The octets of the natural numbers

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Abstract

"Every natural number, with the exception of 0 and 1, can be written in a unique way as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1". From this theorem, four fundamental properties of natural numbers are implied: the conjugate of an odd number, the L/R symmetry, the transpose of a natural number and the octets of natural numbers. Using these properties we get a factorization algorithm for natural numbers.

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1. Introduction

In this article, we start by proving the theorem: "Every natural number, with the exception of 0 and 1, can be written in a unique way as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1" (theorem 2.1). It's easy to show that a theorem of this kind doesn't hold for any other natural number different than 2.

The main difference between our theorem and the known arithmetic systems (binary, decimal etc.) is that the coefficients of the linear combination can take the negative value. These negative values highlight a property of natural numbers, the "conjugate" of an odd number (definition 2.1).

Another property of natural numbers is that of the "transpose" of an odd number (definition 4.1). The transpose of an odd number can be defined in any arithmetic system and its value depends on the system used. However, the main mathematical object to which we come to, "the odd number octet", is defined from a combination of conjugates and transposes of odd numbers (equation (5.1)). Thus, the odd number octet emerges only by using theorem 2.1.

In chapter 6 we categorize the odd numbers, so indirectly, we categorize the even numbers too since every even is the product of a power of 2 with an odd number. This categorization is achieved by using the octets of odd numbers and the "L/R symmetry" (definition 3.1). In chapter 7 a factorization algorithm for odd numbers is given.

At the end of the introduction, I would like to make a comment for the octets of odd numbers. This mathematical object plays a central role in factorization of odd numbers and also has a rare mathematical beauty.

2. Natural numbers as linear combinations of consecutive powers of 2

We prove the following theorem.

Theorem 2.1. *Every natural number, with the exception of* 0 *and* 1, *can be uniquely written as a linear combination of consecutive powers of* 2, *with the coefficients of the linear combination being* -1 *or* +1.

Proof. Let the odd number Π as given from equation

$$\begin{aligned}&\Pi=\Pi\left(\nu,\beta_i\right)=2^{\nu+1}+2^{\nu}\pm 2^{\nu-1}\pm 2^{\nu-2}\pm........\pm 2^1\pm 2^0=2^{\nu+1}+2^{\nu}+\sum_{i=0}^{\nu-1}\beta_i 2^i\\&\beta_i=\pm 1, i=0,1,2,........,\nu-1\\&\nu\in\mathbb{N}\end{aligned}\quad. \qquad (2.1)$$

From equation (2.1) for $\nu=0$ we obtain

$$\Pi=2^1+2^0=2+1=3.$$

We now examine the case where $\nu\in\mathbb{N}^*=\{1,2,3,...\}$. The lowest value that the odd number Π of equation (2.1) can obtain is

$$\Pi_{\min}=\Pi\left(\nu\right)=2^{\nu+1}+2^{\nu}-2^{\nu-1}-2^{\nu-1}-........2^1-1$$

$$\Pi_{\min}=\Pi\left(\nu\right)=2^{\nu+1}+1. \qquad (2.2)$$

The largest value that the odd number Π of equation (2.1) can obtain is

$$\Pi_{\max}=\Pi\left(\nu\right)=2^{\nu+1}+2^{\nu}+2^{\nu-1}+........2^1+1$$

$$\Pi_{\max}=\Pi\left(\nu\right)=2^{\nu+2}-1. \qquad (2.3)$$

Thus, for the odd numbers $\Pi=\Pi\left(\nu,\beta_i\right)$ of equation (2.1) the following inequality holds

$$\Pi_{\min}=2^{\nu+1}+1\leq\Pi\left(\nu,\beta_i\right)\leq 2^{\nu+2}-1=\Pi_{\max}\,. \qquad (2.4)$$

The number $N\left(\Pi\left(\nu,\beta_i\right)\right)$ of odd numbers in the closed interval $\left[2^{\nu+1}+1,2^{\nu+2}-1\right]$ is

$$N\left(\Pi\left(\nu,\beta_i\right)\right)=\frac{\Pi_{\max}-\Pi_{\min}}{2}+1=\frac{\left(2^{\nu+2}-1\right)-\left(2^{\nu+1}+1\right)}{2}+1$$

$$N\left(\Pi\left(\nu,\beta_i\right)\right)=2^{\nu}\,. \qquad (2.5)$$

The integers $\beta_i, i=0,1,2,........,\nu-1$ in equation (2.1) can only take two values, $\beta_i=-1\vee\beta_i=+1$, thus equation (2.1) gives exactly $2^{\nu}=N\left(\Pi\left(\nu,\beta_i\right)\right)$ odd numbers.

Therefore, for every $\nu \in \mathbb{N}^*$ equation (2.1) gives all odd numbers in the interval $\left[2^{\nu+1}+1, 2^{\nu+2}-1\right]$.

We now prove the theorem for the even numbers. Every even number α which is a power of 2 can be uniquely written in the form of $\alpha = 2^{\nu}, \nu \in \mathbb{N}^*$. We now consider the case where the even number α is not a power of 2. In that case, the even number α is written in the form of

$$\alpha = 2^{l}\Pi, \Pi = \text{odd}, \Pi \neq 1, l \in \mathbb{N}^* . \tag{2.6}$$

We now prove that the even number α can be uniquely written in the form of equation (2.6). If we assume that the even number α can be written in the form of

$$\begin{gathered}\alpha = 2^{l}\Pi = 2^{l'}\Pi' \\ l \neq l' (l > l') \\ \Pi \neq \Pi' \\ l, l' \in \mathbb{N}^* \\ \Pi, \Pi' = odd\end{gathered} \tag{2.7}$$

then we have

$$2^{l}\Pi = 2^{l'}\Pi'$$

$$2^{l-l'}\Pi = \Pi'$$

which is impossible, since the first part of this equation is even and the second odd. Thus, it is $l = l'$ and we get that $\Pi = \Pi'$ from equation (2.7). Therefore, every even number α that is not a power of 2 can be uniquely written in the form of equation (2.6). The odd number Π of equation (2.6) can be uniquely written in the form of equation (2.1), thus from equation (2.6) it is derived that every even number α that is not a power of 2 can be uniquely written in the form of equation

$$\begin{gathered}\alpha = \alpha\left(l, \nu, \beta_i\right) = 2^{l}\left(2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1}\beta_i 2^{i}\right) \\ l \in \mathbb{N}^*, \nu \in \mathbb{N} \\ \beta_i = \pm 1, i = 0, 1, 2, ..., \nu - 1\end{gathered} \tag{2.8}$$

and equivalently

$$\begin{gathered}\alpha = \alpha\left(l, \nu, \beta_i\right) = 2^{l+\nu+1} + 2^{l+\nu} + \sum_{i=0}^{\nu-1}\beta_i 2^{l+i} \\ l \in \mathbb{N}^*, \nu \in \mathbb{N} \qquad . \\ \beta_i = \pm 1, i = 0, 1, 2, ..., \nu - 1\end{gathered} \tag{2.9}$$

For 1 we get

$$1 = 2^0$$

$$1 = 2^1 - 2^0$$

thus, it can be written in two ways in the form of equation (2.1). Both the odds of equation (2.1) and the evens of the equation (2.8) are positive. Thus, 0 cannot be written either in the form of equation (2.1) or in the form of equation (2.8). □

In order to write an odd number $\Pi \neq 1,3$ in the form of equation (2.1) we initially define the $\nu \in \mathbb{N}^*$ from inequality (2.4). Then, we calculate the sum

$2^{\nu+1} + 2^{\nu}$.

