The Inverse Tangent and Cotangent Functions, their Addition Formulas and their Values on their Branch Cuts

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Abstract

The principal inverse tangent and cotangent functions for complex arguments can be defined as formulas involving principal natural logarithms, but these are not odd on the imaginary axis, which they must be according to their definitions as inverse functions. These formulas are therefore modified in such a way that they become odd on the imaginary axis, and the corresponding addition formulas for complex and real arguments are derived. With these addition formulas their values on their branch cuts are determined, confirming these modified formulas. Some new formulas for the (hyperbolic) inverse tangent function for complex arguments and some new addition formulas for the (hyperbolic) inverse tangent and cotangent functions for real arguments are derived.

1 Definitions and Basic Identities

A complex x can be represented by its absolute value $r = |x|$ and its principal angle with the positive real axis in the complex plane $\phi = \text{Arg}(x)$ where $-\pi < \text{Arg}(x) \leq \pi$:

$$x = re^{i\phi} \tag{1.1}$$

The principal square root of a complex x is then defined by [1, 9]:

$$\sqrt{x} = \sqrt{r}\, e^{i\phi/2} \tag{1.2}$$

A definition which is used in this paper and in an earlier paper [4] is the function $\text{sg}(x)$ for complex x [8].

Definition 1.1. *For complex x, let $\sqrt{x}$ be the principal square root of x, then:*

$$\text{sg}(x) = \begin{cases} \frac{\sqrt{x^2}}{x} = \frac{x}{\sqrt{x^2}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \tag{1.3}$$

The result of this definition is:

$$\text{sg}(x) = \begin{cases} 1 & \text{if } \text{Re}(x) > 0 \\ -1 & \text{if } \text{Re}(x) < 0 \\ 1 & \text{if } \text{Re}(x) = 0 \text{ and } \text{Im}(x) \geq 0 \\ -1 & \text{if } \text{Re}(x) = 0 \text{ and } \text{Im}(x) < 0 \end{cases} \tag{1.4}$$

For complex x: $1/\text{sg}(x) = \text{sg}(x)$ and for complex $x \neq 0$: $\text{sg}(-x) = -\text{sg}(x)$.
For real x, $\text{sg}(ix) = \text{sg}(x)$.
For real x the function $\text{sg}(x)$ reduces to:

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \tag{1.5}$$

From these identities follows that for complex x: $\text{sg}(x)\sqrt{x^2} = x$, $\text{sg}(x)x = \sqrt{x^2}$, and for real x, $\text{sg}(x)|x| = x$ and $\text{sg}(x)x = |x|$. For complex x, $\text{sg}(\sqrt{x}) = 1$, and for complex x, $(\sqrt{x})^2 = x$.
Let $f(x)$ and $g(x)$ be complex functions, and let:

$$g(x) = f^2(x) \tag{1.6}$$

Then:

$$\sqrt{g(x)} = \pm f(x) \tag{1.7}$$

Because for complex x: $\sqrt{x^2} = \text{sg}(x)x$, the sign is $\text{sg}(f(x))$:

$$\sqrt{g(x)} = \text{sg}(f(x))f(x) \tag{1.8}$$

For real x, $\text{sg}(x)x = |x|$, so for real $f(x)$ and $g(x)$: $\sqrt{g(x)} = |f(x)|$.
For real x, y [1, 9]:

$$\sqrt{x+iy} = \sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + i\,\text{sg}(y)\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}} \tag{1.9}$$

For complex x and real nonnegative α:

$$\sqrt{\alpha x} = \sqrt{\alpha}\sqrt{x} \tag{1.10}$$

The Iverson bracket notation [3, 7] is defined.

Definition 1.2. *Let S be a logical expression, then:*

$$[S] = \begin{cases} 1 & \text{if } S \text{ is true} \\ 0 & \text{if } S \text{ is false} \end{cases} \tag{1.11}$$

The function $\text{sg}(x)$ for complex x can be written as:

$$\text{sg}(x) = [\text{Re}(x) > 0] - [\text{Re}(x) < 0] + [\text{Re}(x) = 0]([\text{Im}(x) \geq 0] - [\text{Im}(x) < 0]) \tag{1.12}$$

For complex $x \neq 0$ it is clear that:

$$\text{sg}(\frac{1}{x}) = \text{sg}(x) - 2[\text{Re}(x) = 0]\text{sg}(\text{Im}(x)) \tag{1.13}$$

For the principal angle with the positive real axis in the complex plane $-\pi < \text{Arg}(x) \leq \pi$, where $\text{Arg}(0) = 0$ is defined, the following addition formula holds. For complex x, y:

$$\text{Arg}(x) + \text{Arg}(y) = \text{Arg}(xy) + \begin{cases} 2\pi & \text{if } \text{Arg}(x) + \text{Arg}(y) > \pi \\ -2\pi & \text{if } \text{Arg}(x) + \text{Arg}(y) \leq -\pi \\ 0 & \text{otherwise} \end{cases} \tag{1.14}$$

The following six identities are evident from the complex plane. For real x, y:

$$\text{Arg}(x + iy) = \arctan(\frac{y}{x}) + \pi[x < 0]\text{sg}(y) \tag{1.15}$$

where when $x = 0$, for real y:

$$\arctan(\frac{y}{0}) = \frac{\pi}{2}([y > 0] - [y < 0]) \tag{1.16}$$

because $\arctan(\infty) = \pi/2$, $\arctan(-\infty) = -\pi/2$ and $\text{Arg}(0) = 0$. For complex $x \neq 0$:

$$\text{Arg}(-x) = \text{Arg}(x) + \pi\,\text{sg}(ix) \tag{1.17}$$

For complex x:

$$[\text{Arg}(x) \geq 0] = \frac{1}{2}(1 + \text{sg}(\text{Im}(x))) \tag{1.18}$$

$$[\text{Arg}(x) > 0] = \frac{1}{2}(1 - \text{sg}(ix)) \tag{1.19}$$

For complex x and real positive c:

$$\text{Arg}(x) + \text{Arg}(\frac{c}{x}) = 2\pi[\text{Im}(x) = 0][\text{Re}(x) < 0] \tag{1.20}$$

$$\text{Arg}(x) + \text{Arg}(-\frac{c}{x}) = \pi\,\text{sg}(\text{Im}(x)) \tag{1.21}$$

The principal natural logarithm function $\ln(x)$ for complex x is defined by [1]:

$$\ln(x) = \ln(|x|) + i\text{Arg}(x) \tag{1.22}$$

For complex x, y, application of (1.14) to this identity gives:

$$\ln(x) + \ln(y) = \ln(xy) + \begin{cases} 2\pi i & \text{if } \text{Arg}(x) + \text{Arg}(y) > \pi \\ -2\pi i & \text{if } \text{Arg}(x) + \text{Arg}(y) \leq -\pi \\ 0 & \text{otherwise} \end{cases} \tag{1.23}$$

For complex x, application of (1.17) gives:

$$\ln(-x) = \ln(x) + \pi i\,\text{sg}(ix) \tag{1.24}$$

For complex x, because $-\pi/2 < \text{Arg}(\sqrt{x}) \leq \pi/2$:

$$2\ln(\sqrt{x}) = \ln(x) \tag{1.25}$$

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2 The Inverse Tangent and Cotangent Functions which Must Be Odd on the Imaginary Axis

The principal values of the inverse tangent and cotangent functions can be defined in the complex plane as the following formulas [1], where $\ln(x)$ is the principal natural logarithm function:

$$\arctan(x) = -\frac{i}{2}\ln(\frac{1+ix}{1-ix}) \tag{2.1}$$

$$\text{arccot}(x) = \arctan(\frac{1}{x}) = -\frac{i}{2}\ln(\frac{ix-1}{ix+1}) \tag{2.2}$$

The power series expansion of the $\arctan(x)$ function is odd:

$$\arctan(x) = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1} \tag{2.3}$$

which indicates that the principal $\arctan(x)$ function should be odd. The function $\tan(x)$ for complex x is defined as [1, 2]:

$$\tan(x) = -i\frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}} \tag{2.4}$$

which is odd in the complex plane. The principal inverse tangent function $\arctan(x)$ for complex x is defined by:

$$\arctan(\tan(x)) = x \tag{2.5}$$

Because both $\tan(x)$ and x are odd functions in the complex plane, from this equation follows that the principal $\arctan(x)$ function must also be odd in the complex plane, and similarly for the principal $\text{arccot}(x)$ function for $x \neq 0$. Replacing x by $-x$ in formulas (2.1) and (2.2) means replacing x by $1/x$ in the principal $\ln(x)$ function. From definition (1.22) and (1.20) it is clear that $\ln(1/x) = -\ln(x)$ except for x on the negative real axis, which is the branch cut of the principal $\ln(x)$ function [1], and where the angle is always π and never $-\pi$. Therefore these formulas are not odd there and must be made odd there explicitly. The following theorem determines for which arguments of the principal $\arctan(x)$ and $\text{arccot}(x)$ functions the arguments of the principal $\ln(x)$ function in (2.1) and (2.2) are on the negative real axis. The branch cuts of the principal $\arctan(x)$ and $\text{arccot}(x)$ functions are defined as in [1, 5] and not to include $\pm i$ which are singular points of these functions.

