The EPR-Bohm Experiment from a Classical Viewpoint

P. Driessen, 190 rue Berthelot, 1190 Forest, Belgium.
email: patrick.driessen@infinitesolutions.be

Abstract

The EPR-Bohm experiment is studied in a classical model with hidden variables. The main hypothesis of Bell about this experiment must be modified in a non-trivial way to allow for the fact that our model is non-linear and so non fully deterministic. With our classical model, we then arrive at the same angular behaviour than the one that is predicted by Quantum Mechanics for the correlated measurement of spin projections on the polarizer axis. If the spin-spin interaction predicted by our particle model, or more precisely in the present case, the spin-polarizer field axis interaction, is not able to align the spin axis with the polarizer axis, then our classical result differs by a factor 3 from the quantum mechanical result. It is surprising to find that the angular behaviour is easier to reproduce by a classical model than the amount of probability. Naively, one would have expected the reverse.

1 Introduction

The EPR-Bohm Paradox is one of the most mysterious feature of Quantum Mechanics (QM). It examines the correlations of spin measurements of two intricated particles simultaneously emitted in a singlet state with opposite spins.

We are going to study this system from a classical standpoint. In particular we will see if we can approach the quantum correlations by using a spin model with hidden variables (HV's) which is at the same time realistic, separable and local but non-linear and non-fully deterministic. Our approach is part of a particle model [1] which posits the existence of a new spin-spin interaction. This will be important for the development below.

2 Quantum Calculation

In QM the combined measurement of spins is expressed as follows :

$$\langle \hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}} \rangle_{\psi} = \langle \psi | \boldsymbol{S_1} \cdot \hat{\boldsymbol{a}} \;\; \boldsymbol{S_2} \cdot \hat{\boldsymbol{b}} | \psi \rangle \tag{2.1}$$

where the observables $\boldsymbol{S_1} \cdot \hat{\boldsymbol{a}}$ and $\boldsymbol{S_2} \cdot \hat{\boldsymbol{b}}$ measure spin projections in directions $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$. With $\boldsymbol{S} = \left(\frac{\hbar}{2}\right) \boldsymbol{\sigma}$ it gives :

$$\langle \hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}} \rangle_{\psi} = \left(\frac{\hbar}{2}\right)^2 \langle \psi | \boldsymbol{\sigma_1} \cdot \hat{\boldsymbol{a}} \;\; \boldsymbol{\sigma_2} \cdot \hat{\boldsymbol{b}} | \psi \rangle \tag{2.2}$$

Particles are produced in a singlet state for the spin wave function:

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}} \{ |S^1{}_{\uparrow}\rangle \otimes |S^2{}_{\downarrow}\rangle - |S^1{}_{\downarrow}\rangle \otimes |S^2{}_{\uparrow}\rangle \} \\ &= \frac{1}{\sqrt{2}} \{ |1, \uparrow\rangle |2, \downarrow\rangle - |1, \downarrow\rangle |2, \uparrow\rangle \} \end{aligned} \tag{2.3}$$

which gives :

$$\langle \hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}} \rangle_{\psi} = \left(\frac{\hbar}{2}\right)^2 \frac{1}{\sqrt{2}} \{\langle 1, \uparrow | \langle 2, \downarrow | - \langle 1, \downarrow | \langle 2, \uparrow | \} \{\boldsymbol{\sigma_1} \cdot \hat{\boldsymbol{a}} \ \boldsymbol{\sigma_2} \cdot \hat{\boldsymbol{b}}\} \frac{1}{\sqrt{2}} \{|1, \uparrow\rangle |2, \downarrow\rangle - |1, \downarrow\rangle |2, \uparrow\rangle\} \tag{2.4}$$

Let us project the vectors in a cartesian basis :

$$\begin{aligned} \boldsymbol{\sigma} &= \sigma_x \boldsymbol{i} + \sigma_y \boldsymbol{j} + \sigma_z \boldsymbol{k} \\ \boldsymbol{a} &= a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k} \\ \boldsymbol{b} &= b_x \boldsymbol{i} + b_y \boldsymbol{j} + b_z \boldsymbol{k} \end{aligned} \tag{2.5}$$

One finds :

$$\langle \hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}} \rangle_{\psi} = -\left(\frac{\hbar}{2}\right)^2 \{a_x b_x + a_y b_y + a_z b_z\} = -\left(\frac{\hbar}{2}\right)^2 \hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}} \tag{2.6}$$

3 Bell’s theorem

Bell’s theorem (see for example [2]) provides an inequality which is common to all local deterministic HV theories, but which is violated by QM measurements. In other words, it implies that no local deterministic HV theory can predict QM measurements.

