The Octets of the Natural Numbers

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Abstract. “Every natural number, with the exception of 0 and 1, can be written in a unique way as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1” From this Theorem, four fundamental properties of odd numbers are implied: the conjugate of an odd number, the L/R symmetry, the transpose of an odd number and the octets of odd numbers. These concepts are used to obtain a classification of odd numbers and an algorithm for finding the factors of composite Fermat numbers.

1. Introduction

In this article, we start by proving the Theorem: “Let Π be an odd number other than 1, and $\left[\frac{\ln \Pi}{\ln 2}\right]$ is the integer part of $\frac{\ln \Pi}{\ln 2} \in \mathbb{R}$. Then Π can be uniquely written in the form $\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i$, $\nu \in \mathbb{N}$, $\nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right]$, $\beta_i = \pm 1, i = 0,1,2,...,\nu-1$” (Theorem 2.1). It’s easy to show that a Theorem of this kind doesn’t hold for any other natural number different than 2.

The main difference between our Theorem and the known arithmetic systems (binary, decimal etc.) is that the coefficients of the linear combination can take the negative value. These negative values highlight two properties of odd numbers, the “conjugate” of an odd number (Definition 2.1) and the “L/R symmetry” (Definition 3.1).

Another property of odd numbers is that of the “transpose” of an odd number (Definition 4.1). The transpose of an odd number can be defined in any arithmetic system and its value depends on the system used. However, the main mathematical object to which we come to, “the odd number octet”, is defined from a combination of conjugates and transposes of odd numbers (equation (5.1)). Thus, the odd number octet emerges only by using Theorem 2.1. Using these notions, we obtain a categorization of odd numbers and an algorithm for finding the factors of composite Fermat numbers.

2. Odd numbers as linear combinations of consecutive powers of 2

We prove the following Theorem:

Theorem 2.1. *Let Π be an odd number other than 1, and $\left[\frac{\ln \Pi}{\ln 2}\right]$ is the integer part of $\frac{\ln \Pi}{\ln 2} \in \mathbb{R}$. Then Π can be uniquely written in the form*

$$\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i , \tag{2.1}$$

$\nu \in \mathbb{N}$, $\nu + 1 = \left[\dfrac{\ln \Pi}{\ln 2} \right]$, $\beta_i = \pm 1, i = 0,1,2,...,\nu - 1$.

Proof. For $\Pi = 3$ we have $\nu + 1 = \left[\dfrac{\ln 3}{\ln 2} \right] \Rightarrow \nu = 0$ and from equation (2.1) we obtain

$\Pi = 3 = 2^1 + 2^0$.

We now examine the case where $\nu \in \mathbb{N}^* = \{1,2,3,...\}$. The lowest value that the odd number Π of equation (2.1) can obtain is

$$\Pi_{\min} = \Pi(\nu) = 2^{\nu+1} + 2^{\nu} - 2^{\nu-1} - 2^{\nu-2} - ... - 2^1 - 1 = 2^{\nu+1} + 1. \tag{2.2}$$

The largest value that the odd number Π of equation (2.1) can obtain is

$$\Pi_{\max} = \Pi(\nu) = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + 2^{\nu-2} + ... + 2^1 + 1 = 2^{\nu+2} - 1. \tag{2.3}$$

Thus, for the odd numbers $\Pi = \Pi(\nu, \beta_i)$ of equation (2.1) the following inequality holds

$$\Pi_{\min} = 2^{\nu+1} + 1 \le \Pi(\nu, \beta_i) \le 2^{\nu+2} - 1 = \Pi_{\max} . \tag{2.4}$$

The number $N(\Pi(\nu, \beta_i))$ of odd numbers in the closed interval $\left[2^{\nu+1} + 1, 2^{\nu+2} - 1\right]$ is

$$N(\Pi(\nu, \beta_i)) = \frac{\Pi_{\max} - \Pi_{\min}}{2} + 1 = \frac{(2^{\nu+2} - 1) - (2^{\nu+1} + 1)}{2} + 1 = 2^{\nu} . \tag{2.5}$$

The integers $\beta_i, i = 0,1,2,...,\nu - 1$ in equation (2.1) can only take two values, $\beta_i = \pm 1$, thus equation (2.1) gives exactly $2^{\nu} = N(\Pi(\nu, \beta_i))$ odd numbers. Therefore, for every $\nu \in \mathbb{N}^*$ equation (2.1) gives all odd numbers in the interval $\Omega_{\nu} = \left[2^{\nu+1}, 2^{\nu+2}\right]$.

From inequality (2.4) we obtain

$$2^{\nu+1} + 1 \le \Pi \le 2^{\nu+2} - 1$$

$$2^{\nu+1} < 2^{\nu+1} + 1 \le \Pi \le 2^{\nu+2} - 1 < 2^{\nu+2}$$

$$2^{\nu+1} < \Pi < 2^{\nu+2}$$

$$(\nu + 1)\ln 2 < \ln \Pi < (\nu + 2)\ln 2$$

from which we get

$$\frac{\ln \Pi}{\ln 2} - 1 < \nu + 1 < \frac{\ln \Pi}{\ln 2}$$

and finally

$$\nu+1=\left[\frac{\ln\Pi}{\ln 2}\right]. \tag{2.6}$$

For $\Pi=1$ we have $\nu+1=\left[\frac{\ln 1}{\ln 2}\right]\Rightarrow\nu=-1\notin\mathbb{N}$.

We prove now that every odd number $\Pi\neq 1$ can be uniquely written in the form of equation (2.1). We write the odd Π as

$$\begin{aligned}&\Pi=2^{\nu+1}+2^{\nu}+\sum_{i=0}^{\nu-1}\beta_i 2^i\\&\beta_i=\pm 1, i=0,1,2,\ldots,\nu-1\\&\nu+1=\left[\frac{\ln\Pi}{\ln 2}\right]\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}&\Pi=2^{\nu+1}+2^{\nu}+\sum_{i=0}^{\nu-1}\gamma_i 2^i\\&\gamma_i=\pm 1, i=0,1,2,\ldots,\nu-1.\\&\nu+1=\left[\frac{\ln\Pi}{\ln 2}\right]\end{aligned} \tag{2.8}$$

From equations (2.7), (2.8) we get

$$\begin{aligned}&(\beta_0-\gamma_0)\cdot 2^0+(\beta_1-\gamma_1)\cdot 2^1+(\beta_2-\gamma_2)\cdot 2^2+\ldots+(\beta_{\nu-1}-\gamma_{\nu-1})\cdot 2^{\nu-1}=0\\&\beta_i=\pm 1, i=0,1,2,\ldots,\nu-1\\&\gamma_i=\pm 1, i=0,1,2,\ldots,\nu-1\\&i\in\{0,1,2,\ldots,\nu-1\}\end{aligned}. \tag{2.9}$$

If, in equation (2.9), there are $i\in\{0,1,2,\ldots,\nu-1\}$ such that $\beta_i\neq\gamma_i$ and let k be the smallest of them, dividing by 2^{k+1} we get an odd number equal to an even number. So, it follows that $\beta_i=\gamma_i\,\forall i=0,1,2,\ldots,\nu-1$. □

In order to write an odd number $\Pi\neq 1,3$ in the form of equation (2.1) we initially define the $\nu\in\mathbb{N}^*$ from equation (2.6). Then, we calculate the sum

$$2^{\nu+1}+2^{\nu}.$$

If $2^{\nu+1}+2^{\nu}<\Pi$ we add $2^{\nu-1}$, whereas if $2^{\nu+1}+2^{\nu}>\Pi$ then we subtract it. By repeating the process exactly ν times we write the odd number Π in the form of equation (2.1). The number ν of steps needed in order to write the odd number Π in the form of equation (2.1) is extremely low compared to the magnitude of the odd number Π, as derived from inequality (2.4).