If $2^{\nu+1} + 2^{\nu} < \Pi$ we add $2^{\nu-1}$, whereas if $2^{\nu+1} + 2^{\nu} > \Pi$ then we subtract it. By repeating the process exactly ν times we write the odd number Π in the form of equation (2.1). The number ν of steps needed in order to write the odd number Π in the form of equation (2.1) is extremely low compared to the magnitude of the odd number Π, as derived from inequality (2.4).

Example 2.1. For the odd number $\Pi = 23$ we obtain from inequality (2.4)

$$2^{\nu+1} + 1 < 23 < 2^{\nu+2} - 1$$

$$2^{\nu+1} + 2 < 24 < 2^{\nu+2}$$

$$2^{\nu} < 12 < 2^{\nu+1}$$

thus $\nu = 3$. Then, we have

$2^{\nu+1} + 2^{\nu} = 2^4 + 2^3 = 24 > 23$ (thus 2^2 is subtracted)

$2^4 + 2^3 - 2^2 = 20 < 23$ (thus 2^1 is added)

$2^4 + 2^3 - 2^2 + 2^1 = 22 < 23$ (thus $2^0 = 1$ is added)

$2^4 + 2^3 - 2^2 + 2^1 + 1 = 23$.

Fermat numbers F_s can be written directly in the form of equation (2.1), since they are of the form $\Pi_{\min}$,

$$\begin{gathered} F_s = 2^{2^s} + 1 = \Pi_{\min}\left(2^s - 1\right) = 2^{2^s} + 2^{2^s-1} - 2^{2^s-2} - 2^{2^s-3} - \dots\dots\dots - 2^1 - 1 \\ s \in \mathbb{N} \end{gathered} \tag{2.10}$$

Mersenne numbers M_p can be written directly in the form of equation (2.1), since they are of the form $\Pi_{\max}$,

$$\begin{gathered} M_p = 2^p - 1 = \Pi_{\max}\left(p-2\right) = 2^{p-1} + 2^{p-2} + 2^{p-3} + \dots\dots\dots + 2^1 + 1 \\ p = prime \end{gathered} \tag{2.11}$$

In order to write an even number a, that is not a power of 2, in the form of equation (2.1), initially it is consecutively divided by 2 until it takes the form of equation (2.6). Then, we write the odd number Π in the form of equation (2.1).

Example 2.2. By consecutively dividing the even number $\alpha = 368$ by 2 we obtain

$\alpha = 368 = 2^4 \cdot 23$.

Then, we write the odd number $\Pi = 23$ in the form of equation (2.1),

$$23 = 2^4 + 2^3 - 2^2 + 2^1 + 1,$$

and we get

$$368 = 2^4\left(2^4 + 2^3 - 2^2 + 2^1 + 1\right)$$

$$368 = 2^8 + 2^7 - 2^6 + 2^5 + 2^4.$$

This equation gives the unique way in which the even number $\alpha = 368$ can be written in the form of equation (2.9).

From inequality (2.4) we obtain

$$2^{\nu+1} + 1 \le \Pi \le 2^{\nu+2} - 1$$

$$2^{\nu+1} < 2^{\nu+1} + 1 \le \Pi \le 2^{\nu+2} - 1 < 2^{\nu+2}$$

$$2^{\nu+1} < \Pi < 2^{\nu+2}$$

$$(\nu+1)\ln 2 < \ln \Pi < (\nu+2)\ln 2$$

from which we get

$$\frac{\ln \Pi}{\ln 2} - 1 < \nu + 1 < \frac{\ln \Pi}{\ln 2}$$

and finally

$$\nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right] \tag{2.12}$$

Where $\left[\frac{\ln \Pi}{\ln 2}\right]$ the integer part of $\frac{\ln \Pi}{\ln 2} \in \mathbb{R}$.

We now give the following definition:

Definition 2.1. *We define as the conjugate of the odd* $\Pi \ge 3$,

$$\begin{aligned} &\Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i \\ &\beta_i = \pm 1, i = 0,1,2,........,\nu - 1 \\ &\nu \in \mathbb{N}^* \end{aligned} \tag{2.13}$$

the odd Π^*,

$$\begin{aligned} &\Pi^* = \Pi^*(\nu, \gamma_j) = 2^{\nu+1} + 2^{\nu} + \sum_{j=0}^{\nu-1} \gamma_j 2^j \\ &\gamma_i = \pm 1,\ j = 0,1,2,........,\nu - 1 \\ &\nu \in \mathbb{N}^* \end{aligned} \tag{2.14}$$

for which it is

$$\gamma_k = -\beta_k \forall k = 0,1,2,........,\nu - 1. \tag{2.15}$$

For conjugate odds, the following corollary holds:

Corollary 2.1. *For the conjugate odds* $\Pi = \Pi(\nu, \beta_i) \geq 3$ *and* $\Pi^* = \Pi^*(\nu, \gamma_i)$ *the following hold:*

1. $(\Pi^*)^* = \Pi$. (2.16)

2. $\Pi^* = 3 \cdot 2^{\nu+1} - \Pi$. (2.17)

3. Π *is divisible by* 3 *if and only if* Π^* *is divisible by* 3.

4. *Two conjugate odd numbers cannot have common factors greater than 3.*

Proof. 1. The 1 of the corollary is an immediate consequence of definition 4.1.

2. From equations (2.13), (2.14) and (2.15) we get

$$\Pi + \Pi^* = (2^{\nu+1} + 2^{\nu}) + (2^{\nu+1} + 2^{\nu})$$

and equivalently

$$\Pi + \Pi^* = 3 \cdot 2^{\nu+1}.$$

3. If the odd Π is divisible by 3 then it is written in the form $\Pi = 3x, x = odd$ and from equation (4.17) we get $3x + \Pi^* = 3 \cdot 2^{\nu+1}$ and equivalently $\Pi^* = 3(2^{\nu+1} - x)$. Similarly we can prove the inverse.

4. If $\Pi = xy, \Pi^* = xz$, x, y, z odd numbers, from equation (2.17) we have $x(y+z) = 3 \cdot 2^{\nu+1}$ and consequently $x = 3$. □

From corollary 2.1 we have that 3 is the only odd number which is equal to its conjugate: $3^* = 3 \cdot 2^{0+1} - 3 = 3$. For $\Pi = 1$ the we define

$$1^* = 1. \tag{2.18}$$

From definition 2.1 it follows that the two conjugates are equidistant from the middle $3 \cdot 2^{\nu}$ of the interval $\Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$. Also, from equation

$$(\Pi + X) + (\Pi^* - X) = 3 \cdot 2^{\nu+1} \tag{2.19}$$

it follows that, if the odds $\Pi + X$ and $\Pi - X$, $X = odd$ belong to the interval $\Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$, then they are conjugates

$$(\Pi + X)^* = \Pi^* - X. \tag{2.20}$$

3. The L/R symmetry

We now give the following definition:

Definition 3.1.