Theorem 2.1. *The argument x of the principal $\ln(x)$ function in (2.1) and (2.2) is on the negative real axis if and only if the argument x of the principal $\arctan(x)$ or $\text{arccot}(x)$ functions is on their branch cuts [1].*

Proof. For the $\arctan(x)$ function (2.1), let t be the argument of the principal $\ln(x)$ function in (2.1) and let t be real, then the following identity is solved:

$$\frac{1+ix}{1-ix} = t \tag{2.6}$$

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which is easily checked to be:

$$x = i\frac{1-t}{1+t} \tag{2.7}$$

which means that x must be on the imaginary axis. Therefore x can be replaced with ix where x is real, and because t must be real and negative, the following identity is solved:

$$\frac{1-x}{1+x} < 0 \tag{2.8}$$

which is fulfilled if and only if $x < -1$ or $x > 1$. When $-1 \le t < 0$, $x > 1$ is on the upper branch cut, and when $t < -1$, $x < -1$ is on the lower branch cut.

For the $\mathrm{arccot}(x)$ function (2.2), the following identity is solved:

$$\frac{ix-1}{ix+1} = t \tag{2.9}$$

which is easily checked to be:

$$x = i\frac{t+1}{t-1} \tag{2.10}$$

which means that x must be on the imaginary axis. Therefore x can be replaced with ix where x is real, and because t must be real and negative, the following identity is solved:

$$\frac{x+1}{x-1} < 0 \tag{2.11}$$

which is fulfilled if and only if $-1 < x < 1$. When $-1 < t < 0$, $-1 < x < 0$ is on the lower branch cut, and when $t \le -1$, $0 \le x < 1$ is on the upper branch cut. $\qquad\square$

The principal $\arctan(x)$ and $\mathrm{arccot}(x)$ formulas (2.1) and (2.2) can be made odd on these branch cuts explicitly by defining the following functions that are π on the lower branch cuts of these functions and zero elsewhere, using the Iverson bracket notation definition 1.2.

$$\mathrm{oddtan}(x) = \pi[\mathrm{Re}(x) = 0][\mathrm{Im}(x) < -1] \tag{2.12}$$

$$\mathrm{oddcot}(x) = \pi[\mathrm{Re}(x) = 0][-1 < \mathrm{Im}(x) < 0] \tag{2.13}$$

The following are the formulas for the principal values of the $\arctan(x)$ and $\mathrm{arccot}(x)$ functions that are odd everywhere in the complex plane including on the imaginary axis, where $\ln(x)$ is the principal natural logarithm function.

Definition 2.1. *For complex x:*

$$\arctan(x) = -\frac{i}{2}\ln(\frac{1+ix}{1-ix}) - \pi[\mathrm{Re}(x) = 0][\mathrm{Im}(x) < -1] \tag{2.14}$$

$$\mathrm{arccot}(x) = -\frac{i}{2}\ln(\frac{ix-1}{ix+1}) - \pi[\mathrm{Re}(x) = 0][-1 < \mathrm{Im}(x) < 0] \tag{2.15}$$

This definition will be confirmed by applying the addition formulas for determining the values of these functions on their branch cuts in section 5. In this paper from here these formulas are always used.

Theorem 2.2. *For real x, y:*

$$\arctan(x + iy) = \frac{i}{4}\ln\left(\frac{x^2 + (y+1)^2}{x^2 + (y-1)^2}\right) + \frac{1}{2}\arctan\left(\frac{2x}{1 - x^2 - y^2}\right)$$
$$+ \frac{\pi}{2}[x^2 + y^2 > 1]\mathrm{sg}(x) - \pi[x = 0][y < -1] \qquad (2.16)$$

$$\mathrm{arccot}(x + iy) = \frac{i}{4}\ln\left(\frac{x^2 + (y-1)^2}{x^2 + (y+1)^2}\right) + \frac{1}{2}\arctan\left(\frac{2x}{x^2 + y^2 - 1}\right)$$
$$+ \frac{\pi}{2}[x^2 + y^2 < 1]\mathrm{sg}(x) - \pi[x = 0][-1 < y < 0] \qquad (2.17)$$

Proof. When evaluating (2.14) with (1.22):
For complex x:

$$\arctan(x) = -\frac{i}{4}\ln\left(\left|\frac{1 + ix}{1 - ix}\right|^2\right) + \frac{1}{2}\mathrm{Arg}\left(\frac{1 + ix}{1 - ix}\right) - \pi[\mathrm{Re}(x) = 0][\mathrm{Im}(x) < -1] \qquad (2.18)$$

In this formula x is replaced by x + iy:
For real x, y:

$$\frac{|1 - y + ix|^2}{|1 + y - ix|^2} = \frac{|(1 - y + ix)(1 + y + ix)|^2}{|(1 + y - ix)(1 + y + ix)|^2} = \frac{|(1 - y)(1 + y) - x^2 + 2ix|^2}{((1 + y)^2 + x^2)^2}$$
$$= \frac{((1 - y)(1 + y) - x^2)^2 + 4x^2}{((1 + y)^2 + x^2)^2} = \frac{(1 - y)^2 + x^2}{(1 + y)^2 + x^2} \qquad (2.19)$$

$$\mathrm{Arg}\left(\frac{1 - y + ix}{1 + y - ix}\right) = \mathrm{Arg}\left(\frac{(1 - y + ix)(1 + y + ix)}{x^2 + (1 + y)^2}\right) = \mathrm{Arg}(1 - x^2 - y^2 + 2ix) \qquad (2.20)$$

With application of (1.15) the first identity is proved, and the proof of the second identity is similar. □

When $x = 0$ and $y = \pm 1$ these formulas yield $\arctan(i) = i\infty$, $\arctan(-i) = -i\infty$, $\mathrm{arccot}(i) = -i\infty$ and $\mathrm{arccot}(-i) = i\infty$.
For real x:

$$\arctan(ix) = \frac{i}{2}\ln\left(\left|\frac{1 + x}{1 - x}\right|\right) + \frac{\pi}{2}([x > 1] - [x < -1]) \qquad (2.21)$$

$$\mathrm{arccot}(ix) = \frac{i}{2}\ln\left(\left|\frac{x - 1}{x + 1}\right|\right) + \frac{\pi}{2}([0 \le x < 1] - [-1 < x < 0]) \qquad (2.22)$$

The sum of the two functions (2.14) and (2.15) is now also odd in the complex plane (except for $x = 0$ as mentioned above).