Example 2.1. For the odd number $\Pi=23$ we obtain from equation (2.6)

$$\nu+1=\left[\frac{\ln 23}{\ln 2}\right]\Rightarrow \nu=3.$$

Then, we have

$2^{\nu+1}+2^{\nu}=2^4+2^3=24>23$ (thus 2^2 is subtracted)

$2^4+2^3-2^2=20<23$ (thus 2^1 is added)

$2^4+2^3-2^2+2^1=22<23$ (thus $2^0=1$ is added)

$2^4+2^3-2^2+2^1+1=23$.

Fermat numbers F_s can be written directly in the form of equation (2.1), since they are of the form $\Pi_{\min}$,

$$\begin{aligned}&F_s=2^{2^s}+1=\Pi_{\min}\left(2^s-1\right)=2^{2^s}+2^{2^s-1}-2^{2^s-2}-2^{2^s-3}-...-2^1-1\\&s\in\mathbb{N}\end{aligned}. \qquad (2.10)$$

Mersenne numbers M_p can be written directly in the form of equation (2.1), since they are of the form $\Pi_{\max}$,

$$\begin{aligned}&M_p=2^p-1=\Pi_{\max}\left(p-2\right)=2^{p-1}+2^{p-2}+2^{p-3}+...+2^1+1\\&p=prime\end{aligned}. \qquad (2.11)$$

We now give the following Definition:

Definition 2.1. *Let* Π *be an odd number greater than* 1, *and consider the representation of* Π *as described in Theorem 2.1. Then the conjugate* Π^* *of* Π *is*

$$\begin{aligned}&\Pi^*=\Pi^*\left(\nu,\gamma_j\right)=2^{\nu+1}+2^{\nu}+\sum_{j=0}^{\nu-1}\gamma_j 2^j\\&\gamma_i=\pm1,\ j=0,1,2,...,\nu-1\\&\nu\in\mathbb{N}^*\end{aligned} \qquad (2.12)$$

for which it is

$$\gamma_k=-\beta_k\forall k=0,1,2,...,\nu-1. \qquad (2.13)$$

For conjugate odds, the following Corollary holds:

Corollary 2.1. *For the conjugate odds* $\Pi=\Pi\left(\nu,\beta_i\right)\geq 3$ *and* $\Pi^*=\Pi^*\left(\nu,\gamma_i\right)$ *the following hold:*

1. $\left(\Pi^*\right)^*=\Pi$. (2.14)

2. $\Pi^*=3\cdot 2^{\nu+1}-\Pi$. (2.15)

3. Π *is divisible by* 3 *if and only if* Π^* *is divisible by* 3.

4. *Two conjugate odd numbers cannot have common factors greater than 3.*

5. *Conjugates* Π *and* Π^* *are equidistant from the middle* $3\cdot 2^{\nu}$ *of the interval 2* $\Omega_{\nu} = \left[2^{\nu+1}, 2^{\nu+2}\right]$.

Proof. 1. The 1 of the Corollary is an immediate consequence of Definition 4.1.

2. From equations (2.1), (2.1) and (2.12) we get

$$\Pi + \Pi^* = \left(2^{\nu+1} + 2^{\nu}\right) + \left(2^{\nu+1} + 2^{\nu}\right)$$

and equivalently

$$\Pi + \Pi^* = 3\cdot 2^{\nu+1}.$$

3. If the odd Π is divisible by 3 then it is written in the form $\Pi = 3x, x = odd$ and from equation (2.15) we get $3x + \Pi^* = 3\cdot 2^{\nu+1}$ and equivalently $\Pi^* = 3\left(2^{\nu+1} - x\right)$. Similarly we can prove the inverse.

4. If $\Pi = xy, \Pi^* = xz$, x, y, z odd numbers, from equation (2.15) we have $x\left(y+z\right) = 3\cdot 2^{\nu+1}$ and consequently $x = 3$.

5. From equation (2.15) we obtain

$$\Pi - 3\cdot 2^{\nu} = 3\cdot 2^{\nu} - \Pi^*$$

$$\left|\Pi - 3\cdot 2^{\nu}\right| = \left|3\cdot 2^{\nu} - \Pi^*\right|. \ \square$$

From Corollary 2.1 we have that 3 is the only odd number which is equal to its conjugate; $3^* = 3\cdot 2^{0+1} - 3 = 3$. For the $\Pi = 1$ we define

$$1^* = 1. \qquad (2.16)$$

Also, from equation

$$\left(\Pi + X\right) + \left(\Pi^* - X\right) = 3\cdot 2^{\nu+1} \qquad (2.17)$$

it follows that, if the odds $\Pi + X$ and $\Pi - X$, $X = even$ belong to the interval $\Omega_{\nu} = \left[2^{\nu+1}, 2^{\nu+2}\right]$, then they are conjugates

$$\left(\Pi + X\right)^* = \Pi^* - X\ . \qquad (2.18)$$

It is easily proven that Theorem (2.1) is also valid for even numbers that are not powers of 2. In order to write an even number E that is not a power of 2 in the form of equation (2.1), initially it is consecutively divided by 2 and it takes of the form of equation

$$\begin{aligned} &E = 2^{l}\cdot\Pi \\ &\Pi = odd, \Pi \neq 1, l \in \mathbb{N}^*\end{aligned}. \qquad (2.19)$$

Then, we write the odd number Π in the form of equation (2.1).

Example 2.2. By consecutively dividing the even number 368 by 2 we obtain

$E = 368 = 2^4 \cdot 23$.

Then, we write the odd number Π =23 in the form of equation (2.1),

$23 = 2^4 + 2^3 - 2^2 + 2^1 + 1$

and we get

$368 = 2^4 \cdot \left(2^4 + 2^3 - 2^2 + 2^1 + 1\right) = 2^8 + 2^7 - 2^6 + 2^5 + 2^4$.

This equation gives the unique way in which the even number 368 can be written in the form of equation (2.1). For even numbers the lowest power of two in equation (2.1) is different from $1 = 2^0$.

3. The L/R symmetry

We now give the following Definition:

Definition 3.1.1. *The odd number* Π *has Left-symmetry* L *when there exists an index* L *such that*

$$\begin{aligned} &\beta_L = +1 \\ &\beta_{L-1} = \beta_{L-2} = \ldots = \beta_1 = \beta_0 = -1. \\ &L \in \{1, 2, 3, \ldots, \nu - 1\} \end{aligned} \tag{3.1}$$

2. *The odd number* Π *has Right-symmetry* R *when there exists an index* R *such that*

$$\begin{aligned} &\beta_R = -1 \\ &\beta_{R-1} = \beta_{R-2} = \ldots = \beta_1 = \beta_0 = +1. \\ &R \in \{1, 2, 3, \ldots, \nu - 1\} \end{aligned} \tag{3.2}$$

Next, we have one example:

Example 3.1. The prime number Q=568630647535356955169033410940867804839360742060818433 is a factor of $F_{12} = 2^{4096} + 1$. From the equation (2.6) we have $\nu + 1 = 178$, and then from equation 2.1 we have