1. *The odd number* Π *in equation* (2.1) *has symmetry L when* $\beta_0 = -1$. *We will symbolize with Q the odds with symmetry L.*

2. *The odd number* Π *in equation* (2.1) *has symmetry R when* $\beta_0 = 1$. *We will symbolize with D the odds with symmetry R.*

It can be easily proved that the *Q* odds are written in the form

$$\begin{aligned} &Q = 2^{L+1} \cdot K + 1 \\ &K = odd \end{aligned} \tag{3.1}$$

and the D odds in the form

$$\begin{aligned} &D = 2^{R+1} \cdot K - 1 \\ &K = odd \end{aligned}. \tag{3.2}$$

The natural numbers L and R give the value of the L and R symmetry respectively.

Using Newton's binomial it is easy to prove the following corollary:

Corollary 3.1. *For the symmetric prime numbers A and B with symmetry L or R we have the following:*

1. L(A)<L(B)=>L(AB)=L(A).

2. L(A)<R(B)=>R(AB)=L(A)

3. R(A)<L(B)=>R(AB)=R(A).

4. R(A)<R(B)=>L(AB)=R(A).

5. Symmetry(A)=Symmetry(B)=>Symmetry(AB)>Symmetry(A)=Symmetry(B).

We give two examples.

Example 3.1. L(641)=6<L(114689)=13 =>L(641×114689)=6.

Example 3.2. R(607)= 4<R(16633)=6 =>L(607×16633)=4.

Corollary 4.1 provides the simplest way for the determination of the symmetry of a number. We give two examples.

Example 3.3. For number 18303 we have

$$\begin{aligned} &18303 - 1 = 2^1 \times 9151 \\ &18303 + 1 = 2^7 \times 143 \end{aligned}.$$

Therefore, $R(18303) = 7 - 1 = 6$.

Example 3.4. For the number C1133 which is composite factor of F_{12} with 1133 digits, we have

$$C1133 - 1 = 2^{14} \cdot K.$$

Therefore, $L(C1133) = 14 - 1 = 13$.

Corollary 3.2. *Every composite number* C *of the form*

$$C = 2^{\nu+1} \pm 1$$

has at least two factors the symmetries of which have equal values.

Proof. Corollary 4.2 is a direct consequence of corollary 4.1. □

From definitions 2.1 and 3.1 it emerges that for every conjugate pair $\left(\Pi, \Pi^*\right)$, one is of form Q and the other of form D. Finally, the following equations are easily proven,

$$Q^* = 2^{L+1} K^* - 1 \tag{3.3}$$

$$D^* = 2^{R+1} K^* + 1 \tag{3.4}$$

$$\left(2Q-3\right)^* = 2^{L+2} K^* + 1 \tag{3.5}$$

$$\left(2D+3\right)^* = 2^{R+2} K^* - 1 . \tag{3.6}$$

4. Transpose of odd number

We now give the following definition.

Definition 4.1.

1. *We write the odd D in the form of equation (2.1),*

$$\begin{aligned} & D = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1} 2^{\nu-1} + \beta_{\nu-2} 2^{\nu-2} + \ldots + \beta_1 2^1 + 1 \\ & \nu + 1 = \left[\frac{\ln D}{\ln 2} \right] \end{aligned} . \tag{4.1}$$

We define as transpose $T\left(D\right)$ *of* D *the odd that emerges by inverting the powers of 2 in equation (4.1) and multiplying the resulting number by* $2^{\nu+1}$,

$$T\left(D\right) = \left(\frac{1}{2^{\nu+1}} + \frac{1}{2^{\nu}} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \ldots + \frac{\beta_1}{2^1} + 1 \right) \cdot 2^{\nu+1} = 2^{\nu+1} + 3 + \sum_{k=1}^{\nu-1} \beta_k \cdot 2^{\nu+1-k} . \tag{4.2}$$

2. *We write the odd Q in the form of equation (2.1),*

$$\begin{aligned} & Q = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1} 2^{\nu-1} + \beta_{\nu-2} 2^{\nu-2} + \ldots + \beta_1 2^1 - 1 \\ & \nu + 1 = \left[\frac{\ln Q}{\ln 2} \right] \end{aligned} . \tag{4.3}$$

We define as transpose $T\left(Q\right)$ *of* Q *the opposite of the odd that emerges by inverting the powers of 2 in equation (4.3) and multiplying the resulting number by* $2^{\nu+1}$,

$$T\left(Q\right) = -\left(\frac{1}{2^{\nu+1}} + \frac{1}{2^{\nu}} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \ldots + \frac{\beta_1}{2^1} - 1 \right) \cdot 2^{\nu+1} = 2^{\nu+1} - 3 - \sum_{k=1}^{\nu-1} \beta_k \cdot 2^{\nu+1-k} . \tag{4.4}$$

3. *We set*

$T(1)=1.$ (4.5)

4. *From equations (4.2), (4.4), (4.5) we get the general equation*

$$T(\Pi)=2^{\nu+1}+\beta_0\cdot\left(3+\sum_{k=1}^{\nu-1}\beta_k\cdot 2^{\nu+1-k}\right)$$

$$\nu+1=\left[\frac{\ln\Pi}{\ln 2}\right]. \qquad (4.6)$$

Algorithm for the calculation of the transpose. For the odd Π we calculate $\nu+1=\left[\frac{\ln\Pi}{\ln 2}\right]$ from equation (2.12). Next, applying the algorithm of example 2.1 we write Π in the form of equation (2.1), we calculate $\beta_i=\pm1, i=1,2,3,...,\nu-1$ and the transpose $T(\Pi)$ of Π from equation (4.6).

We now prove three theorems about the transpose of an odd number.

Theorem 4.1.

1. $T(\Pi)=1 \Leftrightarrow \Pi=2^{\nu}-3, \nu\geq 2, \nu\in\mathbb{N}$. (4.7)

2.

$$\left.\begin{array}{l} T(D)=D \\ \nu+1=\left[\frac{\ln D}{\ln 2}\right]\end{array}\right\}\Leftrightarrow\begin{cases}\beta_1=1\\ \beta_{\nu-k}=\beta_{k+1}\\ k=1,2,3,...,\frac{\nu-2}{2}, \nu=even\\ k=1,2,3,...,\frac{\nu-1}{2}, \nu=odd\end{cases} \qquad (4.8)$$

$$\left.\begin{array}{l} T(Q)=Q \\ \nu+1=\left[\frac{\ln Q}{\ln 2}\right]\end{array}\right\}\Leftrightarrow\begin{cases}\beta_1=-1\\ \beta_{\nu-k}=-\beta_{k+1}\\ k=1,2,3,...,\frac{\nu-2}{2}, \nu=even\\ k=1,2,3,...,\frac{\nu-1}{2}, \nu=odd\end{cases} \qquad (4.9)$$

Proof. 1. For $\Pi=2^{\nu}-3$ we get

$$\Pi=2^{\nu}-3$$
$$\Pi=\left(2^{\nu}-1\right)-2=\left(2^{\nu-1}+2^{\nu-2}+2^{\nu-2}+...2^1+1\right)-2$$
$$\Pi=2^{\nu-1}+2^{\nu-2}+2^{\nu-2}+...2^1-1=Q$$

and from equation (4.4) we get $T(\Pi)=1$.