Bell [3,4] supposes the existence of HV’s designated collectively by λ, so that the measurement of spin $A(\hat{\boldsymbol{a}}, \lambda)$ in direction $\hat{\boldsymbol{a}}$ for the particles received by Alice do depend only on $\hat{\boldsymbol{a}}$ and λ. The measurements effected by Bob on the second beam are similarly referred to as $B(\hat{\boldsymbol{b}}, \lambda)$. The separability principle posits that there can be no dependence between two systems which do not interact. One can formulate this principle mathematically as :

$$P(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \lambda) = A(\hat{\boldsymbol{a}}, \lambda) B(\hat{\boldsymbol{b}}, \lambda) \tag{3.1}$$

where A et B are two deterministic functions. For spinors in QM the result of a local spin measurement is $\pm\frac{\hbar}{2}$. In classical theory, one can always measure the spin length along an axis, and the result must be the same except for a constant. One has thus :

$$A(\hat{\boldsymbol{a}}, \lambda) = \pm\beta \frac{\hbar}{2} \quad ; \quad B(\hat{\boldsymbol{b}}, \lambda) = \pm\beta \frac{\hbar}{2} \tag{3.2}$$

where β is a constant. The λ dependence in the right-hand side is contained in the signs. If p_λ is the probability definition of λ, then the simultaneous measurements of spins by Alice and Bob can be computed by an expression of the type :

$$(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}) = \int d\lambda \, p_\lambda \, A(\hat{\boldsymbol{a}}, \lambda) B(\hat{\boldsymbol{b}}, \lambda) \tag{3.3}$$

Indeed, since λ is hidden, one cannot measure its precise value. One can then reach the measurement of an observable only by integrating on all possible values of λ. The ponderation coefficient p_λ represents the probability to observe the configuration characterized by a particular value of λ. The total probability must be normalized to unity and thus :

$$\int d\lambda \;\; p_\lambda = 1 \tag{3.4}$$

When $\widehat{\boldsymbol{a}}$ and $\widehat{\boldsymbol{b}}$ are parallel, QM gives the following result :

$$\langle \widehat{\boldsymbol{a}} \cdot \widehat{\boldsymbol{a}} \rangle_\psi = -\left(\frac{\hbar}{2}\right)^2 \widehat{\boldsymbol{a}} \cdot \widehat{\boldsymbol{a}} = -\left(\frac{\hbar}{2}\right)^2 \tag{3.5}$$

that is; perfect anti-correlation. This is a verified experimental effect, and hence one can expect that this should still be true for a classical model (possibly up to a multiplicative constant β^2). But one can have $(\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{a}}) = -\beta^2 \left(\frac{\hbar}{2}\right)^2$ only if $A(\widehat{\boldsymbol{a}}, \lambda) = -B(\widehat{\boldsymbol{a}}, \lambda)$ for all direction $\widehat{\boldsymbol{a}}$. One can then write, starting from (3.3) :

$$\left(\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}\right) = -\int d\lambda \, p_\lambda \, A(\widehat{\boldsymbol{a}}, \lambda) A\left(\widehat{\boldsymbol{b}}, \lambda\right) \tag{3.6}$$

Let us now consider two possible orientations $\widehat{\boldsymbol{b}}$ and $\widehat{\boldsymbol{c}}$ for Bob's apparatus. One computes :

$$\begin{aligned}\left(\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}\right) - (\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{c}}) &= -\int d\lambda \, p_\lambda \, A(\widehat{\boldsymbol{a}}, \lambda) A\left(\widehat{\boldsymbol{b}}, \lambda\right) + \int d\lambda \, p_\lambda \, A(\widehat{\boldsymbol{a}}, \lambda) A(\widehat{\boldsymbol{c}}, \lambda) \\ &= -\int d\lambda \, p_\lambda \, A(\widehat{\boldsymbol{a}}, \lambda) A\left(\widehat{\boldsymbol{b}}, \lambda\right) \left(1 - A(\widehat{\boldsymbol{c}}, \lambda) A\left(\widehat{\boldsymbol{b}}, \lambda\right) \left(\frac{2}{\beta\hbar}\right)^2\right)\end{aligned} \tag{3.7}$$

