On Representations of the Lorentz Group

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Abstract

The Lorentz group is a non-compact group. Consequently, it's representations cannot be expected to be equivalent to representations of a unitary group. Actually, they consist of a time-like component and space-like components, in some sense analogous to 4-vectors. In contrast to representations of compact groups they carry the actual value of the non-compact variables, the boost-vector. In the non-boosted state the space-components vanish and the time-component transforms according to a representation of the rotation subgroup. Application of a boost then generates space-components, a process that preserves norms. However, the norm now has a growing positive contribution from the time-component and a negative contribution from the space-components, growing absolutely to keep the total unchanged. General transformations are described in detail. The freedom to assign boost directions to space components leads to a topological symmetry with flavor-generating representations for two-sheeted representations.

Symmetry *SO(4)*

There are six infinitesimal rotations for the group SO(4):

$$A_j = x_l \partial_k - x_k \partial_l \quad , \quad B_j = x_j \partial_0 - x_0 \partial_j \quad , \text{(j, k, l)} = (1, 2, 3) \text{ and cyclic permuted.} \qquad (1)$$

They span the associated Lie algebra, which separates into a sum of two independent sub-algebras, generated by

$$J_j = \frac{1}{2}\left(A_j + B_j\right) \quad , \quad K_j = \frac{1}{2}\left(A_j - B_j\right) \quad , \qquad (2)$$

which both obey the commutation relations of the Lie-algebra *o(3)* generating the group of orthogonal transformations in three dimensions.

$$[J_j, J_k] = J_l \quad , \text{(j, k, l) as above.} \qquad (3)$$

Correspondingly, *SO(4)* is a direct product of two groups, each homologous to *SO(3)* [1,2].

Therefore, its representations can be labeled by two integers or half-integers (*j, k*), which characterize the two factors of a particular representation. Due to the closed nature of *SO(4)* these representations are closed as well. Thus, one can apply the standard exponentiation procedure of Lie-groups to obtain an arbitrary transformation of a representation.

Every representation has a chirality parameter (*k-j*) chosen such that pure *j*-representations (*j*>0) have a

negative value (left-handed chirality). Achiral representations have $j = k$.

Remark on the Dimension of Representations of SO(3)

Representations of *SO(3)* must be considered real. This can be seen by choosing a 'magnetic' basis in which a component, say J_3 is diagonal. A representation labeled by a half integer j is $(2j+1)$-dimensional and has the eigenvalues (of J_3) $m_j = -j, 1-j, ..., j-1, j$ with eigenfunctions $e^{im_j\phi}$, where φ is the rotation angle about the 3-axis. Equivalently, we may choose the completely real functions $(e^{im_j\phi}+e^{-im_j\phi})/2$ and $(e^{im_j\phi}-e^{-im_j\phi})/(2i)$. The appearance of complex conjugated functions is owed to the fact that rotations about an axis are an Abelian subgroup with exclusively one dimensional representations. The eigenvalues of a transformation, which is real, are either real (1 or -1) or occur in complex conjugated pairs. Thus, two complex conjugated representations can be combined to a completely real two-dimensional representation now reducible, however.

If the vector spaces are considered as a field over the Euclidean *R4* we can extend them to a Hilbert space by introducing a local norm as the scalar product with the conjugated vector. Clearly, this Hilbert space is over the field of reals, and this holds true for the product representations of *SO(4)* as well. The group *SO(4)* preserves the Euclidean metric, and thus, is the isotropic symmetry group of *R4*.

A consequence of the field of reals is that a function $|\psi>$, when multiplied by the imaginary unit i leads to a negative norm $i^2<\psi|\psi>$, as we will encounter in the following. This situation is somewhat confusing when working with the complex conjugated representations mentioned above. In other words, the freedom of choosing among equivalent quantization axis must not be confused with a complex Hilbert space.

