A PROOF OF POLIGNAC'S CONJECTURE AND INFINITE TWIN PRIMES

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ABSTRACT. We derive a generalised proof of Polignac's Conjecture by regarding a prime in terms of its whole indivisibility and the consequent probability of prime intervals by applying the second Borel–Cantelli lemma. The specific case of the Twin Prime Conjecture is proven as an example.

INTRODUCTION AND PRELIMINARIES

Polignac's Conjecture [1] states that *for every even natural number k, there are infinitely many consecutive prime pairs p and p′ such that* $p' - p = k$ [1]. In the case of $k = 2$, this is known as the Twin Prime Conjecture. In Lemma 1, we define the probability of a division of naturals producing a remainder. In Lemma 2, we define a prime's probability of occurring as a product of being wholly indivisible by all smaller primes. In Lemma 3, we elaborate how this proves the conjecture.

- $\mathbb{P}$ is the set of all prime numbers
- $\mathbb{P}(n)$ is the n^{th} prime
- ε is any arbitrarily small number

LEMMA 1

For any natural numerator *a* and any natural denominator *b*, the probability of producing a nonzero remainder can be expressed as:

$$\Pr(a \bmod b > 0) = \frac{b-1}{b}$$

EXAMPLE. The probability that a natural *a* divided by 4 produces a nonzero remainder can be expressed as:

$$b = 4$$
$$\Pr(a \bmod 4 > 0) = \frac{3}{4}$$

LEMMA 2

A prime can be defined as *a natural number wholly indivisible except by 1 and itself*. That is, a prime *p* must produce a remainder > 0 for any division where the divisor *d* is not 1 or itself.

$$p \bmod d > 0 : p \in \mathbb{P}, \mathrm{d} \neq 1, \mathrm{d} \neq \mathrm{p}$$

The probability that a natural odd *m* is prime can be expressed as the probability that for all divisions of *m* by all primes < *m*, no remainders of zero will occur. Therefore, the probability *m* being prime can be expressed per Lemma 1 and the product rule:

$$S = \{n \colon n \in \mathbb{P}, n < m\}$$

$$\Pr(m \in \mathbb{P}) = \prod_{n=1}^{|S|} \frac{S(n) - 1}{S(n)}$$

EXAMPLE. Consider the probability of 7 being prime in terms of its whole indivisibility:

$$\begin{aligned} m &= 7 \\ S &= \{2,3,5\} \\ \Pr(m \in \mathbb{P}) &= \prod_{\mathrm{n}=1}^{|S|} \frac{S(n) - 1}{S(n)} \\ &= \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} \\ &= 0.2\overline{6} \end{aligned}$$

LEMMA 3

The probability of any odd natural *m* being prime per Lemma 2 converges toward zero but never intersects zero as m → ∞:

$$\lim_{m \to \infty} \Pr(m \in \mathbb{P}) = 0$$

As *m* can be arbitrarily large and offer infinite events, we can apply the second Borel–Cantelli lemma [2]: *if the sum of probabilities of events diverges to infinite, then the probability that infinitely many of them occur is 1*. That is, any event of arbitrarily small probability > 0 will have infinitely many occurrences given an infinite sample:

$$\begin{aligned} \Pr(m \in \mathbb{P}) &= \epsilon \\ E_n &= \langle \epsilon \rangle_{n=1}^{\infty} \\ \Pr\left(\limsup_{n \to \infty} E_n\right) &= 1 \end{aligned}$$

Given any odd *m* and any even *k* > *m*, the probability of *m* + *k* being prime is > 0. Therefore, all even intervals between primes have infinitely many occurrences. Necessarily, Polignac's Conjecture must be true.

EXAMPLE. Consider the Twin Prime Conjecture where *m* is any arbitrarily large prime, and *k* = 2. The probability that *m* + 2 is also prime never intersects zero and must have infinitely many occurrences:

$$\begin{aligned} k &= 2 \\ S &= \{n : n \in \mathbb{P}, n < (m+2)\} \\ \Pr\big((m+2) \in \mathbb{P}\big) &= \prod_{\mathrm{n}=1}^{|S|} \frac{S(n) - 1}{S(n)} \\ &= \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} \ldots \times \frac{\max S - 1}{\max S} \\ &= \epsilon \\ \Pr\left(\limsup_{n \to \infty} \langle \epsilon \rangle_{n=1}^{\infty}\right) &= 1 \\ m \in \mathbb{N}, |\mathbb{N}| &= \infty \end{aligned}$$

Discussion

This deduction inadvertently proves a stronger formulation of Polignac's Conjecture that *for every odd m and any prime k, there are infinitely many even intervals between m and k.* The conjecture remains true even if *m* is merely odd and not necessarily prime.

It may be noteworthy that Lemma 2 provides a novel proof of Euclid's Theorem; the probability of any arbitrarily large number being prime never intersects zero.

References

1. de Polignac, A. "Recherches nouvelles sur les nombres premiers" (1849)
2. Émile Borel, M. "Les probabilités dénombrables et leurs applications arithmétiques" (1909)

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