2} \text{ whice are not the multiples of } \mathrm{p_1} \\ &- \text{number of composite multiples of } \mathrm{p_3} \text{ whice are not the multiples of } \mathrm{p_1}, \mathrm{p_2} - \cdots \\ &- \text{number of composite multiples of } \mathrm{p_q} \text{ whice are not the multiples of } \mathrm{p_1}, \mathrm{p_2}, \dots, p_{q-1}\end{aligned}$$

$$= k - 1 - \frac{k}{p_1} + 1 - \frac{K(P_1-1)}{P_1P_2} + 1 - \frac{K(P_1-1)(P_2-1)}{P_1P_2P_3} + 1 - \cdots - \frac{K(P_1-1)(P_2-1)\dots\left(p_{q-1}-1\right)}{P_1P_2P_3P_4\dots p_q} + 1$$

Now we can write,

$$\pi(k) = k - 1 + n(p) - \frac{k}{p_1} - k\sum_{r=1}^{n(p)-1}\frac{\prod_{a=1}^{r}(p_a-1)}{\prod_{a=1}^{r+1}p_a}$$

This formula can be used in small range of positive integers, because the operation is very complicated. But we can derive a easy form of this formula. Suppose, there is one prime number p_1 less than or equal to $\sqrt{n}$. So,

$$\pi(k) = k - 1 + n(p) - \frac{k}{p_1}$$

Again suppose, there are two prime numbers p_1 and p_2 less than or equal to $\sqrt{n}$. So,

$$\pi(k) = k - 1 + n(p) - \frac{k}{p_1} - \frac{K(P_1 - 1)}{P_1 P_2}$$

$$= k - 1 + n(p) - k\left\{\frac{1}{p_1} + \left(\frac{(P_1 - 1)}{P_1} - \frac{(P_1 - 1)(P_2 - 1)}{P_1 P_2}\right)\right\}$$

$$= k - 1 + n(p) - k\left(\frac{1 + P_1 - 1}{p_1} - \frac{(P_1 - 1)(P_2 - 1)}{P_1 P_2}\right)$$

$$= k - 1 + n(p) - k\left(1 - \frac{(P_1 - 1)(P_2 - 1)}{P_1 P_2}\right)$$

$$= n(p) - 1 + k\frac{(P_1 - 1)(P_2 - 1)}{P_1 P_2} = n(p) - 1 + k\left(1 - \frac{1}{P_1}\right)\left(1 - \frac{1}{P_2}\right)$$

Again suppose, there are three prime numbers p_1, p_2 and p_3 less than or equal to $\sqrt{n}$. So,

$$\pi(k) = k - 1 + n(p) - \frac{k}{p_1} - \frac{K(P_1 - 1)}{P_1 P_2} - \frac{K(P_1 - 1)(P_2 - 1)}{P_1 P_2 P_3}$$

$$= k - 1 + n(p) - k\left\{\frac{1}{p_1} + \frac{(P_1 - 1)}{P_1 P_2} + \left(\frac{(P_1 - 1)(P_2 - 1)}{P_1 P_2} - \frac{(P_1 - 1)(P_2 - 1)(P_3 - 1)}{P_1 P_2 P_3}\right)\right\}$$

$$= k - 1 + n(p) - k\left(\frac{P_2 + P_1 - 1 + P_1 P_2 - P_1 - P_2 + 1}{P_1 P_2} - \frac{(P_1 - 1)(P_2 - 1)(P_3 - 1)}{P_1 P_2 P_3}\right)$$

$$= n(p) - 1 + k\frac{(P_1 - 1)(P_2 - 1)(P_3 - 1)}{P_1 P_2 P_3} = n(p) - 1 + k\left(1 - \frac{1}{P_1}\right)\left(1 - \frac{1}{P_2}\right)\left(1 - \frac{1}{P_3}\right)$$

Again suppose, there are three prime numbers p_1, p_2, p_3 and p_4 less than $\sqrt{n}$. By using the same method we can show that,

$$\pi(k) = n(p) - 1 + k\frac{(P_1 - 1)(P_2 - 1)(P_3 - 1)(P_4 - 1)}{P_1 P_2 P_3 p_4} = n(p) - 1 + k\left(1 - \frac{1}{P_1}\right)\left(1 - \frac{1}{P_2}\right)\left(1 - \frac{1}{P_3}\right)\left(1 - \frac{1}{P_4}\right)$$