where we have used the fact that $\left(A\left(\widehat{\boldsymbol{b}}, \lambda\right)\right)^2 = \left(\beta \frac{\hbar}{2}\right)^2$. Let us then compute the absolute value of both members :

$$\left|\left(\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}\right) - (\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{c}})\right| = \left|\int d\lambda \, p_\lambda \, A(\widehat{\boldsymbol{a}}, \lambda) A\left(\widehat{\boldsymbol{b}}, \lambda\right) \left(1 - A(\widehat{\boldsymbol{c}}, \lambda) A\left(\widehat{\boldsymbol{b}}, \lambda\right) \left(\frac{2}{\beta\hbar}\right)^2\right)\right| \tag{3.8}$$

But one can also write, with (3.2) :

$$A(\widehat{\boldsymbol{a}}, \lambda) A\left(\widehat{\boldsymbol{b}}, \lambda\right) = \pm \left(\beta \frac{\hbar}{2}\right)^2 \tag{3.9}$$

And so (3.8) becomes :

$$\begin{aligned}\left|\left(\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}\right) - (\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{c}})\right| &\leq \int d\lambda \, p_\lambda \left|A(\widehat{\boldsymbol{a}}, \lambda) A\left(\widehat{\boldsymbol{b}}, \lambda\right)\right| \left|\left(1 - A(\widehat{\boldsymbol{c}}, \lambda) A\left(\widehat{\boldsymbol{b}}, \lambda\right) \left(\frac{2}{\beta\hbar}\right)^2\right)\right| \\ &\leq \left(\beta \frac{\hbar}{2}\right)^2 \int d\lambda \, p_\lambda \left|\left(1 - A(\widehat{\boldsymbol{c}}, \lambda) A\left(\widehat{\boldsymbol{b}}, \lambda\right) \left(\frac{2}{\beta\hbar}\right)^2\right)\right| \\ &\leq \left(\beta \frac{\hbar}{2}\right)^2 - \int d\lambda \, p_\lambda \left(A(\widehat{\boldsymbol{c}}, \lambda) A\left(\widehat{\boldsymbol{b}}, \lambda\right)\right) \\ &\leq \left(\beta \frac{\hbar}{2}\right)^2 + \left(\widehat{\boldsymbol{b}}, \widehat{\boldsymbol{c}}\right)\end{aligned} \tag{3.10}$$

To simplify the writing, let us take $\beta \frac{\hbar}{2}$ as length unity, that is $\beta \frac{\hbar}{2} \equiv 1$. One has thus :

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$$\boxed{\left|\left(\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}\right)-\left(\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}\right)\right|\leq 1+\left(\hat{\boldsymbol{b}},\hat{\boldsymbol{c}}\right)} \tag{3.11}$$

It is under this form that Bell's inequality is usually given.

4 Classical Models of Spin Measurements

Several authors have tried to construct classical models which approach as closely as possible the quantum correlations. For example [5] contains several adjustable parameters allowing to approach the theoretical curve as close as possible (see picture below). The agreement is rather impressive.

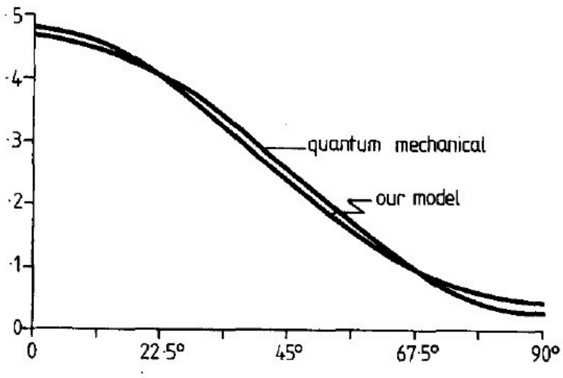

Figure 1: Adjustment of Marshall's classical calculation [5] with QM prediction.

I will base the following computation on an entirely classical spin model [1] and classical spin measurement model. First, I will suppose that the spin effectively corresponds to a proper rotation of the particle around its axis. In the classical model, the spinning particle behaves like a spinning-top and tends to conserve its spin orientation by the gyroscopic effect, as long as no couple is applied.