Proper Orthochronous Lorentz Group, *SO⁺(3,1)*

For the Lorentz group, the 'isotropic' symmetry group of the Minkowski space, there are several substantial modifications, due to the non-compact nature of the group *SO⁺(3,1)*, introduced by the infinitesimal boost operators B. Obviously, we cannot expect to encounter unitary representations (which are compact). To see the problems, let us consider a special representation $(j,0)$ in which the second factor is the identity representation. We use the basis of magnetic quantum numbers in *b*-direction (boost). Starting from a non-boosted state, a boost transformation along the *b*-axis corresponds in *SO(4)* to a rotation in the two-dimensional space spanned by ψ and ${}^{j}D_{\hat{b}}\psi$, where

$$ {}^{j}D_{\hat{b}}=e^{i\pi\sigma_b} \quad . \tag{4} $$

This becomes now a discrete proper transformation (see below), which corresponds to a time reversal and an inversion of the *b*-axis. It mediates between 'separated' parts of the representation. The term separated needs some explication. It corresponds to time- and space-nature of a *4*-vector in space-time. There exists no Lorentz transformation that converts a time-like vector in a space-like one. In this sense time- and space-components must be considered 'separated'.

Clearly, for $(0,k)$-representations we must choose

$$ {}^{k}D_{\hat{b}}=e^{-i\pi\sigma_b} \quad , \tag{5} $$

in accord with time direction in *K* (expression 2).

Writing the a $(0,b)$-rotation from *SO(4)* as

$$ e^{i\phi\sigma_b j}\psi_{m_j}=\cos\left(m_j\frac{\phi}{2}\right){}^{j}\psi_{m_j}+\sin\left(m_j\frac{\phi}{2}\right){}^{j}D_{\hat{b}}{}^{j}\psi_{m_j} \quad , \tag{6} $$

we can make the transition from *SO(4)* to *SO⁺(3,1)* by recalling that we can first perform the rotation part $_rD$, followed by the pure-boost part $_bD$ due to commutation of the two operators. For the latter we start from (6) to arrive at

$$^jD_b(\phi)^j\psi_{m_j}=\cosh\left(m_j\frac{\phi}{2}\right)^j\psi_{m_j}-i\sinh\left(m_j\frac{\phi}{2}\right)^jD_{\hat{b}}{}^j\psi_{m_j} \quad, \tag{7}$$

by the replacement $\varphi \rightarrow i\varphi$. We see that the 'separated' part of the representation (space components) has a *3*-vector nature due to the vector nature of (4), and it has only non-vanishing components for boosted states. As an aside, we must have a simultaneous back-transformation:

$$^jD_b(\phi)^jD_{\hat{b}}{}^j\psi_{m_j}=\cosh\left(m_j\frac{\phi}{2}\right)^jD_{\hat{b}}{}^j\psi_{m_j}+i\sinh\left(m_j\frac{\phi}{2}\right)(^jD_{\hat{b}})^2{}^j\psi_{m_j} \quad, \quad (^jD_{\hat{b}})^2=\pm1 \quad, \tag{8}$$

where the + sign applies for integer *j*, the – sign for half-integer *j* (two-valued representations).

To see that we can extend the concept of Hilbert space, we consider (7) further and write down the norm of the original vector and the transformed one:

$$\langle{}^j\psi_{m_j}{}^j\hat{D}_b(\phi)|^jD_b(\phi)^j\psi_{m_j}\rangle =$$

$$\cosh\left(m_j\frac{\phi}{2}\right)\cosh\left(m_j\frac{\phi}{2}\right)\langle{}^j\psi_{m_j}|^j\psi_{m_j}\rangle-\sinh\left(m_j\frac{\phi}{2}\right)\sinh\left(m_j\frac{\phi}{2}\right)\langle{}^j\psi_{m_j}{}^j\hat{D}_{\hat{b}}|^jD_{\hat{b}}{}^j\psi_{m_j}\rangle \quad. \tag{9}$$

The operator $^jD_{\hat{b}}$ is unitary and by multiplication with its conjugate transpose yields the unit operator, hence

$$\langle{}^j\psi_{m_j}{}^j\hat{D}_b(\phi)|^jD_b(\phi)^j\psi_{m_j}\rangle=\left[\cosh^2\left(m_j\frac{\phi}{2}\right)-\sinh^2\left(m_j\frac{\phi}{2}\right)\right]\langle{}^j\psi_{m_j}|^j\psi_{m_j}\rangle \quad. \tag{10}$$

However, the operator $^jD_b(\phi)$ is the exponential of a Hermitean and thus non-unitary. Nevertheless, we can extend the concept of a norm by recognizing that, for each m_j-component of ψ, we have an expression 'cosh²(*x*) - sinh²(*x*)' which equals one and is unchanged under a pure boost transformation. Obviously, the vectors of the non-boosted representation extend under boosts to generalized 4-vectors, each component of which is a *(2j+1)*-vector.