We can see, here we get a nice pattern when the number of prime numbers less than or equal to $\sqrt{n}$ is greater than 1. Suppose, the number of prime numbers less than or equal to $\sqrt{n}$ is $q > 1$ and the prime numbers are $p_1, p_2, p_3, \dots, p_q$. Let $n(p)$ is the number of prime numbers less than or equal to $\sqrt{n}$. So, the number of prime numbers less than or equal to k will be

$$\pi(k) = n(p) - 1 + k \prod_{r=1}^{n(p)<\sqrt{n}} \left(1 - \frac{1}{p_r}\right)$$

The operation is still very complicated. But this formula will give us better approximation than the other prime counting functions like prime number theorem.

3. The heuristic and approximate solution to the oppermann's conjecture

Now we have to prove that there approximately exists at least one prime between $n(n-1)$ and n^2, and at least another prime between n^2 and $n(n+1)$ where $n > 1$. We can see that when $n = 2,3,4$ the oppermann's conjecture is true. Now, we just have to show that this conjecture can be true when $n > 4$. Firstly we will show that there approximately exists at least one prime between $n(n-1)$ and n^2.

Suppose, the number of prime numbers less than $n(n-1)$ is q and the qth prime number is p_q. Now, we can write, $n(n-1) = p_q + x$ where x is a positive integer. So, $n^2 = p_q + x + n$. Again,

$$n(n-1) < n^2 < \{n(n-1)\}^2 \text{ when } n > 4$$

So, according to our formula,

$$\pi(n^2) - \pi(n^2 - n) = q - 1 + n^2 \prod_{r=1}^{q} \left(1 - \frac{1}{p_r}\right) - q$$

$$= -1 + \frac{(p_q + x + n)(p_1 - 1)(p_2 - 1)(p_3 - 1) \dots (p_q - 1)}{P_1 P_2 P_3 \dots P_q}$$

$$= -1 + \left(1 + \frac{x}{p_q} + \frac{n}{p_q}\right) \frac{(p_1 - 1)(p_2 - 1)(p_3 - 1) \dots (p_q - 1)}{P_1 P_2 P_3 \dots P_{q-1}}$$

Here, $(p_1 - 1) = 1$, $(p_2 - 1) > P_1$,$(p_3 - 1) > P_2$,…, $(p_q - 1) > P_{q-1}$. So,

$$\frac{(p_1 - 1)(p_2 - 1)(p_3 - 1) \dots (p_q - 1)}{P_1 P_2 P_3 \dots P_{q-1}} > 1$$

Again,

$$\frac{(p_1 - 1)(p_2 - 1)}{P_1} < \frac{(p_1 - 1)(p_2 - 1)(p_3 - 1)}{P_1 P_2} < \dots < \frac{(p_1 - 1)(p_2 - 1)(p_3 - 1) \dots (p_q - 1)}{P_1 P_2 P_3 \dots P_{q-1}}$$

Again, when $n = 4$, the prime numbers less than $4(4-1)$ are 2,3,5,7,11. That means $p_1 = 2, p_2 = 3, p_3 = 5$, $p_4 = 7, p_5 = 11$. Here,

$$\frac{(2-1)(3-1)(5-1)(7-1)(11-1)}{2 \times 3 \times 5 \times 7} > 2$$

So, when $n > 4$,

$$\frac{(p_1 - 1)(p_2 - 1)(p_3 - 1) \dots (p_q - 1)}{P_1 P_2 P_3 \dots P_{q-1}} > 2$$

Again,

$$\left(1 + \frac{x}{p_q} + \frac{n}{p_q}\right) > 1$$

So,

$$\left(1+\frac{x}{p_q}+\frac{n}{p_q}\right)\frac{(p_1-1)(p_2-1)(p_3-1)\dots(p_q-1)}{P_1P_2P_3\dots P_{q-1}}>2$$

$$\Rightarrow -1+\left(1+\frac{x}{p_q}+\frac{n}{p_q}\right)\frac{(p_1-1)(p_2-1)(p_3-1)\dots(p_q-1)}{P_1P_2P_3\dots P_{q-1}}>1$$