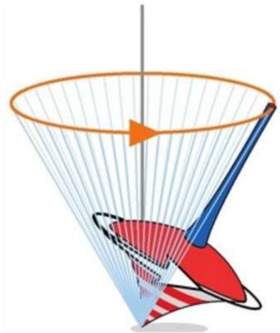

Figure 2: Due to the dipolar magnetic interaction the spin precess around the field axis.

In the quantum framework the spin measurement always gives $\pm\hbar/2$, whatever the measurement axis. To explain that, one has just to imagine that the measuring apparatus (for example the magnetic field in the Stern-Gerlach device) breaks the symmetry of space and is able to act on the spin direction via a couple which forces it to precess around the apparatus symmetry axis. It is the dipolar magnetic interaction between the magnetic moment associated with the spinor and the magnetic field which forces the spin to precess. Thus the direction of spin makes an angle with the field axis which is such that the spin length

$$S = \sqrt{s(s+1)}\hbar = \sqrt{\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)}\hbar = \frac{\hbar}{2}\sqrt{3} \tag{4.1}$$

is compatible with the projection $\hbar/2$ on the field axis.

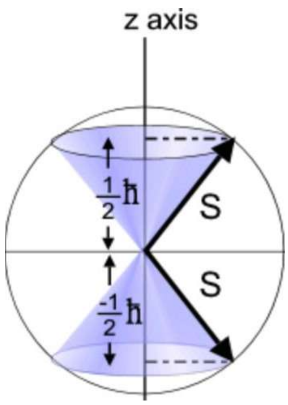

Figure 3: The spin and its projection on the field axis.

We suppose that there is another non-linear spin-spin or spin-magnetic field interaction, yet to be discovered, which would be able to align the spin-direction with the field axis direction. In this case the spin projection would be $\frac{\hbar}{2}\sqrt{3}$. The classical interpretation of the Aspect experiment would then be: the two particles with spin ½ are simultaneously produced with anti-correlated spins whose common axis can take any direction. This orientation is conserved (conservation of the total angular momentum) whatever the particle separation so long as they propagate in the vacuum. When particle 1 arrives at the measuring apparatus *A*, it has a spin whose direction makes an angle with the polarizer axis, say α_A. Due to the action of the polarizer, the spin aligns itself with the polarizer axis or precesses around it.

5 Correlation Measurements revisited

Bell's theory rests on a very simple law :

$$(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}) = \int d\lambda\, p_\lambda\, A(\hat{\boldsymbol{a}}, \lambda) B(\hat{\boldsymbol{b}}, \lambda) \tag{5.1}$$

where $(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})$ is the classical average value of the correlated measurements of the spins along the polarizer's direction and where λ represents the set of hidden variables. But this way of representing $(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})$ is correct only for the class of classical linear theories which are purely deterministic. But it seems to be too restrictive for the larger class of non-linear theories.

In such non-linear systems, very sensible to perturbations, one can only give a probability p_i for the system to be in state *i*. Hence the equation giving the average value of correlated measurements (5.1) becomes more complex :

$$(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}) = \int d\lambda\, p_\lambda \sum_{i=1}^{2} \sum_{j=1}^{2} p_i^1(\lambda)\, S_i^1(\hat{\boldsymbol{a}})\, p_j^2(\lambda)\, S_j^2(\hat{\boldsymbol{b}})$$

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$$= \sum_{i=1}^{2} \sum_{j=1}^{2} S_i^1(\widehat{\boldsymbol{a}})\, S_j^2(\widehat{\boldsymbol{b}}) \int d\lambda\, p_\lambda\, p_i^1(\lambda)\, p_j^2(\lambda) \tag{5.2}$$