If we consider ψ a field in Minkowski space, we now have a generalized Hilbert space with a norm that has a positive contribution from the time-component and a negative contribution from the space components. This norm is unchanged under local boosts and rotations.

A further modification, compared with *SO(4)*, is that the variable φ is now not periodic any more, but extends from zero to plus infinity. Negative values are taken care of by 'inverting' the boost direction (actually by a π-rotation, to stay with proper transformations). We emphasize this by replacing φ by the boost variable

$$|b|=\frac{\phi}{2}=\frac{1}{2}\text{artanh}(v) \quad, \tag{11}$$

where $0\leq v<1$ is the speed of the boost. This allows us to give the vector *b* an extended meaning such that its norm characterizes the value (11) to which the state is boosted. Consequently, we can omit the variable φ in the operator ^jD_b. Furthermore, we can define the cosh- and sinh-operators ^jC_b and ^jS_b, which are diagonal in the frame in which the *b*-component of the spin operator is diagonal, and which have the diagonal elements cosh(*mb*) and sinh(*mb*), respectively. Thus we may write for the boosted field

$$^j\psi_{m_j}(b)=^jD_b{}^j\psi_{m_j}=^jC_b\psi_{m_j}\oplus{}^jD_{\hat{b}}{}^jS_b{}^j\psi_{m_j} \quad, \tag{12}$$

where the circled plus-sign indicates that the first term is in the time-component and the second in the space-component, commonly called large and small, respectively, where the transfer is effected by $^{j}D_{\hat{b}}$. We can now write the norm of the field as

$$\langle ^{j}\psi_{m_{j}}(b)|^{j}\psi_{m_{j}}(b)\rangle=\langle ^{j}\psi_{m_{j}}{}^{j}\hat{C}_{b}|^{j}C_{b}{}^{j}\psi_{m_{j}}\rangle-\langle ^{j}\psi_{m_{j}}{}^{j}\hat{S}_{b}{}^{j}\hat{D}_{\hat{b}}|^{j}D_{\hat{b}}{}^{j}S_{b}{}^{j}\psi_{m_{j}}\rangle \quad . \tag{13}$$

Here we can, furthermore, introduce the operator product

$$^{j}G_{\hat{p},\hat{b}}=^{j}\hat{D}_{\hat{p}}{}^{j}D_{\hat{b}} \quad , \tag{14}$$

which is unitary. In particular it reduces to the unit operator for co-linear b and p.

This construction allows us to generalize the norm (13) to a full bilinear form:

$$\langle ^{j}\psi(p)|^{j}\chi(b)\rangle=\langle ^{j}\psi{}^{j}\hat{C}_{p}|^{j}C_{b}{}^{j}\chi\rangle-\langle ^{j}\psi{}^{j}\hat{S}_{p}{}^{j}G_{\hat{p},\hat{b}}{}^{j}S_{b}{}^{j}\chi\rangle \quad . \tag{15}$$

We have the additional effect, that the m_{j}-component of fields acquire an additional phase of $\exp(im_{j}b)$, because, for a $(j,0)$ representation, a boost is always coupled with a synchronous coaxial rotation (see 2). When the speed of the boost approaches a value of one, this phase oscillates at ever increasing rate. For $m_{j}=0$ the spatial component stays at zero and there is no oscillation.

At this point a comparison with the *SO(4)* case is in order. There, the second term in (15) disappears while C is just the unit matrix. With boosted states and the concurrent non-unit C-matrix we loose the freedom of choosing equivalent quantization axes.