$$Q = 2^{178} + 2^{177} - 2^{176} + 2^{175} + 2^{174} + 2^{173} + 2^{172} - 2^{171} + 2^{170} + 2^{169} + 2^{168} + 2^{167} + 2^{166}$$
$$+2^{165} - 2^{164} + 2^{163} - 2^{162} - 2^{161} - 2^{160} - 2^{159} + 2^{158} + 2^{157} + 2^{156} - 2^{155} - 2^{154} - 2^{153} - 2^{152}$$
$$-2^{151} + 2^{150} - 2^{149} + 2^{148} - 2^{147} - 2^{146} + 2^{145} - 2^{144} + 2^{143} - 2^{142} - 2^{141} - 2^{140} + 2^{139} + 2^{138}$$
$$-2^{137} - 2^{136} + 2^{135} - 2^{134} - 2^{133} + 2^{132} - 2^{131} + 2^{130} - 2^{129} + 2^{128} - 2^{127} + 2^{126} - 2^{125} - 2^{124}$$
$$-2^{123} - 2^{122} - 2^{121} + 2^{120} - 2^{119} + 2^{118} - 2^{117} + 2^{116} - 2^{115} + 2^{114} - 2^{113} - 2^{112} - 2^{111} - 2^{110}$$
$$-2^{109} - 2^{108} + 2^{107} - 2^{106} + 2^{105} - 2^{104} + 2^{103} - 2^{102} + 2^{101} - 2^{100} + 2^{99} + 2^{98} - 2^{97} + 2^{96} - 2^{95}$$
$$-2^{94} + 2^{93} - 2^{92} + 2^{91} + 2^{90} - 2^{89} + 2^{88} - 2^{87} + 2^{86} + 2^{85} + 2^{84} - 2^{83} + 2^{82} - 2^{81} + 2^{80} + 2^{79}$$
$$-2^{78} - 2^{77} - 2^{76} - 2^{75} + 2^{74} + 2^{73} - 2^{72} - 2^{71} - 2^{70} + 2^{69} + 2^{68} + 2^{67} + 2^{66} + 2^{65} + 2^{64} - 2^{63}$$
$$-2^{62} + 2^{61} - 2^{60} - 2^{59} - 2^{58} - 2^{57} - 2^{56} + 2^{55} - 2^{54} - 2^{53} - 2^{52} - 2^{51} - 2^{50} - 2^{49} + 2^{48} + 2^{47}$$
$$-2^{46} + 2^{45} + 2^{44} + 2^{43} + 2^{42} - 2^{41} - 2^{40} + 2^{39} - 2^{38} - 2^{37} - 2^{36} + 2^{35} - 2^{34} - 2^{33} + 2^{32} + 2^{31}$$
$$-2^{30} + 2^{29} + 2^{28} + 2^{27} + 2^{26} + 2^{25} + 2^{24} - 2^{23} + 2^{22} + 2^{21} + 2^{20} - 2^{19} - 2^{18} - 2^{17} - 2^{16} + 2^{15}$$
$$+2^{14} - 2^{13} - 2^{12} - 2^{11} - 2^{10} - 2^{9} - 2^{8} - 2^{7} - 2^{6} - 2^{5} - 2^{4} - 2^{3} - 2^{2} - 2^{1} - 1$$

So the factor 568630647535356955169033410940867804839360742060818433 of F_{12} has symmetry L(568630647535356955169033410940867804839360742060818433)=14.

As a consequence of Definition 3.1, for odd numbers of the form Q is $\beta_0 = -1$ and for odd numbers of the form D is $\beta_0 = +1$. Also, from equation (2.1) and Definitions 3.1 it can be easily proved that the Q odds are written in the form

$$\begin{aligned} &Q = 2^{L+1} \cdot K + 1 \\ &K = odd \end{aligned} \tag{3.3}$$

and the D odds in the form

$$\begin{aligned} &D = 2^{R+1} \cdot K - 1 \\ &K = odd \end{aligned}. \tag{3.4}$$

Proof. We prove equation (3.3) and (3.4) is proven similarly. From Definition 3.1 we get

$$Q = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + ... + \beta_{L+1}2^{L+1} + 2^{L} - 2^{L-1} - 2^{L-2} - ... - 2^{2} - 2^{1} - 1$$
$$\nu + 1 = \left[\frac{\ln Q}{\ln 2}\right]$$

$$Q = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + ... + \beta_{L+1}2^{L+1} + 2^{L} - \left(2^{L-1} + 2^{L-2} + ... + 2^{2} + 2^{1} + 1\right)$$

$$Q = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + ... + \beta_{L+1}2^{L+1} + 2^{L} - \left(2^{L} - 1\right)$$

$$Q = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + ... + \beta_{L+1}2^{L+1} + 1$$

$$Q = 2^{L+1}\left(2^{\nu-L} + 2^{\nu-L-1} + \beta_{\nu-1}2^{\nu-L-2} + \beta_{\nu-2}2^{\nu-L-3} ... + \beta_{L+1}\right) + 1$$

$$Q = 2^{L+1}K + 1$$

$K = 2^{\nu-L} + 2^{\nu-L-1} + \beta_{\nu-1}2^{\nu-L-2} + \beta_{\nu-2}2^{\nu-L-3} ... + \beta_{L+1}$. □

It follows from Lucas Theorem for the Fermat numbers (see, [1,2]) that the factors of the Fermat numbers are of the form (3.3). Equations (3.3), (3.4) provides the simplest way for the determination of the symmetry of a number. An odd number with positive L-symmetry would necessarily have 0 R-symmetry and vice versa. We give two examples.

Example 3.2. For odd number 18303 we have

$$\begin{aligned} 18303 - 1 &= 2^1 \times 9151 \\ 18303 + 1 &= 2^7 \times 143 \end{aligned}.$$

Therefore, $L(18303) = 0$ and $R(18303) = 7 - 1 = 6$. Indeed, from equation (2.6) we get $\nu = 13$ and from equation (2.1) we obtain

$$18303 = 2^{14} + 2^{13} - 2^{12} - 2^{11} - 2^{10} + 2^9 + 2^8 + 2^7 - 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 1.$$

Example 3.3. For the number C1133 which is composite factor of F_{12} with 1133 digits, we have

$$C1133 - 1 = 2^{14} \cdot K.$$

Therefore, $L(C1133) = 14 - 1 = 13$.

It is easy to prove the following Corollary:

Corollary 3.1.

1. $Q_1 Q_2 = Q$.
2. $D_1 D_2 = Q$.
3. $Q_1 D_1 = D$.
4. $L(Q_1) < L(Q_2) \Rightarrow L(Q_1 Q_2) = L(Q_1)$.
5. $L(Q) < R(D) \Rightarrow R(QD) = L(Q)$.
6. $R(D) < L(Q) \Rightarrow R(QD) = R(D)$.
7. $R(D_1) < R(D_2) \Rightarrow L(D_1 D_2) = R(D_1)$.
8. $Symmetry(\Pi_1) = Symmetry(\Pi_2) \Rightarrow Symmetry(\Pi_1 \Pi_2) > Symmetry(\Pi_1) = Symmetry(\Pi_2)$.

We give two examples:

Example 3.4. L (641)=6< L (114689)=13 => L (641×114689)=6.

Example 3.5. R (607)= 4< R (16633)=6 => L (607×16633)=4.

We now prove the following Corollary:

Corollary 3.2. *Every composite number* C *of the form*

$C = 2^{\nu+1} \pm 1$

has at least two factors the symmetries of which have equal values.

Proof. Corollary 3.2 is a direct consequence of Corollary 3.1. □

From Definitions 2.1 and 3.1 it emerges that for every conjugate pair (Π, Π^*), one is of form Q and the other of form D.

4. Transpose of odd number

We now give the following Definition:

Definition 4.1. 1. *We write the odd D in the form of equation (2.1),*

$$\begin{aligned} &D = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1} 2^{\nu-1} + \beta_{\nu-2} 2^{\nu-2} + \ldots + \beta_1 2^1 + 1 \\ &\nu + 1 = \left[\frac{\ln D}{\ln 2} \right] \end{aligned} . \tag{4.1}$$

We define the transpose $T(D)$ *of* D,

$$T(D) = \left(\frac{1}{2^{\nu+1}} + \frac{1}{2^{\nu}} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \ldots + \frac{\beta_1}{2^1} + 1 \right) \cdot 2^{\nu+1} = 2^{\nu+1} + 3 + \sum_{k=1}^{\nu-1} \beta_k \cdot 2^{\nu+1-k} . \tag{4.2}$$

2. *We write the odd* Q *in the form of equation (2.1),*

$$\begin{aligned} &Q = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1} 2^{\nu-1} + \beta_{\nu-2} 2^{\nu-2} + \ldots + \beta_1 2^1 - 1 \\ &\nu + 1 = \left[\frac{\ln Q}{\ln 2} \right] \end{aligned} . \tag{4.3}$$

We define as transpose $T(Q)$ *of* Q,

$$T(Q) = -\left(\frac{1}{2^{\nu+1}} + \frac{1}{2^{\nu}} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \ldots + \frac{\beta_1}{2^1} - 1 \right) \cdot 2^{\nu+1} = 2^{\nu+1} - 3 - \sum_{k=1}^{\nu-1} \beta_k \cdot 2^{\nu+1-k} . \tag{4.4}$$

3. *We set*

$$T(1) = 1. \tag{4.5}$$

4. *From equations (4.2), (4.4), (4.5) we get the general equation*

$$\begin{aligned} &T(\Pi) = 2^{\nu+1} + \beta_0 \cdot \left(3 + \sum_{k=1}^{\nu-1} \beta_k \cdot 2^{\nu+1-k} \right) \\ &\nu + 1 = \left[\frac{\ln \Pi}{\ln 2} \right] \end{aligned} . \tag{4.6}$$

Algorithm for the calculation of the transpose. For the odd Π we calculate $\nu + 1 = \left[\frac{\ln \Pi}{\ln 2} \right]$ from equation (2.6). Next, applying the algorithm of example 2.1 we write Π in the form of equation (2.1), we calculate $\beta_i = \pm 1, i = 1,2,3,...,\nu - 1$ and the transpose $T(\Pi)$ of Π from equation (4.6).