Now, let $T\left(\Pi\right)=1$. The odd Π is either of the form D or of the form Q. We prove the theorem for $\Pi=Q$, and the proof is similar for $\Pi=D$.

For $\Pi=Q$ we get

$$\begin{aligned} &\Pi=Q=2^{n+1}+2^{n}+\beta_{n-1}2^{n-1}+\beta_{n-2}2^{n-2}+...+\beta_{2}2^{2}+\beta_{1}2^{1}-1 \\ &n+1=\left[\frac{\ln\Pi}{\ln 2}\right] \end{aligned} \tag{4.10}$$

and from equation (4.4) we get

$$T\left(\Pi\right)=-1-2^{1}-\beta_{n-1}2^{2}-...-\beta_{2}2^{n}+2^{n+1}$$

so we get

$$\begin{aligned} &T\left(\Pi\right)=1 \\ &-1-2^{1}-\beta_{n-1}2^{2}-...-\beta_{2}2^{n}+2^{n+1}=1 \\ &-1-2^{1}-\beta_{n-1}2^{2}-...-\beta_{2}2^{n}+2^{n+1}+2^{n+2}=2^{n+2}+1 \\ &-1-2^{1}-\beta_{n-1}2^{2}-...-\beta_{2}2^{n}+2^{n+1}+2^{n+2}=2^{n+2}+2^{n}-2^{n-1}-2^{n-2}-...-2^{2}-2^{1}-1 \end{aligned}$$

and taking into account that every odd Π is written in a unique way in the form of equation (2.1) we get

$$\beta_{2}=\beta_{3}=\beta_{3}=...=\beta_{n-1}=+1$$

and from equation (4.10) we get

$$\begin{aligned} &\Pi=2^{n+1}+2^{n}+2^{n-1}...+2^{1}-1 \\ &\Pi=2\left(2^{n}+2^{n-1}+2^{n-2}+...+2^{1}+1\right)-1 \\ &\Pi=2\left(2^{n+1}-1\right)-1 \\ &\Pi=2^{n+2}-3 \end{aligned}$$

and setting $n+2=\nu$ we obtain $\Pi=2^{\nu}-3$.

2. We prove the equivalence (4.8), and (4.9) is similarly proven. We write the odd D in the form of equation (2.1),

$$\begin{aligned} &D=2^{\nu+1}+2^{\nu}+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+...+\beta_{1}2^{1}+1 \\ &\nu+1=\left[\frac{\ln D}{\ln 2}\right] \end{aligned} \quad . \tag{4.11}$$

From equations (4.11), (4.2) we get

$$T\left(D\right)=1+2+\beta_{\nu-1}2^{2}+\beta_{\nu-2}2^{3}+...+\beta_{2}2^{\nu}+2^{\nu+1}. \tag{4.12}$$

We next get

$$T(D)=D \Leftrightarrow$$
$$1+2+\beta_{\nu-1}2^2+\beta_{\nu-2}2^3+\ldots+\beta_2 2^\nu+2^{\nu+1}=2^{\nu+1}+2^\nu+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+\ldots+\beta_2 2^2+\beta_1 2^1+1$$

and taking into account that the odd D is written in a unique way in the form of equation (2.1) we get equivalence (4.8).

Theorem 4.2.

1. $T(D)-T(D^*)=6$. (4.13)

2. $T(Q)-T(Q^*)=-6$. (4.14)

Proof. We prove equation (4.13) and (4.14) is proven similarly. From equation (4.11) we get

$$D^*=2^{\nu+1}+2^\nu-\beta_{\nu-1}2^{\nu-1}-\beta_{\nu-2}2^{\nu-2}-\ldots-\beta_1 2^1-1=Q. \quad (4.15)$$

From equation (4.4) we get

$$T(D^*)=-1-2+\beta_{\nu-1}2^2+\beta_{\nu-2}2^3+\ldots\beta_1 2^\nu+2^{\nu+1}. \quad (4.16)$$

From equation (4.12), (4.16) we obtain $T(D)-T(D^*)=6$. □

Theorem 4.3. *For every odd* Π, $\nu+1=\left[\dfrac{\ln\Pi}{\ln 2}\right]$, $\Pi\in\Omega_\nu=\left[2^{\nu+1},2^{\nu+2}\right]$ *the following inequality holds*

$$T(\Pi)<2^{\nu+2}. \quad (4.17)$$

Proof. We prove inequality (4.17) for the D odds and the proof is similar for the Q odds. From equation (4.12) and taking into account that $\beta_i=\pm1, i=0,1,2,\ldots,\nu-1$ we obtain

$$\begin{gathered}T(D)=1+2+\beta_{\nu-1}2^2+\beta_{\nu-2}2^3+\ldots+\beta_2 2^\nu+2^{\nu+1}\le 1+2+2^2+\ldots+2^\nu+2^{\nu+1}=2^{\nu+2}-1\\ T(D)\le 2^{\nu+2}-1<2^{\nu+2}\end{gathered}. \ □$$

From inequality (4.17) it follows that if an odd Π belongs to the interval $\Omega_\nu=\left[2^{\nu+1},2^{\nu+2}\right]$, its transpose $T(\Pi)$ can be found in intervals $\Omega_n, n\le\nu$ but cannot be found in intervals $\Omega_N, N>\nu$.

5. The odd number octet

We now give the following definitions.

Definition 5.1. *We define as the octet of odd number* Π *the non ordered octet*

$$\left(\Pi,T(\Pi),(T(\Pi))^*,T\left((T(\Pi))^*\right),\Pi^*,T(\Pi^*),(T(\Pi^*))^*,T(T(\Pi^*))\right). \quad (5.1)$$

Definition 5.2.

1. *We define as the* 8^* *transformation of the (5.1) octet, the octet that contains the conjugates of the (5.1) octet.*

2. *We define as the* T_8 *transformation of the (5.1) octet, the octet that contains the transposes of the (5.1) octet.*

Definition 5.3. *We define as symmetric every octet that remains unchanged under the* T_8 *transformation.*

Properties of the symmetric octet. As a consequence of definition 5.3 and the theorems of the previous chapter, every symmetric octet has the following properties:

1. It consists of pairs of conjugate odds (common property of all octets).

2. It is unchanged under the 8^* transformation (common property of all octets).

3. It is unchanged under the T_8 transformation (definition): for every odd Π of the symmetric octet, it is $T\left(T\left(\Pi\right)\right)=\Pi$.