Theorem 2.3. *For complex $x \ne \pm i$:*

$$\mathrm{arccot}(x) + \arctan(x) = \frac{\pi}{2}\mathrm{sg}(x) \qquad (2.23)$$

where $\mathrm{sg}(x)$ *is defined by (1.4).*

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Proof. Substituting formulas (2.14) and (2.15) and using (1.23) and $\ln(-1) = \pi i$:

$$
\begin{aligned}
&\operatorname{arccot}(x) + \arctan(x) \\
&= -\frac{i}{2}[\ln(\frac{1+ix}{1-ix}) + \ln(\frac{ix-1}{ix+1})] - \operatorname{oddtan}(x) - \operatorname{oddcot}(x) \\
&= \frac{\pi}{2} - \operatorname{oddtan}(x) - \operatorname{oddcot}(x) +
\begin{cases}
\pi & \text{if } \operatorname{Arg}(\frac{1+ix}{1-ix}) + \operatorname{Arg}(\frac{ix-1}{ix+1}) > \pi \\
-\pi & \text{if } \operatorname{Arg}(\frac{1+ix}{1-ix}) + \operatorname{Arg}(\frac{ix-1}{ix+1}) \leq -\pi \\
0 & \text{otherwise}
\end{cases}
\end{aligned}
\tag{2.24}
$$

In the last identity (1.21) with $c = 1$ can be applied, for which the imaginary part of $(1+ix)/(1-ix)$ can be determined by substituting $x = a + bi$:

$$
\frac{1+i(a+bi)}{1-i(a+bi)} = \frac{(1-b+ia)(1+b+ia)}{(1+b-ia)(1+b+ia)} = \frac{1-a^2-b^2+2ia}{(1+b)^2+a^2}
\tag{2.25}
$$

From this follows that $\operatorname{Im}((1+ix)/(1-ix)) \geq 0$ if and only if $\operatorname{Re}(x) \geq 0$, and application of (1.21) with $c = 1$ yields:

$$
\operatorname{Arg}(\frac{1+ix}{1-ix}) + \operatorname{Arg}(\frac{ix-1}{ix+1}) = \pi \operatorname{sg}(\operatorname{Re}(x))
\tag{2.26}
$$

and consequently from (2.24) follows:

$$
\operatorname{arccot}(x) + \arctan(x) = \frac{\pi}{2} - \operatorname{oddtan}(x) - \operatorname{oddcot}(x) - \pi[\operatorname{Re}(x) < 0]
\tag{2.27}
$$

which for $x \neq \pm i$ is exactly $\frac{\pi}{2}\operatorname{sg}(x)$ with $\operatorname{sg}(x)$ defined in (1.4). $\qquad \square$

This result is different from [1] eq. 4.4.5 when $\operatorname{Re}(x) = 0$, because as mentioned the function definitions in this reference are (2.1) and (2.2) which are not odd on the imaginary axis.

Theorem 2.4. *For complex x:*

$$
\arctan(\frac{1}{x}) = \operatorname{arccot}(x) + \pi[\operatorname{Re}(x) = 0]([-1 < \operatorname{Im}(x) < 0] - [0 < \operatorname{Im}(x) < 1])
\tag{2.28}
$$

$$
\operatorname{arccot}(\frac{1}{x}) = \arctan(x) + \pi[\operatorname{Re}(x) = 0]([\operatorname{Im}(x) < -1] - [\operatorname{Im}(x) > 1])
\tag{2.29}
$$

Proof. Applying the definitions (2.14) and (2.15):

$$
\begin{aligned}
&\arctan(\frac{1}{x}) - \operatorname{arccot}(x) \\
&= -\frac{i}{2}[\ln(\frac{1+i\frac{1}{x}}{1-i\frac{1}{x}}) - \ln(\frac{ix-1}{ix+1})] + \operatorname{oddcot}(x) - \operatorname{oddtan}(\frac{1}{x}) \\
&= -\frac{i}{2}[\ln(\frac{ix-1}{ix+1}) - \ln(\frac{ix-1}{ix+1})] + \operatorname{oddcot}(x) - \operatorname{oddtan}(\frac{1}{x}) \\
&= \operatorname{oddcot}(x) - \operatorname{oddtan}(\frac{1}{x})
\end{aligned}
\tag{2.30}
$$

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Because the lower branch cuts are on the imaginary axis, the following identity holds:

$$\text{oddtan}(\frac{1}{x}) = \text{oddcot}(-x) \tag{2.31}$$

and the first identity in the theorem is proved. The proof of the second identity is similar.
□

Both sides of these identities are odd for $x \neq 0$. For real x, $\text{arccot}(x) = \arctan(1/x)$. The following is a consequence of this theorem, using theorem 2.3. For complex x:

$$\arctan(x) + \arctan(\frac{1}{x}) = \frac{\pi}{2}\text{sg}(x) + \pi[\text{Re}(x) = 0]([-1 < \text{Im}(x) < 0] - [0 < \text{Im}(x) < 1]) \tag{2.32}$$

$$\text{arccot}(x) + \text{arccot}(\frac{1}{x}) = \frac{\pi}{2}\text{sg}(x) + \pi[\text{Re}(x) = 0]([\text{Im}(x) < -1] - [\text{Im}(x) > 1]) \tag{2.33}$$

Theorem 2.5. *For complex x:*

$$\arctan(x) = -i\ln(\frac{1 + ix + \sqrt{1 + x^2}}{1 - ix + \sqrt{1 + x^2}}) \tag{2.34}$$

which is identical to (2.14), and therefore odd in the complex plane.

Proof. From definition (2.14) and (1.25) follows:

$$\arctan(x) = -i\ln(\sqrt{\frac{1 + ix}{1 - ix}}) - \pi[\text{Re}(x) = 0][\text{Im}(x) < -1] \tag{2.35}$$

Let:

$$F(x) = \frac{1 + ix + \sqrt{1 + x^2}}{1 - ix + \sqrt{1 + x^2}} \tag{2.36}$$

Then:

$$F^2(x) = \frac{2(1 + ix)(1 + \sqrt{1 + x^2})}{2(1 - ix)(1 + \sqrt{1 + x^2})} = \frac{1 + ix}{1 - ix} \tag{2.37}$$

Using (1.8) the result is:

$$\sqrt{\frac{1 + ix}{1 - ix}} = \text{sg}(F(x))F(x) \tag{2.38}$$

Evaluation of $F(x)$ with (1.9) and:

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac + bd + i(bc - ad)}{c^2 + d^2} \tag{2.39}$$

yields for real x, y:

$$F(x + iy) = \frac{f(x, y) + ig(x, y)}{h(x, y)} \tag{2.40}$$

where:

$$z(x, y) = \sqrt{(1 + x^2 - y^2)^2 + (2xy)^2} = \sqrt{(x^2 + y^2 - 1)^2 + 4x^2} \tag{2.41}$$

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$$f(x, y) = 1 - x^2 - y^2 + z(x, y) + 2\sqrt{(z(x, y) + 1 + x^2 - y^2)/2} \tag{2.42}$$

$$\begin{aligned} g(x, y) =& 2x(1 + \sqrt{(z(x, y) + 1 + x^2 - y^2)/2}) \\ &+ 2y \operatorname{sg}(xy)\sqrt{(z(x, y) - 1 - x^2 + y^2)/2} \end{aligned} \tag{2.43}$$

$$\begin{aligned} h(x, y) =& x^2 + (1 + y)^2 + z(x, y) + 2(1 + y)\sqrt{(z(x, y) + 1 + x^2 - y^2)/2} \\ &- 2x \operatorname{sg}(xy)\sqrt{(z(x, y) - 1 - x^2 + y^2)/2} \end{aligned} \tag{2.44}$$

where $h(x, y) \geq 0$. Because $z(x, y) \geq |x^2 + y^2 - 1|$ and $x + |x| \geq 0$, it is concluded that $f(x, y) \geq 0$, so $\operatorname{sg}(F(x + iy)) = 1$ except when $f(x, y) = 0$ and $g(x, y) < 0$. Furthermore $f(x, y) = 0$ if and only if $x = 0$ and $y^2 \geq 1$. When $x = 0$, $g(x, y) < 0$ if and only if $y < 0$, so $\operatorname{sg}(F(x + iy)) = -1$ if and only if $x = 0$ and $y \leq -1$. Therefore the following results. For complex x:

$$\sqrt{\frac{1 + ix}{1 - ix}} = (1 - 2[\operatorname{Re}(x) = 0][\operatorname{Im}(x) \leq -1])F(x) \tag{2.45}$$

With (2.35) and (1.24) the theorem is proved. □

3 Addition Formulas for the Inverse Tangent and Cotangent Functions for Complex Arguments

For the addition formulas of the principal $\arctan(x)$ and $\operatorname{arccot}(x)$ functions for complex arguments, application of (1.23) to the arguments of (2.14) and (2.15) gives the following theorems.