We now prove five Theorems about the transpose of an odd number:

Theorem 4.1.

1. $T(\Pi) = 1 \Leftrightarrow \Pi = 2^\nu - 3, \nu \geq 2, \nu \in \mathbb{N}$. (4.7)

2.

$$\left.\begin{matrix} T(D) = D \\ \nu + 1 = \left[\dfrac{\ln D}{\ln 2}\right] \end{matrix}\right\} \Leftrightarrow \begin{cases} \beta_1 = 1 \\ \beta_{\nu-k} = \beta_{k+1} \\ k = 1,2,3,...,\dfrac{\nu-2}{2}, \nu = even \\ k = 1,2,3,...,\dfrac{\nu-1}{2}, \nu = odd \end{cases} \tag{4.8}$$

$$\left.\begin{matrix} T(Q) = Q \\ \nu + 1 = \left[\dfrac{\ln Q}{\ln 2}\right] \end{matrix}\right\} \Leftrightarrow \begin{cases} \beta_1 = -1 \\ \beta_{\nu-k} = -\beta_{k+1} \\ k = 1,2,3,...,\dfrac{\nu-2}{2}, \nu = even \cdot \\ k = 1,2,3,...,\dfrac{\nu-1}{2}, \nu = odd \end{cases} \tag{4.9}$$

Proof. 1. For $\Pi = 2^\nu - 3$ we get

$$\Pi = 2^\nu - 3 = \left(2^\nu - 1\right) - 2 = \left(2^{\nu-1} + 2^{\nu-2} + 2^{\nu-2} + ... + 2^1 + 1\right) - 2$$
$$= 2^{\nu-1} + 2^{\nu-2} + 2^{\nu-2} + ... + 2^1 - 1 = Q$$

and from equation (4.4) we get $T(\Pi) = 1$.

Now, let $T(\Pi) = 1$. The odd Π is either of the form D or of the form Q. We prove the Theorem for $\Pi = Q$, and the proof is similar for $\Pi = D$.

For $\Pi = Q$ we get

$$\begin{aligned} &\Pi = Q = 2^{n+1} + 2^n + \beta_{n-1}2^{n-1} + \beta_{n-2}2^{n-2} + ... + \beta_2 2^2 + \beta_1 2^1 - 1 \\ &n + 1 = \left[\frac{\ln \Pi}{\ln 2}\right] \end{aligned} \tag{4.10}$$

and from equation (4.4) we get

$$T(\Pi) = -1 - 2^1 - \beta_{n-1}2^2 - ... - \beta_2 2^n + 2^{n+1}$$

so we get

$$T(\Pi)=1$$
$$-1-2^1-\beta_{n-1}2^2-\ldots-\beta_2 2^n+2^{n+1}=1$$
$$-1-2^1-\beta_{n-1}2^2-\ldots-\beta_2 2^n+2^{n+1}+2^{n+2}=2^{n+2}+1$$
$$-1-2^1-\beta_{n-1}2^2-\ldots-\beta_2 2^n+2^{n+1}+2^{n+2}=2^{n+2}+2^n-2^{n-1}-2^{n-2}-\ldots-2^2-2^1-1$$

and taking into account that every odd Π is written in a unique way in the form of equation (2.1) we get

$$\beta_2=\beta_3=\beta_3=\ldots=\beta_{n-1}=+1$$

and from equation (4.10) we get

$$\Pi=2^{n+1}+2^n+2^{n-1}\ldots+2^1-1=2\left(2^n+2^{n-1}+2^{n-2}+\ldots+2^1+1\right)-1$$
$$=2\left(2^{n+1}-1\right)-1=2^{n+2}-3$$

and setting $n+2=\nu$ we obtain $\Pi=2^\nu-3$.

2. We prove the equivalence (4.8), and (4.9) is similarly proven. We write the odd D in the form of equation (2.1),

$$D=2^{\nu+1}+2^\nu+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+\ldots+\beta_1 2^1+1$$
$$\nu+1=\left[\frac{\ln D}{\ln 2}\right] . \quad (4.11)$$

From equations (4.11), (4.2) we get

$$T(D)=1+2+\beta_{\nu-1}2^2+\beta_{\nu-2}2^3+\ldots+\beta_2 2^\nu+2^{\nu+1} . \quad (4.12)$$

We next get

$$T(D)=D\Leftrightarrow$$
$$1+2+\beta_{\nu-1}2^2+\beta_{\nu-2}2^3+\ldots+\beta_2 2^\nu+2^{\nu+1}=2^{\nu+1}+2^\nu+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+\ldots+\beta_2 2^2+\beta_1 2^1+1$$

and taking into account that the odd D is written in a unique way in the form of equation (2.1) we get equivalence (4.8). □

Theorem 4.2. 1. For odd numbers of the form D, the equation holds

$$T(D)-T\left(D^*\right)=6 . \quad (4.13)$$

2. For odd numbers of the form Q, the equation holds

$$T(Q)-T\left(Q^*\right)=-6 . \quad (4.14)$$

Proof. We prove equation (4.13) and (4.14) is proven similarly. From equation (4.11) we get

$$D^*=2^{\nu+1}+2^\nu-\beta_{\nu-1}2^{\nu-1}-\beta_{\nu-2}2^{\nu-2}-\ldots-\beta_1 2^1-1=Q . \quad (4.15)$$

From equation (4.4) we get

$$T\left(D^*\right)=-1-2+\beta_{\nu-1}2^2+\beta_{\nu-2}2^3+...+\beta_1 2^\nu+2^{\nu+1}. \qquad (4.16)$$

From equation (4.12), (4.16) we obtain $T\left(D\right)-T\left(D^*\right)=6$. □

Theorem 4.3. *For every odd* Π, $\nu+1=\left[\frac{\ln\Pi}{\ln 2}\right]$, $\Pi\in\Omega_\nu=\left[2^{\nu+1},2^{\nu+2}\right]$ *the following inequality holds*

$$T\left(\Pi\right)<2^{\nu+2}. \qquad (4.17)$$

Proof. We prove inequality (4.17) for the D odds and the proof is similar for the Q odds. From equation (4.12) and taking into account that $\beta_i=\pm1, i=0,1,2,...,\nu-1$ we obtain

$$\begin{aligned}&T\left(D\right)=1+2+\beta_{\nu-1}2^2+\beta_{\nu-2}2^3+...+\beta_2 2^\nu+2^{\nu+1}\le 1+2+2^2+...+2^\nu+2^{\nu+1}=2^{\nu+2}-1\\&T\left(D\right)\le 2^{\nu+2}-1<2^{\nu+2}\end{aligned}. \ □$$

From inequality (4.17) it follows that if an odd Π belongs to the interval $\Omega_\nu=\left[2^{\nu+1},2^{\nu+2}\right]$, its transpose $T\left(\Pi\right)$ can be found in intervals $\Omega_n, n\le\nu$ but cannot be found in intervals $\Omega_N, N>\nu$.