4. It consists of pairs of odds that differ by 6 (consequence of theorem 4.2).

5. It consists of 8 or 4 or 2 different odds.

6. The numbers of the symmetric octet belong to the same interval $\Omega_\nu=\left[2^{\nu+1},2^{\nu+2}\right]$, $\nu+1=\left[\dfrac{\ln\Pi_k}{\ln 2}\right]$, where $\Pi_k, k=1,2,3...,8$ the octet numbers.

7. The smallest odd m of a symmetric octet is always of the form Q, $m=Q$ and the largest M is its conjugate, $M=Q^*$.

8. An odd Π can belong to exactly one symmetric octet (while every odd which belongs to a symmetric octet, also belongs to infinite non symmetric octets, as we shall see later).

In order to determine the octet of an odd Π we use the algorithm for the calculation of the transpose, and equation (2.17) for the calculation of the conjugate. We now give an example which also shows the ways in which we can write a symmetric octet.

Example 5.1. From equation (5.1) we get the symmetric octet in which $\Pi=889$ belongs, (889, 529, 1007, 895, 647, 535, 1001, 641).

In order to discern the pairs of transposes and of conjugates, we write the octet in the form

$$\begin{array}{ccccccc} 889 & \xleftrightarrow{T} & 529 & \xleftrightarrow{*} & 1007 & \xleftrightarrow{T} & 895 \\ *\updownarrow & & & & & & \updownarrow * \\ 647 & \xleftrightarrow{T} & 535 & \xleftrightarrow{*} & 1001 & \xleftrightarrow{T} & 641 \end{array}.$$

Because of equation (2.16) $\left(\Pi^*\right)^*=\Pi$ two conjugates are always connected, in all symmetries, by the symbol $\xleftrightarrow{*}$, $\Pi\xleftrightarrow{*}\Pi^*$. With $\Pi_1\xleftrightarrow{T}\Pi_2$ we denote that

$T(\Pi_1)=\Pi_2$ and $T(\Pi_2)=\Pi_1$. If $T(\Pi_1)=\Pi_2$ and $T(\Pi_2)\neq\Pi_1$, we write $\Pi_1 \xrightarrow{T} \Pi_2$. In our example, $\Pi_1 \xleftrightarrow{T} \Pi_2$ for all of the octet numbers, therefore, it is unchanged under the T_8 transformation and consequently it is symmetric (definition). The 8^* and T_8 transformations only change the relative position of the numbers within the symmetric octet. The octet symmetries are easily seen when we place the numbers on the corners of a regular octagon,

$$\begin{tikzcd}
 & 889 \arrow[r, leftrightarrow, "T"] & 529 \arrow[dr, leftrightarrow, "*"] & \\
647 \arrow[ur, leftrightarrow, "*"] \arrow[d, leftrightarrow, "T"'] & & & 1007 \arrow[d, leftrightarrow, "T"] \\
535 \arrow[dr, leftrightarrow, "*"'] & & & 895 \\
 & 1001 \arrow[r, leftrightarrow, "T"] & 641 \arrow[ur, leftrightarrow, "*"'] &
\end{tikzcd}. \tag{5.2}$$

A symmetric octet can be composed of 8 different numbers, like the one of the previous example, or of 4 different numbers (with the exception of the degenerate octet $(1,1,1,1,1,1,1,1)$ of 1), or of 2 different numbers. From the definitions of the conjugate and the transpose, the following equations are easily proven

$$\begin{aligned}
&2^{\nu}+1 \xleftrightarrow{*} 2^{\nu+1}-1\\
&2^{\nu}+7 \xleftrightarrow{*} 2^{\nu+1}-7\\
&2^{\nu}+1 \xleftrightarrow{T} 2^{\nu+1}-7\\
&2^{\nu}+7 \xleftrightarrow{T} 2^{\nu}+7\\
&2^{\nu+1}-1 \xleftrightarrow{T} 2^{\nu+1}-1
\end{aligned} \tag{5.3}$$

Considering equations (5.3) we get the symmetric octets

$$\begin{aligned}
&\left(2^{\nu+1}+1, 2^{\nu+2}-7, 2^{\nu+1}+7, 2^{\nu+1}+7, 2^{\nu+2}-1, 2^{\nu+2}-1, 2^{\nu+1}+1, 2^{\nu+2}-7\right).\\
&\nu\geq 3, \nu\in\mathbb{N}
\end{aligned} \tag{5.4}$$

The (5.4) symmetric octets consist of 4 different numbers. Fermat numbers for $\nu+1=2^S$ and Mersenne numbers for $\nu+2=p=prime$ belong to the (5.4) symmetric octets. The symmetric octet $(9,9,15,15,15,15,9,9)$ of conjugates $(\Pi,\Pi^*)=(9,15)$ consists of 2 numbers.

Definition 5.4. *We define as non symmetric or asymmetric every octet that contains a pair of conjugates* (Π_1,Π_1^*) *for which*

$$\begin{gathered}\Pi_1 \xrightarrow{T} \Pi_2 \wedge \Pi_2 \xleftrightarrow{T} \Pi_3 \neq \Pi_1 \\ \Pi_1^* \xrightarrow{T} \Pi_4 \wedge \Pi_4 \xleftrightarrow{T} \Pi_5 \neq \Pi_1^*\end{gathered}. \tag{5.5}$$

Asymmetric octets as generators of symmetric octets. An asymmetric octet contains pairs of conjugates that belong in different intervals Ω_ν. At the position of these pairs, "bifurcations" emerge outside the (5.1) octet. At each of these bifurcations, an asymmetric octet "produces" a symmetric one.

We now present one example of an asymmetric octet in which one can see the way in which we can write it so that the asymmetry is evident and so are the symmetric octet that it produces.

Example 5.2. The conjugate pair $(91,101)$ gives the asymmetric octet

$$\begin{array}{ccccccc} & & 47 & & & & \\ & & \updownarrow T & & & & \\ 91 & \xrightarrow{T} & 55 & \xleftrightarrow{*} & 41 & \xleftrightarrow{T} & 41 \\ {}^*\updownarrow & & & & & & \updownarrow^* \\ 101 & \xrightarrow{T} & 49 & \xleftrightarrow{*} & 47 & \xleftrightarrow{T} & 55 \\ & & \updownarrow T & & & & \\ & & 49 & & & & \end{array}.$$

The asymmetric octet has produces the symmetric octet

$$\begin{array}{ccccccc} 47 & \xleftrightarrow{T} & 55 & \xleftrightarrow{*} & 41 & \xleftrightarrow{T} & 41 \\ {}^*\updownarrow & & & & & & \updownarrow^* \\ 49 & \xleftrightarrow{T} & 49 & \xleftrightarrow{*} & 47 & \xleftrightarrow{T} & 55 \end{array}$$

which emerges by replacing the pair of conjugates $(91,101)$ by the pair of conjugates $(47,49)$.

We now prove the following theorem:

Theorem 5.1. (Fundamental octet theorem)

A. *The octets of the numbers*

$$\begin{gathered}\Pi = D = 2^2 \cdot K - 1 \\ \Pi^* = Q = 2^2 \cdot \left(3 \cdot 2^{\nu-1} - K\right) + 1 \\ \nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right] \geq 3, K = odd\end{gathered} \tag{5.6}$$

are asymmetric.