Theorem 3.1. *For complex $x \neq \pm i$ and $y \neq \pm i$:*

$$\arctan(x) + \arctan(y) = \begin{cases} 0 & \text{if } y = -x \\ \frac{\pi}{2}\operatorname{sg}(x) + \operatorname{oddcot}(x) - \operatorname{oddcot}(-x) & \text{if } y = 1/x \\ \arctan(\dfrac{x + y}{1 - xy}) + \operatorname{addtan}(x, y) + \operatorname{oddtan}(x, y) & \text{otherwise} \end{cases} \tag{3.1}$$

where:

$$\operatorname{addtan}(x, y) = \begin{cases} \pi & \text{if } \operatorname{Arg}(\frac{1+ix}{1-ix}) + \operatorname{Arg}(\frac{1+iy}{1-iy}) > \pi \\ -\pi & \text{if } \operatorname{Arg}(\frac{1+ix}{1-ix}) + \operatorname{Arg}(\frac{1+iy}{1-iy}) \leq -\pi \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

$$\operatorname{oddtan}(x, y) = \operatorname{oddtan}(\frac{x + y}{1 - xy}) - \operatorname{oddtan}(x) - \operatorname{oddtan}(y) \tag{3.3}$$

Proof. For $xy \neq 1$, application of (1.23) to (2.14):

$$\begin{aligned} \arctan&(x) + \arctan(y) \\ =& -\frac{i}{2}[\ln(\frac{1 + ix}{1 - ix}) + \ln(\frac{1 + iy}{1 - iy})] - \operatorname{oddtan}(x) - \operatorname{oddtan}(y) \\ =& -\frac{i}{2}\ln(\frac{(1 + ix)(1 + iy)}{(1 - ix)(1 - iy)}) + \operatorname{addtan}(x, y) - \operatorname{oddtan}(x) - \operatorname{oddtan}(y) \\ =& -\frac{i}{2}\ln(\frac{1 + iz}{1 - iz}) + \operatorname{addtan}(x, y) - \operatorname{oddtan}(x) - \operatorname{oddtan}(y) \end{aligned} \tag{3.4}$$

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where z has to be solved. First the following equation is solved:

$$\frac{1+iz}{1-iz} = t \tag{3.5}$$

which is easily checked to be:

$$z = i\frac{1-t}{1+t} \tag{3.6}$$

The following t is now substituted:

$$t = \frac{(1+ix)(1+iy)}{(1-ix)(1-iy)} \tag{3.7}$$

and the solution is:

$$z = i\frac{1 - \frac{(1+ix)(1+iy)}{(1-ix)(1-iy)}}{1 + \frac{(1+ix)(1+iy)}{(1-ix)(1-iy)}} = i\frac{(1-ix)(1-iy) - (1+ix)(1+iy)}{(1-ix)(1-iy) + (1+ix)(1+iy)} = \frac{x+y}{1-xy} \tag{3.8}$$

Substituting (2.14):

$$-\frac{i}{2}\ln(\frac{1+iz}{1-iz}) = \arctan(z) + \mathrm{oddtan}(z) \tag{3.9}$$

and the theorem is proved for $xy \neq 1$. For $xy = 1$, that is $y = 1/x$, (2.32) is used. $\qquad \square$

When writing this theorem as $f(x,y) = g(x,y)$, because $f(-x,-y) = -f(x,y)$, the symmetry identity $g(-x,-y) = -g(x,y)$ must hold. When none of the inverse tangent arguments is on a branch cut, the arguments of the $\mathrm{Arg}(x)$ functions in $\mathrm{addtan}(x,y)$ are not on the negative real axis, and then by (1.20) $\mathrm{Arg}(1/x) = -\mathrm{Arg}(x)$, and this symmetry identity holds. When one of x or y is on a branch cut, then only the first or the third case in $\mathrm{addtan}(x)$ is possible, so $\mathrm{addtan}(x,y)$ changes from 0 to π or vice versa, and $\mathrm{oddtan}(x,y)$ changes from 0 to $-\pi$ or vice versa. In all of these cases the same identity holds. When both x and y are on a branch cut, then $\mathrm{addtan}(x,y) = \pi$ and $\mathrm{oddtan}(x,y)$ changes from 0 to -2π or vice versa or from $-\pi$ to $-\pi$, and the same identity holds. When only z is on a branch cut, then (3.7) holds with t real and negative, and using (1.21) then $\mathrm{addtan}(x,y)$ changes from 0 to $-\pi$ or vice versa, and $\mathrm{oddtan}(x,y)$ from 0 to π or vice versa, and the same identity holds. When z and x or y is on a branch cut, then $\mathrm{addtan}(x,y) = 0$ and $\mathrm{oddtan}(x,y)$ changes from 0 to 0 or from π to $-\pi$ or vice versa, and the same identity holds. The arguments x, y and z cannot all be on a branch cut, because the product of two real negative t values cannot be negative.

Theorem 3.2. *For complex $x \neq \pm i$ and $y \neq \pm i$:*

$$\mathrm{arccot}(x) + \mathrm{arccot}(y) = \begin{cases} \pi & \text{if } y = x = 0 \\ 0 & \text{if } y = -x \neq 0 \\ \frac{\pi}{2}\mathrm{sg}(x) + \mathrm{oddtan}(x) - \mathrm{oddtan}(-x) & \text{if } y = 1/x \\ \mathrm{arccot}(\frac{xy-1}{x+y}) + \mathrm{addcot}(x,y) + \mathrm{oddcot}(x,y) & \text{otherwise} \end{cases} \tag{3.10}$$

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where:

$$\text{addcot}(x, y) = \begin{cases} \pi & \text{if } \text{Arg}(\frac{ix-1}{ix+1}) + \text{Arg}(\frac{iy-1}{iy+1}) > \pi \\ -\pi & \text{if } \text{Arg}(\frac{ix-1}{ix+1}) + \text{Arg}(\frac{iy-1}{iy+1}) \le -\pi \\ 0 & \text{otherwise} \end{cases} \qquad (3.11)$$

$$\text{oddcot}(x, y) = \text{oddcot}(\frac{xy-1}{x+y}) - \text{oddcot}(x) - \text{oddcot}(y) \qquad (3.12)$$

Proof. For $xy \ne 1$, application of (1.23) to (2.15):

$$\begin{aligned} &\text{arccot}(x) + \text{arccot}(y) \\ &= -\frac{i}{2}[\ln(\frac{ix-1}{ix+1}) + \ln(\frac{iy-1}{iy+1})] - \text{oddcot}(x) - \text{oddcot}(y) \\ &= -\frac{i}{2}\ln(\frac{(ix-1)(iy-1)}{(ix+1)(iy+1)}) + \text{addcot}(x, y) - \text{oddcot}(x) - \text{oddcot}(y) \\ &= -\frac{i}{2}\ln(\frac{iz-1}{iz+1}) + \text{addcot}(x, y) - \text{oddcot}(x) - \text{oddcot}(y) \end{aligned} \qquad (3.13)$$

where z has to be solved. First the following equation is solved:

$$\frac{iz-1}{iz+1} = t \qquad (3.14)$$

which is easily checked to be:

$$z = i\frac{t+1}{t-1} \qquad (3.15)$$

The following t is now substituted:

$$t = \frac{(ix-1)(iy-1)}{(ix+1)(iy+1)} \qquad (3.16)$$

and the solution is:

$$z = i\frac{\frac{(ix-1)(iy-1)}{(ix+1)(iy+1)}+1}{\frac{(ix-1)(iy-1)}{(ix+1)(iy+1)}-1} = i\frac{(ix-1)(iy-1)+(ix+1)(iy+1)}{(ix-1)(iy-1)-(ix+1)(iy+1)} = \frac{xy-1}{x+y} \qquad (3.17)$$

Substituting (2.15):

$$-\frac{i}{2}\ln(\frac{iz-1}{iz+1}) = \text{arccot}(z) + \text{oddcot}(z) \qquad (3.18)$$

and the theorem is proved for $xy \ne 1$. For $xy = 1$, that is $y = 1/x$, (2.33) is used. $\qquad \square$

When $x \ne 0$ and $y \ne 0$, writing this theorem as $f(x, y) = g(x, y)$, because then $f(-x, -y) = -f(x, y)$, also $g(-x, -y) = -g(x, y)$, and the same reasoning as for the previous theorem can be given. When $y = 0$ this theorem and (2.31) and (2.33) yields:
For complex x:

$$\frac{1}{2}(1 + \text{sg}(x)) + [\text{Re}(x) = 0]([-1 < \text{Im}(x) < 0] - [\text{Im}(x) > 1]) = [\text{Arg}(\frac{ix-1}{ix+1}) > 0] \quad (3.19)$$

and replacing x by $1/x$ and using (1.13) and (2.31):
For complex x:

$$\frac{1}{2}(1 + \text{sg}(x)) + [\text{Re}(x) = 0]([\text{Im}(x) < -1] - [0 \le \text{Im}(x) \le 1]) = [\text{Arg}(\frac{1+ix}{1-ix}) > 0] \quad (3.20)$$

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4 Addition Formulas for the Inverse Tangent and Cotangent Functions for Real Arguments

For the addition formulas of the principal $\arctan(x)$ and $\text{arccot}(x)$ functions for real arguments, theorems 3.1 and 3.2 are applied for real x and y.