Theorem 4.4. *For the consecutive numbers* $D-2$, D *of the same interval* Ω_ν, *the equation applies*

$$\begin{aligned}&T\left(D-2\right)+T\left(D\right)=T\left(Q\right)+T\left(Q+2\right)=2^{\nu+2}\\&\nu+1=\left[\frac{\ln D}{\ln 2}\right]=\left[\frac{\ln Q}{\ln 2}\right]\end{aligned}. \qquad (4.18)$$

Proof. The smallest odd number in the form of D of the interval $\Omega_\nu=\left[2^{\nu+1},2^{\nu+2}\right]$ is $D_{\min}=2^{\nu+1}+3$. Thus, the following equivalence holds: $D\in\Omega_\nu\Leftrightarrow\left(D-2\right)\in\Omega_\nu$ ($D>3$). The largest odd number in the form of Q of the interval Ω_ν is $Q_{\min}=2^{\nu+1}+1$. Thus, the following equivalence holds: $Q\in\Omega_\nu\Leftrightarrow\left(Q+2\right)\in\Omega_\nu$ ($Q>3$). Then, the Theorem is a consequence of equations (4.1), (4.3) ($D-2=Q,\ Q+2=D$) and (4.2), (4.4). □

Theorem 4.5. 1. *Let* Q *be odd number belonging to the interval* $\Omega_\nu=\left[2^{\nu+1},2\wedge\nu+2\right]$, *then the following equivalence holds,*

$$T\left(Q\right)=Q\Leftrightarrow\frac{Q^*+T\left(Q^*\right)}{2}=3\cdot\left(2^\nu+1\right). \qquad (4.19)$$

2. *Let* D *be odd number belonging to the interval* $\Omega_\nu=\left[2^{\nu+1},2\wedge\nu+2\right]$, *then the following equivalence holds,*

$$T\left(D\right)=D\Leftrightarrow\frac{D^*+T\left(D^*\right)}{2}=3\cdot\left(2^\nu-1\right). \qquad (4.20)$$

Proof. We prove 1 of the theorem and similarly the proof of 2 is done. Let Q be odd number for which $T(Q)=Q$ holds. Taking into account equation (4.14) we have $Q^*+T(Q^*)=Q^*+T(Q)+6$ and for $T(Q)=Q$ we get $Q^*+T(Q^*)=Q^*+Q+6$. and with equation (2.15) we obtain $Q^*+T(Q^*)=3\cdot 2^{\nu+1}+6$ and equivalently we get

$$\frac{Q^*+T(Q^*)}{2}=3\cdot\left(2^\nu+1\right).$$

We now prove the converse. Let Q be odd number for which

$$\frac{Q^*+T(Q^*)}{2}=3\cdot\left(2^\nu+1\right)$$

holds. Taking into account equation (4.14) we get

$$\frac{Q^*+T(Q)+6}{2}=3\cdot\left(2^\nu+1\right)$$

and equivalently we get

$$T(Q)=3\cdot 2^{\nu+1}-Q^*$$

and with equation (2.15) we obtain

$$T(Q)=Q. \ \square$$

5. The odd number octet

We now give the following Definitions:

Definition 5.1. *We define as the octet of odd number* Π *the non ordered octet*

$$\left(\Pi,T(\Pi),(T(\Pi))^*,T\left((T(\Pi))^*\right),\Pi^*,T(\Pi^*),(T(\Pi^*))^*,T\left((T(\Pi^*))^*\right)\right). \tag{5.1}$$

Definition 5.2. 1. *We define as the* 8^* *transformation of the (5.1) octet, the octet that contains the conjugates of the (5.1) octet.*

2. *We define as the* T_8 *transformation of the (5.1) octet, the octet that contains the transposes of the (5.1) octet.*

Definition 5.3. *We define as symmetric every octet that remains unchanged under the* T_8 *transformation.*

Properties of the symmetric octet. As a consequence of Definition 5.3 and the Theorems of the previous section, every symmetric octet has the following properties:

1. It consists of pairs of conjugate odds (common property of all octets).

2. It is unchanged under the 8^* transformation (common property of all octets).

3. It is unchanged under the T_8 transformation (Definition): for every odd Π of the symmetric octet, it is $T\left(T\left(\Pi\right)\right)=\Pi$.

4. It consists of pairs of odds that differ by 6 (consequence of Theorem 4.2).

5. It consists of 8 or 4 or 2 different odds.

6. The numbers of the symmetric octet belong to the same interval $\Omega_\nu=\left[2^{\nu+1},2^{\nu+2}\right]$,

$\nu+1=\left[\frac{\ln\Pi_k}{\ln 2}\right]$, where $\Pi_k, k=1,2,3,...,8$ the octet numbers.

7. The smallest odd m of a symmetric octet is always of the form Q, $m=Q$ and the largest M is its conjugate, $M=Q^*$.

8. An odd Π can belong to exactly one symmetric octet (while every odd which belongs to a symmetric octet, also belongs to infinite non symmetric octets, as we shall see later).

In order to determine the octet of an odd Π we use the algorithm for the calculation of the transpose, and equation (2.17) for the calculation of the conjugate. We now give an example which also shows the ways in which we can write a symmetric octet.

Example 5.1. From equation (5.1) we get the symmetric octet in which $\Pi=889$ belongs,

(889, 529, 1007, 895, 647, 535, 1001, 641).

In order to discern the pairs of transposes and of conjugates, we write the octet in the form

$$\begin{array}{ccccccc} 889 & \xleftrightarrow{T} & 529 & \xleftrightarrow{*} & 1007 & \xleftrightarrow{T} & 895 \\ {}^{*}\updownarrow & & & & & & \updownarrow^{*} \\ 647 & \xleftrightarrow{T} & 535 & \xleftrightarrow{*} & 1001 & \xleftrightarrow{T} & 641 \end{array}.$$

Because of equation (2.16) $\left(\Pi^*\right)^*=\Pi$ two conjugates are always connected, in all symmetries, by the symbol $\xleftrightarrow{*}$, $\Pi\xleftrightarrow{*}\Pi^*$. With $\Pi_1\xleftrightarrow{T}\Pi_2$ we denote that $T\left(\Pi_1\right)=\Pi_2$ and $T\left(\Pi_2\right)=\Pi_1$. If $T\left(\Pi_1\right)=\Pi_2$ and $T\left(\Pi_2\right)\neq\Pi_1$, we write $\Pi_1\xrightarrow{T}\Pi_2$. In our example, $\Pi_1\xleftrightarrow{T}\Pi_2$ for all of the octet numbers, therefore, it is unchanged under the T_8 transformation and consequently it is symmetric (Definition). The 8^* and T_8 transformations only change the relative position of the numbers within the symmetric octet. The octet symmetries are easily seen when we place the numbers on the corners of a regular octagon,

$$\begin{tikzcd}
& 889 \arrow[r, leftrightarrow, "T"] \arrow[dl, leftrightarrow, "*"'] & 529 \arrow[dr, leftrightarrow, "*"] & \\
647 \arrow[d, leftrightarrow, "T"'] & & & 1007 \arrow[d, leftrightarrow, "T"] \\
535 \arrow[dr, leftrightarrow, "*"] & & & 895 \\
& 1001 \arrow[r, leftrightarrow, "T"] & 641 \arrow[ur, leftrightarrow, "*"'] &
\end{tikzcd} \qquad (5.2)$$

A symmetric octet can be composed of 8 different numbers, like the one of the previous example, or of 4 different numbers or of 2 different numbers (with the exception of the degenerate octets $(1,1,1,1,1,1,1,1)$ of 1 and $(3,3,3,3,3,3,3,3)$ of 3). From the Definitions of the conjugate and the transpose, the following equations are easily proven

$$\begin{aligned}
&2^{\nu}+1 \xleftrightarrow{*} 2^{\nu+1}-1 \\
&2^{\nu}+7 \xleftrightarrow{*} 2^{\nu+1}-7 \\
&2^{\nu}+1 \xleftrightarrow{T} 2^{\nu+1}-7 \quad . \\
&2^{\nu}+7 \xleftrightarrow{T} 2^{\nu}+7 \\
&2^{\nu+1}-1 \xleftrightarrow{T} 2^{\nu+1}-1
\end{aligned} \qquad (5.3)$$

Considering equations (5.3) we get the symmetric octets

$$\begin{aligned}
&\left(2^{\nu+1}+1, 2^{\nu+2}-7, 2^{\nu+1}+7, 2^{\nu+1}+7, 2^{\nu+2}-1, 2^{\nu+2}-1, 2^{\nu+1}+1, 2^{\nu+2}-7\right) \\
&\nu \geq 3, \nu \in \mathbb{N}
\end{aligned}. \qquad (5.4)$$

The (5.4) symmetric octets consist of 4 different numbers. Fermat numbers for $\nu+1=2^S, S \in \mathbb{N}$ and Mersenne numbers for $\nu+2=p=prime$ belong to the (5.4) symmetric octets. The symmetric octet $(9,9,15,15,15,15,9,9)$ of conjugates $(\Pi,\Pi^*)=(9,15)$ consists of 2 numbers.