B.

1. *The asymmetric octets are given by the odds*

$$\begin{aligned} D_1 &= 11+8\lambda, \lambda \in \mathbb{N} \\ Q_1 &= 13+8\mu, \mu \in \mathbb{N} \end{aligned} \tag{5.7}$$

except from the asymmetric octet $(5,1,1,1,7,7,5,1)$.

2. *The symmetric octets are given by the odds*

$$\begin{aligned} D_2 &= 7+8l, l \in \mathbb{N}^* \\ Q_2 &= 9+8m, m \in \mathbb{N} \end{aligned}. \tag{5.8}$$

Proof. A. From 2. of theorem 4.1. it follows that for the odds Π, for which $\beta_0\beta_1 = -1$, it holds that $T(\Pi) \neq \Pi$,

$$\beta_0\beta_1 = -1 \Rightarrow T(\Pi) \neq \Pi. \tag{5.9}$$

$D = 2^3 + 2^2 - 2 + 1 = 11$ and $Q = 2^3 + 2^2 + 2 - 1 = 13$ are the smallest odds for which it can be $\beta_0\beta_1 = -1$. Therefore, the theorem holds for $\nu + 1 \geq 3$.

We prove the theorem for $\Pi = D$ and it is similarly proven for $\Pi = Q$. From equation (4.11) for $\beta_0\beta_1 = -1$ we get

$$\begin{aligned} \Pi = D &= 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \ldots + \beta_2 2^2 - 2 + 1 \\ &= 2^2 \cdot \left(2^{\nu-1} + 2^{\nu-2} + \beta_{\nu-1}2^{\nu-3} + \beta_{\nu-2}2^{\nu-3} + \ldots + \beta_2\right) - 2 + 1 \\ &= 2^2 \cdot \left(2^{\nu-1} + 2^{\nu-2} + \beta_{\nu-1}2^{\nu-3} + \beta_{\nu-2}2^{\nu-3} + \ldots + \beta_2\right) - 1 \end{aligned}$$

and setting

$$K = 2^{\nu-1} + 2^{\nu-2} + \beta_{\nu-1}2^{\nu-3} + \beta_{\nu-2}2^{\nu-3} + \ldots + \beta_2 \tag{5.10}$$

we get $D = 2^2 \cdot K - 1$. Considering equation (5.10) we calculate the conjugate of D which is $\Pi^* = Q = 2^2 \cdot \left(3 \cdot 2^{\nu-1} - K\right) + 1$.

B. Equations (5.6) give the consecutive pairs $(D,Q) = (11,13),(19,21),(27,29),\ldots$, that is, equations (5.7). Taking into account the fourth property of the symmetric octets we conclude that the intermediate pairs of odds give the symmetric octets. These pairs are given by equations (5.8). Additionally, equations (5.7) give all odds that produce asymmetric octets (with the exception of the asymmetric octet $(5,1,1,1,7,7,5,1)$) and the equations give all the odds that produce symmetric octets (otherwise the fourth property of the symmetric octets would not hold, which cannot be true due to theorem 4.2)

The octet

$$(5,1,1,1,7,7,5,1) \tag{5.11}$$

of the conjugate pair $(\Pi,\Pi^*) = (5,7)$ is the only asymmetric octet that does not belong to the (5.7) octets, since for this conjugate pair it is

$$\left[\frac{\ln 5}{\ln 2}\right]=\left[\frac{\ln 7}{\ln 2}\right]=\nu+1=2<3.$$

The asymmetric octet (5.11) is given by the terms of the a_n sequence of odds,

$$\begin{aligned}&\gamma_n=3+2^{n+1}\\&n=1,2,3,\ldots\end{aligned}. \tag{5.12}$$

The (5.11) octet emerges from the terms of the γ_n sequence because of equation

$$\begin{aligned}&T(\gamma_n)=T\left(2^{n+1}+3\right)=\gamma_1=7\\&n=1,2,3,\ldots\end{aligned}. \tag{5.13}$$

The terms of the γ_n sequence are of the form D. From equation (4.13) we get

$$T\left(2^{n+1}+3\right)-T\left(\left(2^{n+1}+3\right)^*\right)=6$$

$$T\left(2^{n+1}+3\right)-T\left(2^{n+2}-3\right)=6$$

and with equation (4.7) we obtain

$$T\left(2^{n+1}+3\right)-1=6$$

$$T\left(2^{n+1}+3\right)=7$$

which is equation (5.13). □

For the odds Q, D that belong to a symmetric octet we make the following conjecture:

Conjecture 5.1.

1. *For every odd Q it holds that*

$$\begin{aligned}&T\left(2^n\cdot(Q+3)-3\right)=T(Q)\\&\forall n\in\mathbb{N}^*\end{aligned}. \tag{5.14}$$

2. *For every odd D it holds that*

$$\begin{aligned}&T\left(2^n\cdot(D-3)+3\right)=T(D)\\&\forall n\in\mathbb{N}^*\end{aligned}. \tag{5.15}$$

From conjecture 5.1 and the definition of the symmetric octet, the following corollary emerges directly.

Corollary 5.1.

1. *For every odd Q that belongs to a symmetric octet the sequence*

$$\begin{aligned}&A_n(Q)=2^n\cdot(Q+3)-3\\&n\in\mathbb{N}^*\end{aligned} \tag{5.16}$$

gives (infinite) asymmetric octets that produce the symmetric octet to which Q belongs.

2. *For every odd D that belongs to a symmetric octet the sequence*

$$\begin{aligned} &B_n(D) = 2^n \cdot (D-3) + 3 \\ &n \in \mathbb{N}^* \end{aligned} \tag{5.17}$$

gives (infinite) asymmetric octets that produce the symmetric octet to which D belongs.

The $\mathrm{A}_n(Q)$ sequence has exactly one term in every interval

$$\Omega_{\mathrm{N}} = \left[2^{\mathrm{N}+1}, 2^{\mathrm{N}+2}\right], \mathrm{N} \in \mathbb{N}, \mathrm{N}+1 > \left[\frac{\ln Q}{\ln 2}\right].$$

The $B_n(D)$ sequence has exactly one term in every interval

$$\Omega_{\mathrm{N}} = \left[2^{\mathrm{N}+1}, 2^{\mathrm{N}+2}\right], \mathrm{N} \in \mathbb{N}, \mathrm{N}+1 > \left[\frac{\ln D}{\ln 2}\right].$$

We give an example:

Example 5.3. We pick a number from the symmetric octet (5.2), for example $647 = D$ and a random $n = 20$. From equation (5.17) we get $B_{20}(647) = 2^{20} \cdot (647-3) + 3 = 675282947$. The odd 675282947 gives the asymmetric octet

$$\begin{tikzcd}
 & 647 \arrow[d, leftrightarrow, "T"] & & \\
675282947 \arrow[r, "T"] \arrow[d, leftrightarrow, "*"'] & 535 \arrow[r, leftrightarrow, "*"] & 1001 \arrow[r, leftrightarrow, "T"] & 641 \arrow[d, leftrightarrow, "*"] \\
935329789 \arrow[r, "T"] & 529 \arrow[r, leftrightarrow, "*"] \arrow[d, leftrightarrow, "T"] & 1007 \arrow[r, leftrightarrow, "T"] & 895 \\
 & 889 & &
\end{tikzcd}$$

which produces the symmetric octet

$$\begin{tikzcd}
647 \arrow[r, leftrightarrow, "T"] \arrow[d, leftrightarrow, "*"'] & 535 \arrow[r, leftrightarrow, "*"] & 1001 \arrow[r, leftrightarrow, "T"] & 641 \arrow[d, leftrightarrow, "*"] \\
889 \arrow[r, leftrightarrow, "T"] & 529 \arrow[r, leftrightarrow, "*"] & 1007 \arrow[r, leftrightarrow, "T"] & 895
\end{tikzcd}$$

which is (5.2). $B_{20}(647) = 675282947 \in \Omega_{28}$ belongs to the interval $\Omega_{28} = \left[2^{29}, 2^{30}\right]$.