Theorem 4.1. *For real x, y:*

$$\arctan(x) + \arctan(y) = \begin{cases} \frac{\pi}{2}\text{sg}(x) & \text{if } y = 1/x \\ \arctan(\dfrac{x+y}{1-xy}) + \pi[xy > 1]\text{sg}(x) & \text{otherwise} \end{cases} \tag{4.1}$$

Proof. Applying theorem 3.1 for x and y real, $\text{oddtan}(x, y) = 0$, and for $\text{addtan}(x, y)$ in (3.2), because x and y are real:

$$\text{Arg}(\frac{1+ix}{1-ix}) + \text{Arg}(\frac{1+iy}{1-iy}) = 2[\text{Arg}(1+ix) + \text{Arg}(1+iy)] \tag{4.2}$$

Applying (1.14) because $-\pi < \text{Arg}(1+ix) + \text{Arg}(1+iy) < \pi$:

$$\text{Arg}(1+ix) + \text{Arg}(1+iy) = \text{Arg}((1+ix)(1+iy)) = \text{Arg}(1 - xy + i(x+y)) \tag{4.3}$$

The $\text{addtan}(x, y)$ in theorem 3.1 for real x, y thus becomes:

$$\text{addtan}(x, y) = \begin{cases} \pi & \text{if } \text{Arg}(1 - xy + i(x+y)) > \pi/2 \\ -\pi & \text{if } \text{Arg}(1 - xy + i(x+y)) \leq -\pi/2 \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

In the complex plane this is easily seen to be equivalent to the theorem, with the fact that in this case $xy \neq 1$. When $xy > 1$, x and y have the same sign, so $\text{sg}(x + y)$ can be replaced by $\text{sg}(x)$. $\qquad\square$

Theorem 4.2. *For real x, y:*

$$\text{arccot}(x) + \text{arccot}(y) = \begin{cases} \pi & \text{if } y = x = 0 \\ 0 & \text{if } y = -x \neq 0 \\ \frac{\pi}{2}\text{sg}(x) & \text{if } y = 1/x \\ \text{arccot}(\dfrac{xy-1}{x+y}) + \pi[|xy| < \text{sg}(x)\text{sg}(y)]\text{sg}(x+y) & \text{otherwise} \end{cases} \tag{4.5}$$

Proof. Applying theorem 3.2 for x and y real, $\text{oddtan}(x, y) = 0$, and for $\text{addtan}(x, y)$ in (3.11), because x and y are real and using $\text{sg}(x)x = |x|$:

$$\frac{ix-1}{ix+1} = \frac{-i\,\text{sg}(x)(ix-1)}{-i\,\text{sg}(x)(ix+1)} = \frac{|x| + i\,\text{sg}(x)}{|x| - i\,\text{sg}(x)} \tag{4.6}$$

$$\text{Arg}(\frac{|x| + i\,\text{sg}(x)}{|x| - i\,\text{sg}(x)}) + \text{Arg}(\frac{|y| + i\,\text{sg}(y)}{|y| - i\,\text{sg}(y)}) = 2[\text{Arg}(|x| + i\,\text{sg}(x)) + \text{Arg}(|y| + i\,\text{sg}(y))] \tag{4.7}$$

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Applying (1.14) because not $x = y = 0$ which is a special case in the theorem, $-\pi < \operatorname{Arg}(|x| + i\operatorname{sg}(x)) + \operatorname{Arg}(|y| + i\operatorname{sg}(y)) < \pi$:

$$\begin{aligned}
&\operatorname{Arg}(|x| + i\operatorname{sg}(x)) + \operatorname{Arg}(|y| + i\operatorname{sg}(y)) \\
=\ &\operatorname{Arg}((|x| + i\operatorname{sg}(x))(|y| + i\operatorname{sg}(y))) \\
=\ &\operatorname{Arg}(|xy| - \operatorname{sg}(x)\operatorname{sg}(y) + i(\operatorname{sg}(x)|y| + \operatorname{sg}(y)|x|))
\end{aligned} \tag{4.8}$$

The $\operatorname{addcot}(x, y)$ in theorem 3.2 for real x, y thus becomes:

$$\operatorname{addcot}(x, y) = \begin{cases} \pi & \text{if } \operatorname{Arg}(|xy| - \operatorname{sg}(x)\operatorname{sg}(y) + i(\operatorname{sg}(x)|y| + \operatorname{sg}(y)|x|)) > \pi/2 \\ -\pi & \text{if } \operatorname{Arg}(|xy| - \operatorname{sg}(x)\operatorname{sg}(y) + i(\operatorname{sg}(x)|y| + \operatorname{sg}(y)|x|)) \le -\pi/2 \\ 0 & \text{otherwise} \end{cases} \tag{4.9}$$

Because $|xy| = \operatorname{sg}(x)\operatorname{sg}(y)$ only occurs when $xy = 1$, in the complex plane this is easily seen to be equivalent to the theorem, with the fact that in this case $xy \neq 1$. Because $|xy| < \operatorname{sg}(x)\operatorname{sg}(y)$ can only occur when $\operatorname{sg}(x) = \operatorname{sg}(y)$, and because $\operatorname{sg}(x)|x| = x$, $\operatorname{sg}(x)|y| + \operatorname{sg}(y)|x|$ can be replaced by $x + y$. $\qquad\square$

Theorem 4.3. *For real x, y:*

$$\operatorname{arccot}(x) + \operatorname{arccot}(y) = \begin{cases} \frac{\pi}{2}\operatorname{sg}(x) & \text{if } y = 1/x \\ \arctan(\dfrac{x + y}{xy - 1}) + \pi[|xy| < \operatorname{sg}(x)\operatorname{sg}(y)]\operatorname{sg}(x + y) & \text{otherwise} \end{cases} \tag{4.10}$$

Proof. From (2.28) follows that for real x: $\operatorname{arccot}(x) = \arctan(1/x)$, so this theorem follows directly from the previous theorem, where the special cases $x = y = 0$ and $y = -x \neq 0$ give identical results. $\qquad\square$

When $x = y$ these addition formulas reduce to the following duplication formulas. For real x:

$$2\arctan(x) = \arctan(\frac{2x}{1 - x^2}) + \pi[x^2 > 1]\operatorname{sg}(x) \tag{4.11}$$

$$2\operatorname{arccot}(x) = \operatorname{arccot}(\frac{x^2 - 1}{2x}) + \pi[x^2 < 1]\operatorname{sg}(x) \tag{4.12}$$

$$2\operatorname{arccot}(x) = \arctan(\frac{2x}{x^2 - 1}) + \pi[x^2 < 1]\operatorname{sg}(x) \tag{4.13}$$

These formulas also follow from theorem 2.2 when taking $y = 0$.