Definition 5.4. *We define as non-symmetric or asymmetric every octet that contains a pair of conjugates* (Π_1,Π_1^*) *for which*

$$\begin{aligned}
&\Pi_1 \xrightarrow{T} \Pi_2 \, and \, \Pi_2 \xleftrightarrow{T} \Pi_3 \neq \Pi_1 \\
&\Pi_1^* \xrightarrow{T} \Pi_4 \, and \, \Pi_4 \xleftrightarrow{T} \Pi_5 \neq \Pi_1^*
\end{aligned}. \qquad (5.5)$$

Asymmetric octets as generators of symmetric octets. If an odd number Π belongs to a symmetric octet, then its conjugate Π^* and its inverse $\mathrm{T}(\Pi)$ belong to the octet. Also, all the numbers in the symmetric octet belong to the same interval Ω_ν. The asymmetric octets result from a pair of conjugates (Π,Π^*) belonging to an interval Ω_μ and their transposes

$\left(T(\Pi),T(\Pi^*)\right)$ in another interval $\Omega_\nu, \nu<\mu$. The octet of the pair $\left(T(\Pi),T(\Pi^*)\right)$ is symmetric and we say that it is produced from the initial asymmetric octet.

We now present one example of an asymmetric octet in which one can see the way in which we can write it so that the asymmetry is evident and so are the symmetric octet that it produces.

Example 5.2. The conjugate pair $(91,101)$ gives the asymmetric octet

$$
\begin{array}{ccccccc}
 & & 47 & & & & \\
 & & \updownarrow T & & & & \\
91 & \xrightarrow{T} & 55 & \xleftrightarrow{*} & 41 & \xleftrightarrow{T} & 41 \\
{}^*\updownarrow & & & & & & \updownarrow^* \\
101 & \xrightarrow{T} & 49 & \xleftrightarrow{*} & 47 & \xleftrightarrow{T} & 55 \\
 & & \updownarrow T & & & & \\
 & & 49 & & & &
\end{array}.
$$

The asymmetric octet has produces the symmetric octet

$$
\begin{array}{ccccccc}
47 & \xleftrightarrow{T} & 55 & \xleftrightarrow{*} & 41 & \xleftrightarrow{T} & 41 \\
{}^*\updownarrow & & & & & & \updownarrow^* \\
49 & \xleftrightarrow{T} & 49 & \xleftrightarrow{*} & 47 & \xleftrightarrow{T} & 55
\end{array}
$$

which emerges by replacing the pair of conjugates $(91,101)$ by the pair of conjugates $(47,49)$.

We now prove the following Theorem:

Theorem 5.1. (Fundamental octet Theorem)

A. *The octets of the numbers*

$$
\begin{aligned}
&\Pi = D = 2^2\cdot K-1 \\
&\Pi^* = Q = 2^2\cdot\left(3\cdot 2^{\nu-1}-K\right)+1 \\
&\nu+1=\left[\frac{\ln\Pi}{\ln 2}\right]\geq 3, K=odd
\end{aligned}
\tag{5.6}
$$

are asymmetric.

B.

1. *The asymmetric octets are given by the odds*

$$
\begin{aligned}
&D_1 = 11+8m, m\in\mathbb{N} \\
&Q_1 = 13+8m, m\in\mathbb{N}
\end{aligned}
\tag{5.7}
$$

except from the asymmetric octet $(5,1,1,1,7,7,5,1)$.

2. *The symmetric octets are given by the odds*

$$\begin{aligned} &D_2 = 7+8m, m \in \mathbb{N}^* \\ &Q_2 = 9+8m, m \in \mathbb{N} \end{aligned} . \tag{5.8}$$

Proof. A. From 2 of Theorem 4.1 it follows that for the odds Π, for which $\beta_0\beta_1 = -1$, it holds that $T(\Pi) \neq \Pi$,

$$\beta_0\beta_1 = -1 \Rightarrow T(\Pi) \neq \Pi . \tag{5.9}$$

$D = 2^3 + 2^2 - 2 + 1 = 11$ and $Q = 2^3 + 2^2 + 2 - 1 = 13$ are the smallest odds for which it can be $\beta_0\beta_1 = -1$. Therefore, the Theorem holds for $\nu + 1 \geq 3$.

We prove the Theorem for $\Pi = D$ and it is similarly proven for $\Pi = Q$. From equation (4.11) for $\beta_0\beta_1 = -1$ we get

$$\begin{aligned} \Pi = D &= 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + ... + \beta_2 2^2 - 2 + 1 \\ &= 2^2 \cdot \left(2^{\nu-1} + 2^{\nu-2} + \beta_{\nu-1}2^{\nu-3} + \beta_{\nu-2}2^{\nu-3} + ... + \beta_2\right) - 2 + 1 \\ &= 2^2 \cdot \left(2^{\nu-1} + 2^{\nu-2} + \beta_{\nu-1}2^{\nu-3} + \beta_{\nu-2}2^{\nu-3} + ... + \beta_2\right) - 1 \end{aligned}$$

and setting

$$K = 2^{\nu-1} + 2^{\nu-2} + \beta_{\nu-1}2^{\nu-3} + \beta_{\nu-2}2^{\nu-3} + ... + \beta_2 \tag{5.10}$$

we get $D = 2^2 \cdot K - 1$. Considering equation (5.10) we calculate the conjugate of D which is $\Pi^* = Q = 2^2 \cdot \left(3 \cdot 2^{\nu-1} - K\right) + 1$.

B. Equations (5.6) give the consecutive pairs $(D,Q) = (11,13), (19,21), (27,29), \dots$, that is, equations (5.7). Taking into account the fourth property of the symmetric octets we conclude that the intermediate pairs of odds give the symmetric octets. These pairs are given by equations (5.8). Additionally, equations (5.7) give all odds that produce asymmetric octets (with the exception of the asymmetric octet $(5,1,1,1,7,7,5,1)$) and the equations give all the odds that produce symmetric octets (otherwise the fourth property of the symmetric octets would not hold, which cannot be true due to Theorem 4.2)

The octet

$$(5,1,1,1,7,7,5,1) \tag{5.11}$$

of the conjugate pair $(\Pi, \Pi^*) = (5,7)$ is the only asymmetric octet that does not belong to the (5.7) octets, since for this conjugate pair it is

$$\left[\frac{\ln 5}{\ln 2}\right] = \left[\frac{\ln 7}{\ln 2}\right] = \nu + 1 = 2 < 3 .$$

The asymmetric octet (5.11) is given by the terms of the ε_n sequence of odds,

$$\begin{aligned}&\varepsilon_n = 3 + 2^{n+1}\\&n = 1, 2, 3, \ldots\end{aligned} \quad . \tag{5.12}$$

The (5.11) octet emerges from the terms of the ε_n sequence because of equation

$$\begin{aligned}&T(\varepsilon_n) = T\left(2^{n+1} + 3\right) = \varepsilon_1 = 7\\&n = 1, 2, 3, \ldots\end{aligned} \quad . \tag{5.13}$$

The terms of the ε_n sequence are of the form D. From equation (4.13) we get

$$T\left(2^{n+1} + 3\right) - T\left(\left(2^{n+1} + 3\right)^*\right) = 6$$

$$T\left(2^{n+1} + 3\right) - T\left(2^{n+2} - 3\right) = 6$$

and with equation (4.7) we obtain

$$T\left(2^{n+1} + 3\right) - 1 = 6$$

$$T\left(2^{n+1} + 3\right) = 7$$

which is equation (5.13). □

For the odds Q, D that belong to a symmetric octet we make the following Conjecture:

Conjecture 5.1. A. 1. *For every odd* Q *it holds that*

$$\begin{aligned}&T\left(2^n \cdot (Q+3) - 3\right) = T(Q)\\&\forall n \in \mathbb{N}\end{aligned} \tag{5.14}$$

$$\begin{aligned}&T\left(2^n \cdot (Q+3) + 3\right) = \left(T(Q)\right)^*\\&\forall n \in \mathbb{N}\end{aligned} \quad . \tag{5.14a}$$

2. *For every odd* D *it holds that*

$$\begin{aligned}&T\left(2^n \cdot (D-3) + 3\right) = T(D)\\&\forall n \in \mathbb{N}\end{aligned} \tag{5.15}$$

$$\begin{aligned}&T\left(2^n \cdot (D-3) - 3\right) = \left(T(D)\right)^*\\&\forall n \in \mathbb{N}\end{aligned} \quad . \tag{5.15a}$$

B. 1. *The sequences*

$$\begin{aligned}&\gamma_n(Q) = 2^n \cdot (Q-3) - 3\\&\Gamma_n(Q) = 2^n \cdot (Q-3) + 3\\&n \in \mathbb{N}^*\end{aligned} \tag{5.16}$$

derive the same symmetric octet for each $n \in \mathbb{N}^*$.

2. *The sequences*

$$\begin{aligned}&\Delta_n(D)=2^n\cdot(D+3)+3\\&\delta_n(Q)=2^n\cdot(D+3)-3\\&n\in\mathbb{N}^*\end{aligned}\tag{5.17}$$

derive the same symmetric octet for each $n\in\mathbb{N}^*$.

From Conjecture 5.1 and the Definition of the symmetric octet, the following Corollary emerges directly:

Corollary 5.1. 1. *For every odd* Q *that belongs to a symmetric octet the sequence*

$$\begin{aligned}&a_n(Q)=2^n\cdot(Q+3)-3\\&A_n(Q)=2^n\cdot(Q+3)+3\\&n\in\mathbb{N}^*\end{aligned}\tag{5.18}$$

gives (infinite) asymmetric octets that produce the symmetric octet to which Q *belongs*.

2. *For every odd* D *that belongs to a symmetric octet the sequence*

$$\begin{aligned}&B_n(D)=2^n\cdot(D-3)+3\\&\beta_n(D)=2^n\cdot(D-3)-3\\&n\in\mathbb{N}^*\end{aligned}\tag{5.19}$$

gives (infinite) asymmetric octets that produce the symmetric octet to which D *belongs*.

The $a_n(Q)$ sequence has exactly one term in every interval

$$\Omega_{\mathrm{N}}=\left[2^{\mathrm{N}+1},2^{\mathrm{N}+2}\right],\mathrm{N}\in\mathbb{N},\mathrm{N}+1>\left[\frac{\ln Q}{\ln 2}\right].$$

The $B_n(D)$ sequence has exactly one term in every interval

$$\Omega_{\mathrm{N}}=\left[2^{\mathrm{N}+1},2^{\mathrm{N}+2}\right],\mathrm{N}\in\mathbb{N},\mathrm{N}+1>\left[\frac{\ln D}{\ln 2}\right].$$

We give an example:

Example 5.3. We pick a number from the symmetric octet (5.2), for example $647=D$ and a random $n=20$. From equation (5.19) we get $B_{20}(647)=2^{20}\cdot(647-3)+3=675282947$. The odd 675282947 gives the asymmetric octet

$$\begin{array}{ccccccc}
 & & 647 & & & & \\
 & & \updownarrow T & & & & \\
675282947 & \xrightarrow{T} & 535 & \xleftrightarrow{*} & 1001 & \xleftrightarrow{T} & 641 \\
{}^{*}\updownarrow & & & & & & \updownarrow^{*} \\
935329789 & \xrightarrow{T} & 529 & \xleftrightarrow{*} & 1007 & \xleftrightarrow{T} & 895 \\
 & & \updownarrow T & & & & \\
 & & 889 & & & &
\end{array}$$

which produces the symmetric octet

$$\begin{array}{ccccccc}
647 & \xleftrightarrow{T} & 535 & \xleftrightarrow{*} & 1001 & \xleftrightarrow{T} & 641 \\
{}^{*}\updownarrow & & & & & & \updownarrow^{*} \\
889 & \xleftrightarrow{T} & 529 & \xleftrightarrow{*} & 1007 & \xleftrightarrow{T} & 895
\end{array}$$

which is (5.2). $B_{20}(647) = 675282947 \in \Omega_{28}$ belongs to the interval $\Omega_{28} = \left[2^{29}, 2^{30}\right]$.
$B_{21}(647) = 2^{21} \cdot (647-3) + 3 = 1350565891 \in \Omega_{29}$ also produces the (5.2) symmetric octet and belongs to the next interval $\Omega_{29} = \left[2^{30}, 2^{31}\right]$,

$$\begin{array}{ccccccc}
 & & 647 & & & & \\
 & & \updownarrow T & & & & \\
1350565891 & \xrightarrow{T} & 535 & \xleftrightarrow{*} & 1001 & \xleftrightarrow{T} & 641 \\
{}^{*}\updownarrow & & & & & & \updownarrow^{*} \\
1870659581 & \xrightarrow{T} & 529 & \xleftrightarrow{*} & 1007 & \xleftrightarrow{T} & 895 \\
 & & \updownarrow T & & & & \\
 & & 889 & & & &
\end{array}.$$

We now prove the following Corollary:

Corollary 5.2. 1. *For every odd* Q *belonging to a symmetric octet, it holds that*

$$T\left(2^n \cdot \left(T(Q)+3\right)-3\right) = T\left(T(Q)\right) = Q. \qquad \forall n \in \mathbb{N} \tag{5.20}$$

2. *For every odd* D *belonging to a symmetric octet, it holds that*

$$T\left(2^n \cdot \left(T(D)-3\right)+3\right) = T\left(T(D)\right) = D. \qquad \forall n \in \mathbb{N} \tag{5.21}$$

Proof. We prove equation (5.20) and the proof of (5.21) is similar. From Definition 5.3 of the symmetric octet it follows that if Q belongs to a symmetric octet, then $T(Q)$ belongs to the same symmetric octet. Therefore, equation (5.14) also holds for $T(Q)$,

$$T\left(2^n\cdot\left(T\left(Q\right)+3\right)-3\right)=T\left(T\left(Q\right)\right)$$
$$\forall n\in\mathbb{N}^*$$

and taking into account that for the symmetric octet it is (Definition) $T\left(T\left(Q\right)\right)=Q$ we get equation (5.20). □

Also, applying Definition 5.1 for *even numbers* $2^n\cdot\Pi, n\in\mathbb{N}^*, \Pi=odd$ we obtain the following Corollary:

Corollary 5.3. *For every odd* Π *it holds that*

$$\begin{gathered}T\left(2^n\cdot\Pi\right)=T\left(\Pi\right)\\ \forall n\in\mathbb{N},\Pi=odd\end{gathered}. \tag{5.22}$$

Proof. We prove equation (5.22) for the D odds and the proof is similar for the Q odds.

$$D=2^{\nu+1}+2^{\nu}+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+...+\beta_1 2^1+1$$

$$\nu+1=\left[\frac{\ln D}{\ln 2}\right]$$

$$2^n D=2^{n+\nu+1}+2^{n+\nu}+\beta_{\nu-1}2^{n+\nu-1}+\beta_{\nu-2}2^{n+\nu-2}+...+\beta_1 2^{n+1}+2^n$$

$$\begin{aligned}T\left(2^n D\right)&=\left(\frac{1}{2^{n+\nu+1}}+\frac{1}{2^{n+\nu}}+\frac{\beta_{\nu-1}}{2^{n+\nu-1}}+\frac{\beta_{\nu-2}}{2^{n+\nu-2}}+...+\frac{\beta_1}{2^{n+1}}+\frac{1}{2^n}\right)\cdot 2^{n+\nu+1}\\ &=\left(\frac{1}{2^{\nu+1}}+\frac{1}{2^{\nu}}+\frac{\beta_{\nu-1}}{2^{\nu-1}}+\frac{\beta_{\nu-2}}{2^{\nu-2}}+...+\frac{\beta_1}{2^1}+\frac{1}{2^0}\right)\cdot 2^{\nu+1}=T\left(D\right)\end{aligned}. \square$$