In the expression above, p_i^1 denotes the probability to measure a particular configuration (parallel or anti-parallel) of the spin 1 with respect to the polarizer axis $\widehat{\boldsymbol{a}}$. Similarly, p_j^2 relates to the spin 2 configuration with respect to $\widehat{\boldsymbol{b}}$. The i index refers to all possible configurations of spin 1 after measurements. As we shall see later, its two values correspond to the following possibilities:

$$\begin{cases} i = 1 & no\ spin\ flip \\ i = 2 & spin\ flip \end{cases}$$

And the same for spin 2. Of course we must have :

$$\sum_{i=1}^{2} p_i^1 = 1 \quad ; \sum_{j=1}^{2} p_j^2 = 1 \tag{5.3}$$

In particular, for the spin measurements which interest us here, the hidden variables are the angles (θ, φ) which fix the initial direction of the spin common axis and one has for (5.2) :

$$(\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\ p(\theta,\varphi) \sum_{i=1}^{2} \sum_{j=1}^{2} p_i^1(\theta,\varphi)\ S_i^1(\widehat{\boldsymbol{a}})\, p_j^2(\theta,\varphi)\, S_j^2(\widehat{\boldsymbol{b}}) \tag{5.4}$$

where $p(\theta,\varphi)$ represents the probability that the axis which supports the two anti-correlated spins be oriented in the direction (θ,φ). But all the axis direction are equiprobable and so $p(\theta,\varphi) = p = constant$. To determine this constant, we remember that the total probability must be normalized, that is :

$$\int d\lambda \quad p(\theta,\varphi) = 1$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\, p = 4\pi p = 1 \quad \rightarrow\ p = \frac{1}{4\pi} \tag{5.5}$$

$S_i^1(\widehat{\boldsymbol{a}})$ is the measure of spin $\hat{s}_1$, in the direction $\widehat{\boldsymbol{a}}$, of the state (designated by index i) which the spin $\hat{s}_1$ takes after interacting with the polarizer. Since in our model the spins may eventually terminate in positions parallel or anti-parallel to the polarizer axis, one has :

$$S_1^1(\widehat{\boldsymbol{a}}) = +\frac{\hbar}{2}\sqrt{3} \quad ; S_2^1(\widehat{\boldsymbol{a}}) = -\frac{\hbar}{2}\sqrt{3}$$

$$S_1^2(\widehat{\boldsymbol{b}}) = +\frac{\hbar}{2}\sqrt{3} \quad ; S_2^2(\widehat{\boldsymbol{b}}) = -\frac{\hbar}{2}\sqrt{3} \tag{5.6.a}$$

If the spins precess but do not align, we have :

$$S_1^1(\widehat{\boldsymbol{a}}) = +\frac{\hbar}{2} \quad ; S_2^1(\widehat{\boldsymbol{a}}) = -\frac{\hbar}{2}$$

(5.6.b)

$$S_1^2(\hat{\boldsymbol{b}}) = +\frac{\hbar}{2} \quad ; \; S_2^2(\hat{\boldsymbol{b}}) = -\frac{\hbar}{2}$$

One difficulty is to find plausible values for the probabilities $p_i^1(\theta,\varphi), p_j^2(\theta,\varphi)$. One can reasonably suppose that $p_i^1(\theta,\varphi)$ depends only on the angle α_A between $\hat{s}_1$ (fixed in direction by (θ,φ)) and $\hat{\boldsymbol{a}}$. It must evolve between 0 and 1 and be equal to 1 when the direction of $\hat{s}_1$coincides with that of $\hat{\boldsymbol{a}}$. One is easily convinced that, when α_A is acute, the following can be used :

$$p_1^1 = \left(\frac{1+\cos\alpha_A}{2}\right) = cos^2\left(\frac{\alpha_A}{2}\right) \tag{5.7}$$

In the obtuse case, the probability becomes :

$$p_2^1 = 1 - \left(\frac{1+\cos\alpha_A}{2}\right) = \left(\frac{1-\cos\alpha_A}{2}\right) = sin^2\left(\frac{\alpha_A}{2}\right) \tag{5.8}$$

We are now able to compute $(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})$.

The state before measurement is shown below :

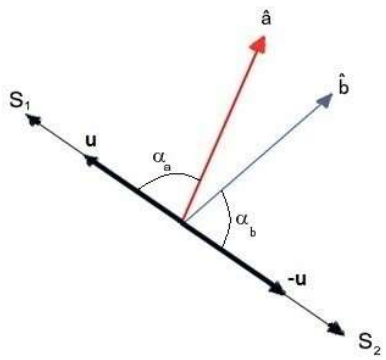

Figure 4: Original positions of the spins and the polarizers axis.