Theorem 4.4. *For real x, y:*

$$\arctan(x) + \arctan(y) = 2\arctan(\frac{x + y}{1 - xy + \sqrt{(1 + x^2)(1 + y^2)}}) \tag{4.14}$$

Proof. From theorem 2.5, let:

$$F(x) = \frac{1 + ix + \sqrt{1 + x^2}}{1 - ix + \sqrt{1 + x^2}} \tag{4.15}$$

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Then:
$$\arctan(x) = -i\ln(F(x)) \tag{4.16}$$

From theorem 2.5, for real x:
$$F(x) = \sqrt{\frac{1+ix}{1-ix}} \tag{4.17}$$

Because for complex x: $-\pi/2 < \mathrm{Arg}(\sqrt{x}) \leq \pi/2$:
$$\arctan(x) + \arctan(y) = -i(\ln(F(x)) + \ln(F(y))) = -i\ln(F(x)F(y))$$
$$= -i\ln(\sqrt{\frac{1+ix}{1-ix}}\sqrt{\frac{1+iy}{1-iy}}) \tag{4.18}$$

From definition 2.14, for real z:
$$2\arctan(z) = -i\ln(\frac{1+iz}{1-iz}) \tag{4.19}$$

From theorem 2.1, the solution of:
$$\frac{1+iz}{1-iz} = t \tag{4.20}$$

is:
$$z = i\frac{1-t}{1+t} \tag{4.21}$$

Substituting for t the result of (4.18), and using (1.10) and $(1-ix)(1+ix) = 1+x^2$:

$$z = i\frac{1 - \sqrt{\frac{1+ix}{1-ix}}\sqrt{\frac{1+iy}{1-iy}}}{1 + \sqrt{\frac{1+ix}{1-ix}}\sqrt{\frac{1+iy}{1-iy}}} = i\frac{\sqrt{(1+x^2)(1+y^2)} - \sqrt{(1+ix)^2}\sqrt{(1+iy)^2}}{\sqrt{(1+x^2)(1+y^2)} + \sqrt{(1+ix)^2}\sqrt{(1+iy)^2}} \tag{4.22}$$

Because for real x, $\mathrm{sg}(1+ix) = 1$:

$$z = i\frac{\sqrt{(1+x^2)(1+y^2)} - (1+ix)(1+iy)}{\sqrt{(1+x^2)(1+y^2)} + (1+ix)(1+iy)} \tag{4.23}$$

Using (2.39):

$$z = \frac{2(x+y)\sqrt{(1+x^2)(1+y^2)}}{(\sqrt{(1+x^2)(1+y^2)} + 1 - xy)^2 + (x+y)^2}$$
$$= \frac{(x+y)\sqrt{(1+x^2)(1+y^2)}}{(1+x^2)(1+y^2) + (1-xy)\sqrt{(1+x^2)(1+y^2)}} \tag{4.24}$$
$$= \frac{x+y}{\sqrt{(1+x^2)(1+y^2)} + 1 - xy}$$

$\square$

Taking $x = 1/a$ and $y = 1/(4a^3 + 3a)$ and using $1 + (4a^3 + 3a)^2 = (1 + a^2)(1 + 4a^2)^2$, from this theorem the following known identity [6] results.
For real a:

$$\arctan(\frac{1}{a}) + \arctan(\frac{1}{4a^3 + 3a}) = 2\arctan(\frac{1}{2a}) \qquad (4.25)$$

Theorem 4.5. *For real x, y:*

$$\text{arccot}(x) + \text{arccot}(y) = \begin{cases} \pi & \text{if } y = x = 0 \\ 0 & \text{if } y = -x \neq 0 \\ 2\,\text{arccot}(\dfrac{xy - 1 + \text{sg}(x)\text{sg}(y)\sqrt{(1 + x^2)(1 + y^2)}}{x + y}) & \text{otherwise} \end{cases}$$

$$(4.26)$$

Proof. This theorem directly follows from the previous theorem because for real x: $\text{arccot}(x) = \arctan(1/x)$. $\qquad\square$

When $y = 0$ these addition formulas reduce to the following bisection formulas.
For real x:

$$\frac{1}{2}\arctan(x) = \arctan(\frac{x}{1 + \sqrt{1 + x^2}}) \qquad (4.27)$$

$$\frac{1}{2}\text{arccot}(x) = \text{arccot}(x + \text{sg}(x)\sqrt{1 + x^2}) \qquad (4.28)$$

For real $x \neq 0$:

$$\frac{1}{2}\text{arccot}(x) = \text{arccot}(\frac{-1 + \text{sg}(x)\sqrt{1 + x^2}}{x}) - \frac{\pi}{4} \qquad (4.29)$$

5 The Principal Values of the Inverse Tangent and Cotangent Functions on the Imaginary Axis

For determining the principal values of the $\arctan(x)$ and $\text{arccot}(x)$ functions on the imaginary axis, and thus confirming the new definitions (2.14) and (2.15), the addition theorems for complex arguments are used to express these values as a sum of two principal $\arctan(x)$ or $\text{arccot}(x)$ terms with arguments that are not on the imaginary axis. The following theorems state that this is possible.

Theorem 5.1. *Let t be on the imaginary axis and $t \neq \pm i$ and let x be real and positive, then from theorem 3.1 follows:*

$$\arctan(t) = \arctan(x) + \arctan(\frac{t - x}{1 + tx}) - \text{oddtan}(t) \qquad (5.1)$$

Proof. In theorem 3.1 the solution of:

$$\frac{x + y}{1 - xy} = t \qquad (5.2)$$

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is easily checked to be:

$$y = \frac{t - x}{1 + tx} \tag{5.3}$$

When t is on the imaginary axis then it can be replaced by $t = iz$ where z is real:

$$y = \frac{iz - x}{1 + izx} = \frac{(iz - x)(1 - izx)}{1 + z^2 x^2} = \frac{x(z^2 - 1) + iz(1 + x^2)}{1 + z^2 x^2} \tag{5.4}$$

Because x is real and positive, y is only on the imaginary axis when $z^2 = 1$, that is when $t = \pm i$ which are singular points and excluded in the theorem. Therefore y is never on the imaginary axis and $\operatorname{oddtan}(y) = 0$. Because x is real, also $\operatorname{oddtan}(x) = 0$, so in theorem 3.1 $\operatorname{oddtan}(x, y) = \operatorname{oddtan}(t)$. For evaluating $\operatorname{addtan}(x, y)$ in that theorem, by substituting (5.3):

$$\frac{1 + iy}{1 - iy} = \frac{1 + it}{1 - it} \cdot \frac{1 - ix}{1 + ix} \tag{5.5}$$

When t is on the imaginary axis the first factor in the right side of this equation is real and is called c, and because $t \neq \pm i$, $c \neq 0$. Then in $\operatorname{addtan}(x, y)$ in theorem 3.1, because x is real and positive, application of (1.20) and (1.21) yields:

$$\operatorname{Arg}(\frac{1 + ix}{1 - ix}) + \operatorname{Arg}(\frac{1 + iy}{1 - iy}) = \operatorname{Arg}(\frac{1 + ix}{1 - ix}) + \operatorname{Arg}(c\frac{1 - ix}{1 + ix}) = \begin{cases} 0 & \text{if } c > 0 \\ \pi & \text{if } c < 0 \end{cases} \tag{5.6}$$

This means that in theorem 3.1 $\operatorname{addtan}(x, y) = 0$ and this theorem follows. $\qquad \square$

Theorem 5.2. *Let t be on the imaginary axis and $t \neq \pm i$ and let x be real and positive, then from theorem 3.2 follows:*

$$\operatorname{arccot}(t) = \operatorname{arccot}(x) + \operatorname{arccot}(\frac{1 + tx}{x - t}) - \operatorname{oddcot}(t) \tag{5.7}$$

Proof. In theorem 3.1 the solution of:

$$\frac{xy - 1}{x + y} = t \tag{5.8}$$

is easily checked to be:

$$y = \frac{1 + tx}{x - t} \tag{5.9}$$

When t is on the imaginary axis then it can be replaced by $t = iz$ where z is real:

$$y = \frac{1 + izx}{x - iz} = \frac{(1 + izx)(x + iz)}{x^2 + z^2} = \frac{x(1 - z^2) + iz(1 + x^2)}{x^2 + z^2} \tag{5.10}$$

Because x is real and positive, y is only on the imaginary axis when $z^2 = 1$, that is when $t = \pm i$ which are singular points and excluded in the theorem. Therefore y is never on the imaginary axis and $\operatorname{oddcot}(y) = 0$. Because x is real, also $\operatorname{oddcot}(x) = 0$, so in theorem 3.2 $\operatorname{oddcot}(x, y) = \operatorname{oddcot}(t)$. For evaluating $\operatorname{addcot}(x, y)$ in that theorem, by substituting (5.9):

$$\frac{iy - 1}{iy + 1} = \frac{it - 1}{it + 1} \cdot \frac{ix + 1}{ix - 1} \tag{5.11}$$

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When t is on the imaginary axis the first factor in the right side of this equation is real and is called c, and because $t \neq \pm i$, $c \neq 0$. Then in addcot(x, y) in theorem 3.2, because x is real and positive and using (4.6), application of (1.20) and (1.21) yields:

$$\mathrm{Arg}(\frac{ix-1}{ix+1}) + \mathrm{Arg}(\frac{iy-1}{iy+1})$$

$$=\mathrm{Arg}(\frac{|x|+i\mathrm{sg}(x)}{|x|-i\mathrm{sg}(x)}) + \mathrm{Arg}(c\frac{|x|-i\mathrm{sg}(x)}{|x|+i\mathrm{sg}(x)}) = \begin{cases} 0 & \text{if } c > 0 \\ \pi & \text{if } c < 0 \end{cases} \tag{5.12}$$

This means that in theorem 3.2 addcot$(x, y) = 0$ and this theorem follows. $\qquad \square$

The two tables below are the computation of principal values of $\arctan(t)$ and $\mathrm{arccot}(t)$ on four points of the imaginary axis, where the principal $\arctan(y)$ and $\mathrm{arccot}(y)$ values can be computed with a computer algebra program. With these two tables it can be checked that for these values on the imaginary axis $\arctan(t)$ and $\mathrm{arccot}(t)$ are odd and that theorem 2.3 is valid. These values are also in agreement with definitions (2.14) and (2.15), which are therefore now confirmed with the addition formulas.