We can repeat this consideration for general (j,k) representations by the observation that one can, in boost direction, choose a magnetic quantum number basis in each factor of the representation which yield, under boost, components with total magnetic quantum numbers of $(m_{j} - m_{k})$. The components of the corresponding *4*-vector now are $(2j+1)(2k+1)$-tuples, of which the spatial ones corresponding to $\Delta m = (m_{j} - m_{k}) = 0$ now stay at a value of zero. For half-integer $j+k$ there are no $\Delta m = 0$ states, for integer $j+k$ we have $z_{jk}=2\min(j,k)+1$ of them. Correspondingly, the representations of $(v \rightarrow 1)$-limiting states are characterized by a direction vector and z_{jk} longitudinal components.

Transformation Behavior

A general state ψ is characterized by a boost vector b and the non-boosted state ψ_{0} from which it is generated by the pure b-boost D_{b}. The subscript *0* indicates that this state is purely time-like, with zero space components.

$$\psi=D_{b}\psi_{0} \tag{16}$$

To keep track of the transformation behavior under general Lorentz transformations we take care that the transformation is such that we have the total pure boost as final transformation. Therefore, to rotate a state with final boost D_{b} by a rotation R we must write $D_{R} = D_{R}D_{b}D_{-b}$ such that we can operate on ψ_{0} with $D_{R}D_{b}$:

$$D_{R}\psi=D_{R}D_{b}D_{-b}\psi=D_{R}D_{b}\psi_{0} \tag{17}$$

We can apply the rotation R on the boost vector b to obtain the new boost vector p. There exist a whole set of rotations to achieve that, but only one of them transforms the pure boost B_{b} to the pure boost B_{p}, which we call

$$R_{p,b}b=p \tag{18}$$

This rotation has the rotation axis parallel to the vector product of b and p.

To proceed we make use of the decomposition possibilities of a general Lorentz transformation:

$$L = R\,B_b = R_{p,b}\,R_b\,B_b = B_p\,R_p\,R_{p,b} \quad , \tag{19}$$

where R_b and R_p are rotations about the axes b and p, respectively, by the same angle. Thus, one obtains R_b from R and $R_{p,b}$, L from the second equation of (19) and R_p from the third one. This completes the transformation (17):

$$D_R\psi = D_R\,D_b\,\psi_0 = D_p\,D_{R_p}\,D_{R_{p,b}}\,\psi_0 \quad . \tag{20}$$

Clearly, the rotated field $\quad D_{R_p}\,D_{R_{p,b}}\,\psi_0\quad$ is again purely time-like of which we know how to apply the pure boost D_p.

A similar consideration can be made for applying a pure boost

$$D_p\psi = D_p\,D_b\,D_{-b}\psi = D_p\,D_b\,\psi_0 \quad . \tag{21}$$

In this case one makes use of the formula for multiplication of two pure boost:

$$L = B_p\,B_b = R_{p\oplus b,\,b\oplus p}\,B_{b\oplus p} = B_{p\oplus b}\,R_{p\oplus b,\,b\oplus p} \quad . \tag{22}$$

Here we have to remember, that combining two boosts corresponds to adding the corresponding velocities according to the relativistic addition formula, symbolized by the circled plus sign, which is not commutative. Formula (22) also illustrates that the product of two pure boosts is not a pure boost (Thomas-Wigner Rotation). It again allows to start from the non-boosted ψ_0 and to apply the rotation before the final pure boost.

To summarize, a (j,k)-representations of the Lorentz group has a $(2j+1)(2k+1)$-dimensional time-like component and three space-like components of the same dimension, which are characterized by a boost vector b. Applying the reverse boost, $-b$, produces a purely time-like situation. The fact that a general Lorentz transformation can always be written as a product of a rotation with a final pure boost allows to perform the rotational part in the time-like component and finally apply the terminal boost (16).