One supposes that in this state, that is just after emission of the particle pair, spin $\hat{s}_1$is oriented along $\hat{u}(\theta,\varphi)$ and spin $\hat{s}_2$ in the anti-parallel direction, namely $\hat{u}(\pi-\theta,\varphi+\pi)$. We call α_A the measure of the angle between $\hat{a}$ and the axis $\hat{u}(\theta,\varphi)$. Similarly, we call α_B the measure of the angle between $\hat{b}$ and the axis $\hat{u}(\pi-\theta,\varphi+\pi) = -\hat{u}(\theta,\varphi)$.

When one interposes the polarizer fields, the respective spins which are initially anti-correlated will reorient themselves to fall in positions where they are aligned with the field axis (or precess around it). But they will fall in one of the two possible orientations (parallel or anti-parallel) with a probability which only depends on the angle between their initial position and the polarizer field axis, as we have shown above. We are thus going to make a great number of measurements with pairs oriented differently, and we will have, after measurements, four possible situations corresponding to the four graphs below :

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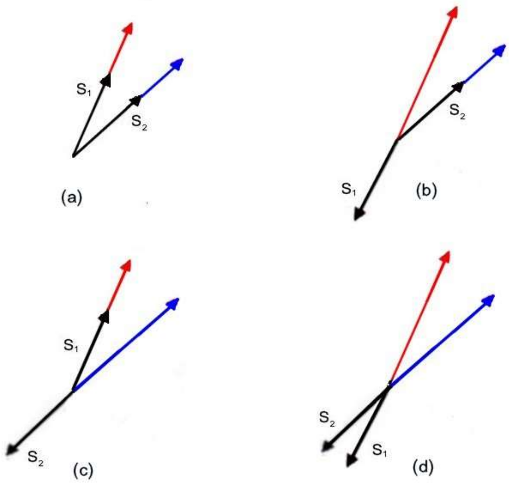

Figure 5: The four possible arrangements of the spins with respect to the polarizers axis after dipole and spin-spin interaction.

They correspond in fact respectively to the following cases :

- Case (a) : no spin flip.
- Case (b) : flip of spin 1.
- Case (c) : flip of spin 2.
- Case (d) : flip of spins 1 and 2.

Let us suppose, without loss of generality, that for the example considered here, both angles α_A and α_B are acute (one can reproduce the calculations by supposing that one of the two angles is obtuse, or both are, and one can check that this has no influence on the final result).

Let us consider initially a single initial position (θ, φ) for the common spin axis and let us sum the results of measurements by affecting to each case the adequate probability coefficient. One finds :

$$\sum_{i=1}^{2} \sum_{j=1}^{2} p_i^1(\theta,\varphi)\ S_i^1(\hat{\boldsymbol{a}})\ p_j^2(\theta,\varphi)\ S_j^2(\hat{\boldsymbol{b}}) =$$

$$\begin{aligned} &\left(\frac{1+\cos\alpha_A}{2}\right)\left(+\sqrt{3}\frac{\hbar}{2}\right)\left(\frac{1+\cos\alpha_B}{2}\right)\left(+\sqrt{3}\frac{\hbar}{2}\right) \\ &+\left(\frac{1-\cos\alpha_A}{2}\right)\left(-\sqrt{3}\frac{\hbar}{2}\right)\left(\frac{1+\cos\alpha_B}{2}\right)\left(+\sqrt{3}\frac{\hbar}{2}\right) \\ &+\left(\frac{1+\cos\alpha_A}{2}\right)\left(+\sqrt{3}\frac{\hbar}{2}\right)\left(\frac{1-\cos\alpha_B}{2}\right)\left(-\sqrt{3}\frac{\hbar}{2}\right) \\ &+\left(\frac{1-\cos\alpha_A}{2}\right)\left(-\sqrt{3}\frac{\hbar}{2}\right)\left(\frac{1-\cos\alpha_B}{2}\right)\left(-\sqrt{3}\frac{\hbar}{2}\right) \end{aligned} \tag{5.9}$$

$$= 3\left(\frac{\hbar}{2}\right)^2 \cos\alpha_A \cos\alpha_B$$

This result clearly demonstrates that, in our case of a non-fully deterministic theory, it is Bell's basic assumption $A(\hat{\boldsymbol{a}}, \lambda) = \pm\beta\frac{\hbar}{2}$ which is violated. We will also show in the next chapter that it is because we are violating this condition that we will be able to reproduce the correct angular behaviour of $(\hat{\boldsymbol{a}},\hat{\boldsymbol{b}})$.