Table 1: Evaluation of $\arctan(t)$ with theorem 5.1

t	$2i$	$\frac{1}{2}i$	$-\frac{1}{2}i$	$-2i$
x	1	1	1	1
$y = \frac{t-x}{1+tx}$	$\frac{3}{5} + \frac{4}{5}i$	$-\frac{3}{5} + \frac{4}{5}i$	$-\frac{3}{5} - \frac{4}{5}i$	$\frac{3}{5} - \frac{4}{5}i$
$\frac{1+ix}{1-ix}$	i	i	i	i
$\frac{1+iy}{1-iy}$	$\frac{1}{3}i$	$-\frac{1}{3}i$	$-3i$	$3i$
$\arctan(x)$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$
$\arctan(y)$	$\frac{\pi}{4} + \frac{1}{2}i\ln(3)$	$-\frac{\pi}{4} + \frac{1}{2}i\ln(3)$	$-\frac{\pi}{4} - \frac{i}{2}\ln(3)$	$\frac{\pi}{4} - \frac{i}{2}\ln(3)$
$-$oddtan(t)	0	0	0	$-\pi$
$\arctan(t)$	$\frac{\pi}{2} + \frac{1}{2}i\ln(3)$	$\frac{1}{2}i\ln(3)$	$-\frac{1}{2}i\ln(3)$	$-\frac{\pi}{2} - \frac{1}{2}i\ln(3)$

Table 2: Evaluation of $\mathrm{arccot}(t)$ with theorem 5.2

t	$2i$	$\frac{1}{2}i$	$-\frac{1}{2}i$	$-2i$
x	1	1	1	1
$y = \frac{1+tx}{x-t}$	$-\frac{3}{5}+\frac{4}{5}i$	$\frac{3}{5}+\frac{4}{5}i$	$\frac{3}{5}-\frac{4}{5}i$	$-\frac{3}{5}-\frac{4}{5}i$
$\frac{ix-1}{ix+1}$	i	i	i	i
$\frac{iy-1}{iy+1}$	$-3i$	$3i$	$\frac{1}{3}i$	$-\frac{1}{3}i$
$\mathrm{arccot}(x)$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$
$\mathrm{arccot}(y)$	$-\frac{\pi}{4}-\frac{1}{2}i\ln(3)$	$\frac{\pi}{4}-\frac{1}{2}i\ln(3)$	$\frac{\pi}{4}+\frac{i}{2}\ln(3)$	$-\frac{\pi}{4}+\frac{i}{2}\ln(3)$
$-\mathrm{oddcot}(t)$	0	0	$-\pi$	0
$\mathrm{arccot}(t)$	$-\frac{1}{2}i\ln(3)$	$\frac{\pi}{2}-\frac{1}{2}i\ln(3)$	$-\frac{\pi}{2}+\frac{1}{2}i\ln(3)$	$\frac{1}{2}i\ln(3)$

6 The Inverse Hyperbolic Tangent and Cotangent Functions and their Addition Formulas

The principal $\mathrm{arctanh}(x)$ and $\mathrm{arccoth}(x)$ functions for complex x are defined by:

$$\mathrm{arctanh}(x) = -i\,\arctan(ix) \tag{6.1}$$

$$\mathrm{arccoth}(x) = i\,\mathrm{arccot}(ix) \tag{6.2}$$

With these definitions from (2.14) and (2.15) follows:
For complex x:

$$\mathrm{arctanh}(x) = -\frac{1}{2}\ln\left(\frac{1-x}{1+x}\right) + \pi i[\mathrm{Im}(x)=0][\mathrm{Re}(x)<-1] \tag{6.3}$$

$$\mathrm{arccoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) - \pi i[\mathrm{Im}(x)=0][-1<\mathrm{Re}(x)<0] \tag{6.4}$$

Theorem 6.1. *For complex x:*

$$\mathrm{arctanh}(x) = \ln\left(\frac{1+x+\sqrt{1-x^2}}{1-x+\sqrt{1-x^2}}\right) \tag{6.5}$$

Proof. This theorem directly follows from theorem 2.5 and definition (6.1). $\qquad\square$

Theorem 6.2. *For real x, y:*

$$\begin{aligned}\mathrm{arctanh}(x+iy) =& \frac{1}{4}\ln\left(\frac{y^2+(x+1)^2}{y^2+(x-1)^2}\right) + \frac{i}{2}\arctan\left(\frac{2y}{1-x^2-y^2}\right)\\ &-\frac{\pi}{2}i[x^2+y^2>1]\mathrm{sg}(-y) + \pi i[y=0][x<-1]\end{aligned} \tag{6.6}$$

$$\begin{aligned}\mathrm{arccoth}(x+iy) =& \frac{1}{4}\ln\left(\frac{y^2+(x+1)^2}{y^2+(x-1)^2}\right) - \frac{i}{2}\arctan\left(\frac{2y}{x^2+y^2-1}\right)\\ &+\frac{\pi}{2}i[x^2+y^2<1]\mathrm{sg}(-y) - \pi i[y=0][-1<x<0]\end{aligned} \tag{6.7}$$

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Proof. These two identities follow from theorem 2.2 with definitions (6.1) and (6.2) by replacing $x + iy$ with $i(x + iy) = -y + ix$, that is replacing x with $-y$ and y with x.

$\square$

When $x = \pm 1$ and $y = 0$ these formulas yield $\operatorname{arctanh}(1) = \infty$, $\operatorname{arctanh}(-1) = -\infty$, $\operatorname{arccoth}(1) = \infty$ and $\operatorname{arccoth}(-1) = -\infty$.

For real x:

$$\operatorname{arctanh}(x) = \frac{1}{2}\ln\left(\left|\frac{1+x}{1-x}\right|\right) - \frac{\pi}{2}i([x > 1] - [x < -1]) \tag{6.8}$$

$$\operatorname{arccoth}(x) = \frac{1}{2}\ln\left(\left|\frac{x+1}{x-1}\right|\right) + \frac{\pi}{2}i([0 \le x < 1] - [-1 < x < 0]) \tag{6.9}$$

For complex $x \neq \pm 1$:

$$\operatorname{arccoth}(x) - \operatorname{arctanh}(x) = \frac{\pi}{2}i\operatorname{sg}(ix) \tag{6.10}$$

For complex x:

$$\operatorname{arctanh}(\frac{1}{x}) = \operatorname{arccoth}(x) + \pi i[\operatorname{Im}(x) = 0]([-1 < \operatorname{Re}(x) < 0] - [0 < \operatorname{Re}(x) < 1]) \tag{6.11}$$

$$\operatorname{arccoth}(\frac{1}{x}) = \operatorname{arctanh}(x) - \pi i[\operatorname{Im}(x) = 0]([\operatorname{Re}(x) < -1] - [\operatorname{Re}(x) > 1]) \tag{6.12}$$

For complex $x \neq \pm 1$:

$$\begin{aligned}
&\operatorname{arctanh}(x) - \operatorname{arctanh}(\frac{1}{x}) \\
&= -\frac{\pi}{2}i\operatorname{sg}(ix) - \pi i[\operatorname{Im}(x) = 0]([-1 < \operatorname{Re}(x) < 0] - [0 < \operatorname{Re}(x) < 1])
\end{aligned} \tag{6.13}$$

$$\operatorname{arccoth}(x) - \operatorname{arccoth}(\frac{1}{x}) = \frac{\pi}{2}i\operatorname{sg}(ix) + \pi i[\operatorname{Im}(x) = 0]([\operatorname{Re}(x) < -1] - [\operatorname{Re}(x) > 1]) \tag{6.14}$$

The addition formulas for these functions for complex x and y can be taken from the addition formulas for complex x and y above by substituting ix for x and iy for y. The addition formulas for real x and y are given, using for real x: $\operatorname{sg}(ix) = \operatorname{sg}(x)$.