$B_{21}(647) = 2^{21} \cdot (647-3) + 3 = 1350565891 \in \Omega_{29}$ also produces the (5.2) symmetric octet and belongs to the next interval $\Omega_{29} = \left[2^{30}, 2^{31}\right]$,

$$\begin{array}{ccccccc}
 & & 647 & & & & \\
 & & \updownarrow T & & & & \\
1350565891 & \xrightarrow{T} & 535 & \xleftrightarrow{*} & 1001 & \xleftrightarrow{T} & 641 \\
{}^{*}\updownarrow & & & & & & \updownarrow^{*} \\
1870659581 & \xrightarrow{T} & 529 & \xleftrightarrow{*} & 1007 & \xleftrightarrow{T} & 895 \\
 & & \updownarrow T & & & & \\
 & & 889 & & & &
\end{array}.$$

We now prove the following corollary.

Corollary 5.2.

1. *For every odd* Q *belonging to a symmetric octet, it holds that*

$$\begin{array}{l} T\left(2^{n}\cdot\left(T\left(Q\right)+3\right)-3\right)=T\left(T\left(Q\right)\right)=Q \\ \forall n\in\mathbb{N}^{*} \end{array}. \tag{5.18}$$

2. *For every odd* D *belonging to a symmetric octet, it holds that*

$$\begin{array}{l} T\left(2^{n}\cdot\left(T\left(D\right)-3\right)+3\right)=T\left(T\left(D\right)\right)=D \\ \forall n\in\mathbb{N}^{*} \end{array}. \tag{5.19}$$

Proof. We prove equation (5.18) and the proof of (5.19) is similar. From definition 5.3 of the symmetric octet it follows that if Q belongs to a symmetric octet, then $T\left(Q\right)$ belongs to the same symmetric octet. Therefore, equation (5.14) also holds for $T\left(Q\right)$,

$$\begin{array}{l} T\left(2^{n}\cdot\left(T\left(Q\right)+3\right)-3\right)=T\left(T\left(Q\right)\right) \\ \forall n\in\mathbb{N}^{*} \end{array}$$

and, taking into account that for the symmetric octet it is (definition) $T\left(T\left(Q\right)\right)=Q$ we get equation (5.18). □

Also, applying definition 5.1 for **even numbers** $2^{n}\cdot\Pi, n\in\mathbb{N}^{*}, \Pi=odd$ we obtain

$$\begin{array}{l} T\left(2^{n}\cdot\Pi\right)=T\left(\Pi\right) \\ \forall n\in\mathbb{N}, \Pi=odd \end{array}. \tag{5.20}$$

We give an example:

Example 5.4. $\Pi=Q=2021=2^{10}+2^{9}+2^{8}+2^{7}+2^{6}+2^{5}+2^{4}-2^{3}-2^{2}+2^{1}-1$

$$\begin{array}{l} 2^{n}\Pi=2^{10+n}+2^{9+n}+2^{8+n}+2^{7+n}+2^{6+n}+2^{5+n}+2^{4+n}-2^{3+n}-2^{2+n}+2^{1+n}-2^{n} \\ n\in\mathbb{N} \end{array}$$

$$T\left(2^{n}\cdot\Pi\right)=-\left(\frac{1}{2^{10+n}}+\frac{1}{2^{9+n}}+\frac{1}{2^{8+n}}+\frac{1}{2^{7+n}}+\frac{1}{2^{6+n}}+\frac{1}{2^{5+n}}+\frac{1}{2^{4+n}}-\frac{1}{2^{3+n}}-\frac{1}{2^{2+n}}+\frac{1}{2^{1+n}}-\frac{1}{2^{n}}\right)\cdot 2^{10+n}$$

$$T\left(2^n \cdot \Pi\right) = -\left(\frac{1}{2^{10}} + \frac{1}{2^9} + \frac{1}{2^8} + \frac{1}{2^7} + \frac{1}{2^6} + \frac{1}{2^5} + \frac{1}{2^4} - \frac{1}{2^3} - \frac{1}{2^2} + \frac{1}{2^1} - 1\right) \cdot 2^{10} = T\left(\Pi\right).$$

We now prove the following corollary.

Corollary 5.4. *For every odd A belonging to an asymmetric octet, it holds that*

$$A \in \Omega_\nu \Rightarrow T\left(A\right) \in \Omega_\mu, \mu < \nu. \tag{5.21}$$

Proof. Corollary 5.4 is a direct consequence of theorem 4.3 and definition 5.4. □

6. A Categorization of Odd Numbers

Theorem 2.1 categorizes odd numbers according to two criteria. An odd number can either be of form Q or of form D. An odd number either belongs to exactly one symmetric octet or it does not belong to any symmetric octet. The categorization of the odds finally comes from theorem 5.1 and equation (5.12). From theorem 5.1 and the first of equations (5.8) it follows that the odd Π is of form D_2 if and only if the difference $\Pi - 7$ is a multiple of 8, $\Pi - 7 = 8l, l \in \mathbb{N}^*$. Similarly, from equations (5.7), (5.8) and (5.12) we determine the category of odds to which Π belongs.