We give an example:

Example 5.4. $\Pi=Q=2021=2^{10}+2^9+2^8+2^7+2^6+2^5+2^4-2^3-2^2+2^1-1$

$$\begin{gathered}2^n\Pi=2^{10+n}+2^{9+n}+2^{8+n}+2^{7+n}+2^{6+n}+2^{5+n}+2^{4+n}-2^{3+n}-2^{2+n}+2^{1+n}-2^n\\ n\in\mathbb{N}\end{gathered}$$

$$\begin{aligned}T\left(2^n\cdot\Pi\right)&=-\left(\frac{1}{2^{10+n}}+\frac{1}{2^{9+n}}+\frac{1}{2^{8+n}}+\frac{1}{2^{7+n}}+\frac{1}{2^{6+n}}+\frac{1}{2^{5+n}}+\frac{1}{2^{4+n}}-\frac{1}{2^{3+n}}-\frac{1}{2^{2+n}}+\frac{1}{2^{1+n}}-\frac{1}{2^n}\right)\cdot 2^{10+n}\\ &=-\left(\frac{1}{2^{10}}+\frac{1}{2^9}+\frac{1}{2^8}+\frac{1}{2^7}+\frac{1}{2^6}+\frac{1}{2^5}+\frac{1}{2^4}-\frac{1}{2^3}-\frac{1}{2^2}+\frac{1}{2^1}-1\right)\cdot 2^{10}=T\left(\Pi\right)\end{aligned}.$$

We now prove the following Corollary:

Corollary 5.4. *For every odd* A *that doesn't belong to a symmetric octet, it holds that*

$$A\in\Omega_\nu\Rightarrow T\left(A\right)\in\Omega_\mu, \mu<\nu. \tag{5.23}$$

Proof. Corollary 5.4 is a direct consequence of Theorem 4.3 and Definition 5.4. □

We complete section 5 with the following Definition:

Definition 5.5. (Categorization of odd numbers) 1. *We define as asymmetric numbers, the numbers that don't belong to a symmetric octet (equations (5.7))*

2. *We define as symmetric numbers, the numbers that belong to a symmetric octet (equations (5.8)).*

6. An algorithm for finding the factors of Fermat numbers

There exists a sequence of odd numbers of the form $Q = 8m+1$, $m = 1,2,3,...$ for which $T(Q) \geq Q$ and $T(Q^*) \geq Q^*$. Fermat numbers and their factors (see, [1-5]) belong to this sequence. Starting from this finding we get an algorithm for finding factors of Fermat numbers in an interval $\Omega_\nu = \left[2^{\nu+1}, 2^{\nu+2}\right]$.

Let

$$Q = 2^n \cdot K + 1, \tag{6.1}$$

where K is an odd number, be a factor of a Fermat number in the interval $\Omega_\nu = \left[2^{\nu+1}, 2^{\nu+2}\right]$. We take the sequence of numbers

$$D = 2^{\nu+1} + 3\cdot 2^n - 1 + 2^{n+1}\cdot\lambda. \tag{6.2}$$

Considering that D belongs to the interval Ω_ν, we get the possible values of λ,

$$\lambda = -1, 0, 1, ..., 2^{\nu-n}. \tag{6.3}$$

If

$$Q \in \left[2^{\nu+1}, 3\cdot 2^\nu\right] \subseteq \left[2^{\nu+1}, 2^{\nu+2}\right] = \Omega_\nu$$

we give values $\lambda = 2^{\nu-n}, 2^{\nu-n}-1, 2^{\nu-n}-2, ..., -1$ in equation (6.2). There is a value of λ for which

$$Q = D^*. \tag{6.4}$$

If

$$Q \in \left[3\cdot 2^\nu, 2^{\nu+2}\right] \subseteq \left[2^{\nu+1}, 2^{\nu+2}\right] = \Omega_\nu$$

we give values $\lambda = -1, 0, 1, ..., 2^{\nu-n}$ in equation (6.2).

We give two examples.

Example 1. For the factor

$$Q = 274177 \in \left[2^{18}, 3\cdot 2^{17}\right] \subseteq \left[2^{18}, 2^{19}\right] = \Omega_{17}$$

of F_6 it is $\nu = 17$ and $n = 6+2 = 8$. From equations (6.2) and (6.3) we get

$$D = 2^{18} + 3\cdot 2^8 - 1 - 2^9\cdot\lambda \tag{6.5}$$

and

$\lambda = -1, 0, 1, ..., 510$.

We give values $\lambda = 510, 509, 508, ..., -1$ in equation (6.5). After 14 tests, for $\lambda = 510, 509, 508, ..., 487$ we get $D = 2^{18} + 3 \cdot 2^8 - 1 - 2^9 \cdot 487 = 512255$ and from equation (6.4) we obtain $Q = D^* = 512255^* = 274177$. Finding $Q = 274177 = 2^8 \cdot 1071 + 1$ from equation (6.1) requires 1071 tests.

Example 2. For the factor

$$Q = 6700417 \in \left[3 \cdot 2^{21}, 2^{23}\right] \subseteq \left[2^{22}, 2^{23}\right] = \Omega_{21}$$

of F_5 it is $\nu = 21$ and $n = 5 + 2 = 7$. From equations (6.2) and (6.3) we get

$$D = 2^{22} + 3 \cdot 2^7 - 1 - 2^8 \cdot \lambda \tag{6.6}$$

and

$\lambda = -1, 0, 1, ..., 16382$.

We give values $\lambda = -1, 0, 1, ..., 16382$ in equation (6.6). After 6595 tests, for $\lambda = -1, 0, 1, ..., 6593$ we get $D = 2^{22} + 3 \cdot 2^7 - 1 - 2^8 \cdot 6593 = 5882495$ and from equation (6.4) we obtain $Q = D^* = 5882495^* = 6700417$. Finding $Q = 6700417 = 2^7 \cdot 52347 + 1$ from equation (6.1) requires 52347 tests.

We ran the algorithm assuming that the interval Ω_{21} to which $Q = 6700417$ belongs is known. In fact, we do not know the interval Ω_ν to which Q belongs. So the algorithm must run on an interval Ω wider than Ω_ν.

Considering that Q is the largest factor of F_5 we get $Q > \sqrt{F_5} = \sqrt{2^{32} + 1}$. Therefore the algorithm must run in the intervals $\Omega_{16}, \Omega_{17}, \Omega_{18}, ...$ until the interval Ω_{21} in which we find $D = 2^{22} + 3 \cdot 2^7 - 1 - 2^8 \cdot 6593 = 5882495$. The maximum number of tests for each interval Ω_ν is $2^{\nu - n} - 2$ (see equation (6.3)). Therefore, the number of structures up to the set Ω_{21} is

$$2^{16-7} - 2 + 2^{17-7} - 2 + 2^{18-7} - 2 + 2^{19-7} - 2 + 2^{20-7} - 2 + 2^{21-7} - 2 = 32244 < 52347.$$

In fact, the number of tests is less than 32244 since in set Ω_{21} 6595 tests are required and not $2^{21-7} - 2 = 16382$. Therefore the required tests are

$$2^{16-7} - 2 + 2^{17-7} - 2 + 2^{18-7} - 2 + 2^{19-7} - 2 + 2^{20-7} - 2 + 6595 = 22455 < 32244.$$

7. Conclusion

If we want to summarize this article in one sentence we would say that we study the symmetries of natural numbers which arise from Theorem (2.1). These symmetries establish a new framework for the study of natural numbers which is entirely different from the context in which they have been studied so far.

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