Theorem 6.3. *For real $x \neq \pm 1$ and $y \neq \pm 1$:*

$$\begin{aligned}
&\operatorname{arctanh}(x) + \operatorname{arctanh}(y) \\
&= \begin{cases} -\frac{\pi}{2}i\operatorname{sg}(x) - \pi i([-1 < x < 0] - [0 < x < 1]) & \text{if } y = -1/x \\ \operatorname{arctanh}(\frac{x+y}{1+xy}) - \pi i(\operatorname{addtanh}(x,y) + \operatorname{oddtanh}(x,y)) & \text{otherwise} \end{cases}
\end{aligned} \tag{6.15}$$

where:

$$\operatorname{addtanh}(x,y) = ([x < -1] + [x > 1])([y < -1] + [y > 1]) \tag{6.16}$$

$$\operatorname{oddtanh}(x,y) = [\frac{x+y}{1+xy} < -1] - [x < -1] - [y < -1] \tag{6.17}$$

Proof. Application of theorem 3.1 by substituting definition (6.1) gives this theorem, where in addtan(x, y) the sum of Arg(x) functions becomes:

$$\text{Arg}(\frac{1-x}{1+x}) + \text{Arg}(\frac{1-y}{1+y}) = \begin{cases} 0 & \text{if } -1 < x < 1 \text{ and } -1 < y < 1 \\ \pi & \text{if one of } -1 < x < 1 \text{ or } -1 < y < 1 \\ 2\pi & \text{if } (x < -1 \text{ or } x > 1) \text{ and } (y < -1 \text{ or } y > 1) \end{cases} \tag{6.18}$$

Only in the last case is addtan(x, y) π and otherwise zero, which is equivalent to addtanh(x, y). □

Theorem 6.4. *For real $x \neq \pm 1$ and $y \neq \pm 1$:*

$$\text{arccoth}(x) + \text{arccoth}(y)$$

$$= \begin{cases} \pi i & \text{if } y = x = 0 \\ 0 & \text{if } y = -x \neq 0 \\ \frac{\pi}{2}i\,\text{sg}(x) + \pi i([x < -1] - [x > 1]) & \text{if } y = -1/x \\ \text{arccoth}(\frac{1+xy}{x+y}) + \pi i(\text{addcoth}(x, y) + \text{oddcoth}(x, y)) & \text{otherwise} \end{cases} \tag{6.19}$$

where:
$$\text{addcoth}(x, y) = [-1 < x < 1][-1 < y < 1] \tag{6.20}$$

$$\text{oddcoth}(x, y) = [-1 < \frac{1+xy}{x+y} < 0] - [-1 < x < 0] - [-1 < y < 0] \tag{6.21}$$

Proof. Application of theorem 3.2 by substituting definition (6.2) gives this theorem, where in addtan(x, y) the sum of Arg(x) functions becomes:

$$\text{Arg}(\frac{x+1}{x-1}) + \text{Arg}(\frac{y+1}{y-1}) = \begin{cases} 0 & \text{if } (x < -1 \text{ or } x > 1) \text{ and } (y < -1 \text{ or } y > 1) \\ \pi & \text{if one of } -1 < x < 1 \text{ or } -1 < y < 1 \\ 2\pi & \text{if } -1 < x < 1 \text{ and } -1 < y < 1 \end{cases} \tag{6.22}$$

Only in the last case is addcot(x, y) π and otherwise zero, which is equivalent to addcoth(x, y). □

Theorem 6.5. *For real $x \neq \pm 1$ and $y \neq \pm 1$:*

$$\text{arccoth}(x) + \text{arccoth}(y) = \begin{cases} \frac{\pi}{2}i\,\text{sg}(x) + \pi i([x < -1] - [x > 1]) & \text{if } y = -1/x \\ \text{arctanh}(\frac{x+y}{1+xy}) + \pi i\,\text{addcoth}(x, y) & \text{otherwise} \end{cases} \tag{6.23}$$

where:

$$\text{addcoth}(x, y) = [\frac{x+y}{1+xy} > 1] + [-1 < x < 1][-1 < y < 1]$$
$$- [-1 < x < 0] - [-1 < y < 0] \tag{6.24}$$

Proof. Using (6.11) and the identity:

$$[0 < \alpha < 1] = [\frac{1}{\alpha} > 1] \tag{6.25}$$

this theorem directly follows from the previous theorem, where the special cases $x = y = 0$ and $y = -x \neq 0$ give identical results. □

When $x = y$ these addition formulas reduce to the following duplication formulas, using $|2x/(1+x^2)| \leq 1$.
For real x:

$$2\operatorname{arctanh}(x) = \operatorname{arctanh}(\frac{2x}{1+x^2}) + \pi i([x < -1] - [x > 1]) \tag{6.26}$$

$$2\operatorname{arccoth}(x) = \operatorname{arccoth}(\frac{1+x^2}{2x}) + \pi i([0 \leq x < 1] - [-1 < x < 0]) \tag{6.27}$$

$$2\operatorname{arccoth}(x) = \operatorname{arctanh}(\frac{2x}{1+x^2}) + \pi i([0 \leq x < 1] - [-1 < x < 0]) \tag{6.28}$$

7 Conclusion

When the principal $\arctan(x)$ and the $\operatorname{arccot}(x)$ functions are defined by (2.14) and (2.15), and the principal $\operatorname{arctanh}(x)$ and $\operatorname{arccoth}(x)$ functions by (6.1) and (6.2), then these functions are related by (2.23) and (6.10) with $\operatorname{sg}(x)$ defined by (1.4).
For complex x:

$$\operatorname{arccot}(x) = \frac{\pi}{2}\operatorname{sg}(x) - \arctan(x) \tag{7.1}$$

$$\operatorname{arccoth}(x) = \frac{\pi}{2}i\operatorname{sg}(ix) + \operatorname{arctanh}(x) \tag{7.2}$$

and not $\operatorname{arccot}(x) = \arctan(1/x)$ and $\operatorname{arccoth}(x) = \operatorname{arctanh}(1/x)$, which are then replaced by (2.28), (2.29), (6.11) and (6.12). This way the functions are odd everywhere in the complex plane (except at $x = 0$ for the $\operatorname{arccot}(x)$ and $\operatorname{arccoth}(x)$ functions) and consistent with the addition formulas as mentioned in section 5. These formulas also have the advantage that no inversion of the arguments is required. For implementation of these functions theorems 2.2 and 6.2 with (1.5) and (1.16) may be used.
The corresponding Mathematica® [10] program:

```
Sg[x_]:=If[Re[x]>0,1,If[Re[x]<0,-1,If[Im[x]>=0,1,-1]]]
```

References

[1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, 1972.

[2] R.V. Churchill, J.W. Brown, *Complex Variables and Applications*, McGraw-Hill, 1984.

[3] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics, A Foundation for Computer Science*, 2nd ed., Addison-Wesley, 1994.

[4] M.J. Kronenburg, Higher Derivatives of the Tangent and Inverse Tangent Functions and Chebyshev Polynomials, `arXiv:2010.09862 [math.GM]`

[5] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.

[6] E.W. Weisstein, *Inverse Tangent.* From Mathworld - A Wolfram Web Resource. `https://mathworld.wolfram.com/InverseTangent.html`

[7] Wikipedia, *Iverson bracket*, `https://en.wikipedia.org/wiki/Iverson_bracket`

[8] Wikipedia, *Sign Function*, `https://en.wikipedia.org/wiki/Sign_function`

[9] Wikipedia, *Square root*, `https://en.wikipedia.org/wiki/Square_root`

[10] S. Wolfram, *The Mathematica Book*, 5th ed., Wolfram Media, 2003.