From corollary 3.1. we have that there exist three kinds of composite odd numbers,

$$\Pi = Q = Q_1 Q_2 \tag{6.1}$$

$$\Pi = Q = D_1 D_2 \tag{6.2}$$

$$\Pi = D = Q_1 D_1. \tag{6.3}$$

Taking into account that the odd numbers in the right sides of equations (6.1), (6.2), (6.3) belong either to symmetric octets or to asymmetric ones we have

$$3 \times \left(2 \times 2\right) = 12 \tag{6.4}$$

possible forms for the composite odd numbers. It follows from Lucas theorem for the Fermat numbers [1] that the factors of the Fermat numbers are of the form (6.1). Also, there are two types of pair conjugates,

$$\begin{aligned} \Pi &= Q = Q_1 Q_2 \\ \Pi^* &= D = Q_3 D_1 \end{aligned} \tag{6.5}$$

$$\begin{aligned} \Pi &= Q = D_1 D_2 \\ \Pi^* &= D = Q_1 D_3 \end{aligned}. \tag{6.6}$$

Entanglement of the odds D_1, Q_1, D_2, Q_2 of theorem 5.1. From theorem 5.1 we obtain

$$\begin{aligned} &\left(5 + 8n\right) Q_1 = Q_2 \\ &n = 0, 1, 2, \ldots \end{aligned} \tag{6.7}$$

$$(3+8n)Q_1 = D_2 \quad n = 0,1,2,... \tag{6.8}$$

$$(5+8n)D_1 = D_2 \quad n = 0,1,2,... \tag{6.9}$$

$$(3+8n)D_1 = Q_2 \quad n = 0,1,2,... \tag{6.10}$$

and

$$(5+8n)Q_2 = Q_1 \quad n = 0,1,2,... \tag{6.11}$$

$$(3+8n)Q_2 = D_1 \quad n = 0,1,2,... \tag{6.12}$$

$$(5+8n)D_2 = D_1 \quad n = 0,1,2,... \tag{6.13}$$

$$(3+8n)D_2 = Q_1. \quad n = 0,1,2,... \tag{6.14}$$

7. An odd number factorization algorithm

We now give an algorithm for the factorization of odd numbers [2-6]. The algorithm always factorizes the pair of conjugate numbers

$$(\Pi, \Pi^*) = (Q, D = Q^*). \tag{7.1}$$

The factorization algorithm for odd numbers is based on the following conjecture.

Conjecture 7.1. There exist factors of

$$A_n(Q) = 2^n \cdot (Q+3) - 3 \quad n \in \mathbb{N}^* \tag{7.2}$$

$$B_n(D) = 2^n \cdot (D-3) + 3 \quad n \in \mathbb{N}^* \tag{7.3}$$

and factors of the odds Π and Π^* which belong in the same octet.

We applied the algorithm in the following two examples.

Example 7.1. We apply the algorithm to factorize the first composite Fermat number, $F_5 = 2^{32} + 1$ which belongs to a symmetric octet (equation (5.4)). The conjugate of $F_5 = Q$ is $F^* = 2^{33} - 1$ so from equations (7.2), (7.3) we obtain

$$A_n\left(2^{32}+1\right)=2^n\cdot\left(2^{32}+1+3\right)-3=2^{n+2}\cdot\left(2^{30}+1\right)-3 \qquad (7.4)$$
$$n=1,2,3,\ldots$$

$$B_n\left(2^{33}-1\right)=2^n\cdot\left(2^{33}-1-3\right)+3=2^{n+2}\cdot\left(2^{31}-1\right)+3. \qquad (7.5)$$
$$n=1,2,3,\ldots$$

For $n=4$ we get

$$A_4\left(2^{32}+1\right)=23^2\times 129904493.$$

The number $\Theta=23^2=529=9+8\times 65$ belongs to the symmetric octet

$$\begin{array}{ccccccc} 529 & \xleftrightarrow{T} & 889 & \xleftrightarrow{*} & 647 & \xleftrightarrow{T} & 535 \\ {}^{*}\updownarrow & & & & & & \updownarrow^{*} \\ 1007 & \xleftrightarrow{T} & 895 & \xleftrightarrow{*} & 641 & \xleftrightarrow{T} & 1001 \end{array}.$$

Trying the numbers $Q=529,889,1001,641$ (the factors of Fermat numbers are of the form Q [1]) we find that the number $Q=641$ is a factor of F_5. Hence we get $F_5=2^{32}+1=641\times 6700417$.

The application of the algorithm in sequence (7.5) gives again the octet of $529=23^2$ for $n=11$, $B_{11}\left(2^{33}-1\right)=11\times 23^2\times 89^2\times 381673$.

Example 7.2. We apply the algorithm to factorize the $\Pi=589$ which belonging to an asymmetric octet (see equation (5.7), $589-13=8\times 72$ or corollary 5.4, $589\in\Omega_8, T\left(589\right)=217\in\Omega_6$). The conjugate of 589 is 947,

$$\left(\Pi,\Pi^*\right)=\left(589,947\right),$$

so from equations (7.2), (7.3) we obtain

$$A_n\left(589\right)=2^n\cdot\left(589+3\right)-3 \qquad (7.6)$$
$$n=1,2,3,\ldots$$

$$B_n\left(947\right)=2^n\cdot\left(947-3\right)+3. \qquad (7.7)$$
$$n=1,2,3,\ldots$$

For $n=1$ in equations (7.6), (7.7) we get

$$A_1\left(589\right)=2^1\cdot\left(589+3\right)-3=1181=\mathit{prime}$$

$$B_1\left(947\right)=2^1\cdot\left(947-3\right)+3=1891=31\times 61.$$

Number $D=31$ is a factor of 589.

For $n=2$ in equations (7.6), (7.7) we get

$$A_2\left(589\right)=2^2\cdot\left(589+3\right)-3=2365 \;=\; 5\times 11\times 43$$

$B_2(947) = 2^2 \cdot (947-3) + 3 = 3779 = prime$.

The number $A = 43$ belongs to the asymmetric octet

$$
\begin{array}{ccccccc}
 & & 23 & & & & \\
 & & \updownarrow T & & & & \\
43 & \xrightarrow{T} & 23 & \xleftrightarrow{*} & 25 & \xleftrightarrow{T} & 17 \\
{}^{*}\updownarrow & & & & & & \updownarrow^{*} \\
53 & \xrightarrow{T} & 17 & \xleftrightarrow{*} & 31 & \xleftrightarrow{T} & 31 \\
 & & \updownarrow T & & & & \\
 & & 25 & & & &
\end{array}.
$$

Trying the numbers of the octet we find that the number $D = 31$ is a factor of 589.

For $n = 4$ in equations (7.6), (7.7) we get

$A_4(589) = 2^4 \cdot (589+3) - 3 = 9469 = 17 \times 557$

$B_4(947) = 2^4 \cdot (947-3) + 3 = 15107 = prime$.

The number $\Theta = 17$ belongs to the symmetric octet

$$
\begin{array}{ccccccc}
17 & \xleftrightarrow{T} & 25 & \xleftrightarrow{*} & 23 & \xleftrightarrow{T} & 23 \\
{}^{*}\updownarrow & & & & & & \updownarrow^{*} \\
31 & \xleftrightarrow{T} & 31 & \xleftrightarrow{*} & 17 & \xleftrightarrow{T} & 25
\end{array}. \tag{7.8}
$$

Trying the numbers of the octet we find that the number $D = 31$ is a factor of 589.

The odd number factorization algorithm emerges from the entanglement (through the sequences (5.16), (5.17)) initially of asymmetric octets, and then of the symmetric ones they produce. Here we used this octet entanglement for the factorization of the odds.

The algorithm finds the factors of the odd Π bypass the factorization process of Π. The factorization of odd numbers is a fundamental topic in number theory. For this reason, we proceed to publication of this paper before the Conjectures 5.1 and 7.1 to be proved. So we don’t know how difficult their proofs are. Another reason for accelerating the process of the publication of this paper is that my computer is not powerful enough to run the algorithm for very large composite numbers.

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