Again suppose, the number of prime numbers less than n^2 is q and the qth prime number is p_q. Now, we can write, $n^2 = p_q + x$ where x is a positive integer. So, $n(n+1) = p_q + x + \sqrt{p_q + x}$. Again,

$$n^2 < n(n+1) < n^4 \text{ when } n > 4$$

So, according to our formula,

$$\pi(n^2+n)-\pi(n^2) = q-1+(n^2+n)\prod_{r=1}^{q}\left(1-\frac{1}{p_r}\right)-q$$

$$= -1+\frac{\left(p_q+x+\sqrt{p_q+x}\right)(p_1-1)(p_2-1)(p_3-1)\dots(p_q-1)}{P_1P_2P_3\dots P_q}$$

$$= -1+\left(1+\frac{x}{P_q}+\frac{\sqrt{p_q+x}}{P_q}\right)\frac{(p_1-1)(p_2-1)(p_3-1)\dots(p_q-1)}{P_1P_2P_3\dots P_{q-1}}$$

So, when $n > 4$,

$$\frac{(p_1-1)(p_2-1)(p_3-1)\dots(p_q-1)}{P_1P_2P_3\dots P_{q-1}}>2$$

Again,

$$\left(1+\frac{x}{P_q}+\frac{\sqrt{p_q+x}}{P_q}\right)>1$$

So,

$$\left(1+\frac{x}{P_q}+\frac{\sqrt{p_q+x}}{P_q}\right)\frac{(p_1-1)(p_2-1)(p_3-1)\dots(p_q-1)}{P_1P_2P_3\dots P_{q-1}}>2$$

$$\Rightarrow -1+\left(1+\frac{x}{P_q}+\frac{\sqrt{p_q+x}}{P_q}\right)\frac{(p_1-1)(p_2-1)(p_3-1)\dots(p_q-1)}{P_1P_2P_3\dots P_{q-1}}>1$$

So, the number of prime numbers between $n(n-1)$ and n^2 and n^2 and $n(n+1)$ are approximately greater than 1. That's why we can say there approximately exists at least one prime between $n(n-1)$ and n^2, and at least another prime between n^2 and $n(n+1)$ where $n > 1$. Another interesting thing we can see here that if we increase the value of *n*, the number of prime numbers between $n(n-1)$ and n^2 and n^2 and $n(n+1)$ will also increase.

4. Applying the same method in case of Bertrand's postulate

Theorem 1.1 (Bertrand's postulate): For every $n > 1$ there is always at least one prime p such that $n < p < 2n$. We can see, when $n = 2$,3 bertrand's postulate is true. Now, we have to prove that there exists at least one prime between n and $2n$ when $n > 3$, because we can use our prime counting formula when the given number of primes is greater than 1. Suppose, the number of prime numbers less than or equal to n is q and the qth prime number is p_q. Now, we can write, $n = p_q + x$ where $x \geq 0$. So, $2n = 2p_q + 2x$. Again,

$$n < 2n < n^2 \quad \text{when } n > 3$$

So, according to our formula,

$$\pi(2n) - \pi(n) = q - 1 + 2n \prod_{r=1}^{q} \left(1 - \frac{1}{p_r}\right) - q$$

$$= -1 + \frac{(2p_q + 2x)(p_1 - 1)(p_2 - 1)(p_3 - 1) \dots (p_q - 1)}{P_1 P_2 P_3 \dots P_q}$$

$$= -1 + \left(2 + \frac{2x}{P_q}\right) \frac{(p_1 - 1)(p_2 - 1)(p_3 - 1) \dots (p_q - 1)}{P_1 P_2 P_3 \dots P_{q-1}}$$

We have shown that,

$$\frac{(p_1 - 1)(p_2 - 1)(p_3 - 1) \dots (p_q - 1)}{P_1 P_2 P_3 \dots P_{q-1}} > 1$$

And,

$$\left(2 + \frac{2x}{P_q}\right) \geq 2$$

So,

$$-1 + \left(2 + \frac{2x}{P_q}\right) \frac{(p_1 - 1)(p_2 - 1)(p_3 - 1) \dots (p_q - 1)}{P_1 P_2 P_3 \dots P_{q-1}} \geq 1$$

So, the number of prime numbers between n and $2n$ are approximately greater than or equal to 1. As we know that Bertrand's postulate is correct and we can prove it in many ways, so we can say that our heuristic and approximate solution to the oppermann's conjecture is logical and it does make sense.

5. Conclusion

In this paper we didn't provide any proper solution to the oppermann's conjecture. But this heuristic and approximate solution shows us that the oppermann's conjecture can be true. We think this paper will help us to think about this conjecture from a new perspective and also help us to find a proper solution.

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7. References

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