The states have a norm (13) which is unchanged under Lorentz Transformations and under multiplication of the states by a arbitrary phase factor. However, one must acknowledge that this phase factor will change with the choice of the origin of the space-time coordinates, because a boost is always accompanied by a synchronous rotation (2). If the field possesses a local boost b, a change in the choice of the (4-)origin δx will produce a concurrent phase of

$$\Phi(\delta x) = e^{\pm\frac{\pi}{2}\,^jB\,\delta x}\,\Phi(0) \quad , \tag{23}$$

where ^jB is the j-specific (antihermitean) boost 4-vector-operator which is a zero-vector (see expressions (8) and (13) of Ref [3]). This new phase varies with the position of the field whenever the field changes it's (local) boost value with position.

It is important to acknowledge that states of a representation carry a parameter b, the characterization of the final pure boost. This is an important difference to representations of compact Lie groups, like e.g. the rotation groups $O(n)$, the states of which have no memory of any applied group elements.

Discrete Transformations

For $O(4)$ the kernel $O(4)/SO(4)$ is the two-element inversion-group I, distinguishing improper transformations from the normal subgroup of proper ones. Under improper transformations the norm of a representation-vector must change it's sign. This can most easily be seen by inverting the quantization axis of spin wave-functions in $O(3)$. The spin operator is a pseudo-vector, the expectation value of

which is not changed by the inversion. However, inversion changes the sign of the magnetic quantum number, and consequently the expectation value of the spin operator, unless the norm changes as well. For integer representations this amounts to multiplying the representation-vector with $\pm i$, and to a sheet change for half-integer ones. Twofold application of the same transformation leads to a change of the phase by *-1*.

For improper transformations the chirality changes sign [2]. In the case of non-chiral representations ($j=k$) one has an even and a odd representation [2]. For S*O(4)* this is the only addition when going to the full group O(4).

For the Lorentz group the situation is somewhat richer because the kernel $O(3,1)/SO^+(3,1)$ now consists of the four elements $\{1, P, T, PT\}$ and is isomorphic to the Abelian Klein four-group, K_4.

Furthermore, due to the 'disconnected' character of time- and space-components of a representation, we have a additional freedom. We have a oriented triple of boost directions ($\hat{b}_1, \hat{b}_2, \hat{b}_3$) and a associated triple of matrices (D_1, D_2, D_3) (4, 5) spanning the space part of a representation. Because of the 'separated' nature of space- and time-components the assignment of the two triples can be arbitrary, as long as the handedness is unchanged. Thus we have the additional symmetry, A_3, of even permutations of mutual assignment of b's and D's. For integer representations, $j+k = n = 0, 1, 2, \ldots$ this yields no additional symmetry because it corresponds to a normal rotation, already included in the Lorentz transformations. This is no longer true for true half-integer representations $j+k = n+\frac{1}{2}$, because these representations are two-sheeted.

Case j = k

In this case a representation acquires a additional signature corresponding to one of the four representations of K_4.

Case j ≠ k, j+k = n (Integer)

Here the representations are chiral and change chirality under improper transformations ($j \leftrightarrow k$). This corresponds to the transition between particle and antiparticle. In addition there exists is a *PT*-parity.

Case j ≠ k, j+k = n + ½

In this situation the additional A_3-symmetry shows up as described above. It gives rise to the three representations known as flavor. As in the previous case one still has chirality as well as *PT*-parity, which is commonly considered a additional flavor characterization.

Conclusion

Representations of the Lorentz group $SO^+(3,1)$ are not unitary. Their states carry a boost parameter (vector). They have a structure consisting of a time-component (large) and a space-component (small). The space-component disappears for zero boost. States possess a norm which is invariant under Lorentz transformations and which has a positive and a negative contribution from time- and space-components, respectively. The two components are 'separated' in the sense, that there is no Lorentz transformation which can turn a time-only component into a space-only component.

The transformation behavior can be obtained by making use of the fact that a general Lorentz transformation can always be composed into a rotation followed by a final pure boost.

Improper transformations (particle-antiparticle relations) change the sign of the norm, such that the contribution of the time- (space-) component is negative (positive). Furthermore, representations carry a *PT*-parity.

In addition, due to the separated nature of time- and space-component, there is a topological symmetry derived from the assignment of boost directions to the three directions in the space-component which leads to a three-valued flavor-signature for half-integer representations.

In this context the concept of little groups must be seriously questioned.

References

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