6 Integration on Hidden Variables

We must next integrate this result on all possible orientations (θ,φ) of the initial spin axis. But to do this we must first express α_A and α_B as a function of (θ,φ). One has :

$$\begin{aligned}\cos\alpha_A &= \hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{u}}(\theta,\varphi) = \left(a_x\boldsymbol{i}+a_y\boldsymbol{j}+a_z\boldsymbol{k}\right)\cdot(\sin\theta\cos\varphi\,\boldsymbol{i}+\sin\theta\sin\varphi\,\boldsymbol{j}+\cos\theta\,\boldsymbol{k})\\ &= \left(a_x\sin\theta\cos\varphi+a_y\sin\theta\sin\varphi+a_z\cos\theta\right)\end{aligned} \tag{6.1}$$

Similarly :

$$\begin{aligned}\cos\alpha_B &= \hat{\boldsymbol{b}}\cdot\hat{\boldsymbol{u}}(\pi-\theta,\varphi+\pi) = -\hat{\boldsymbol{b}}\cdot\hat{\boldsymbol{u}}(\theta,\varphi)\\ &= -\left(b_x\boldsymbol{i}+b_y\boldsymbol{j}+b_z\boldsymbol{k}\right)\cdot(\sin\theta\cos\varphi\,\boldsymbol{i}+\sin\theta\sin\varphi\,\boldsymbol{j}+\cos\theta\,\boldsymbol{k})\\ &= -\left(b_x\sin\theta\cos\varphi+b_y\sin\theta\sin\varphi+b_z\cos\theta\right)\end{aligned} \tag{6.2}$$

The statistical result of correlated measurements of the spins along the polarizers field axis will thus be, with (5.9), (6.1) and (6.2) :

$$\begin{aligned}(\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}) &= \frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\,\sin\theta\;3\left(\frac{\hbar}{2}\right)^2\cos\alpha_A\cos\alpha_B\\
&= -3\left(\frac{\hbar}{2}\right)^2\frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\,\sin\theta\;\left(a_x\sin\theta\cos\varphi+a_y\sin\theta\sin\varphi+a_z\cos\theta\right)\\
&\qquad\times\left(b_x\sin\theta\cos\varphi+b_y\sin\theta\sin\varphi+b_z\cos\theta\right)\\
&= -3\left(\frac{\hbar}{2}\right)^2\frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\,\sin\theta\\
&\quad\{\ \left(a_xb_x\sin^2\theta\cos^2\varphi+a_xb_y\sin^2\theta\cos\varphi\sin\varphi+a_xb_z\sin\theta\cos\theta\cos\varphi\right)\\
&\quad+\left(a_yb_x\sin^2\theta\cos\varphi\sin\varphi+a_yb_y\sin^2\theta\sin^2\varphi+a_yb_z\sin\theta\cos\theta\sin\varphi\right)\\
&\quad+\left(a_zb_x\sin\theta\cos\theta\cos\varphi+a_zb_y\sin\theta\cos\theta\sin\varphi+a_zb_z\cos^2\theta\right)\ \}\end{aligned} \tag{6.3}$$

After integration, we notice that all non-diagonal terms disappear and that all diagonal terms have the same coefficient. We thus obtain :

$$\begin{aligned}(\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}) &= -3\left(\frac{\hbar}{2}\right)^2\frac{1}{4\pi}\frac{4\pi}{3}\{a_xb_x+a_yb_y+a_zb_z\}\\ &= -\left(\frac{\hbar}{2}\right)^2\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}}\end{aligned} \tag{6.4.a}$$

If the spins precess but do not align, our classical model gives :

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$$(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}) = -\left(\frac{\hbar}{2}\right)^2 \left(\frac{1}{3}\right) \hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}} \tag{6.4.b}$$

which differs from the QM result by a factor $1/3$.

Notice that we obtain the correct angular dependence $\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}}$ precisely because our assumptions violate Bell's hypothesis $A(\hat{\boldsymbol{a}}, \lambda) = \pm\beta \frac{\hbar}{2}$.

7 Discussion

If the spin-spin interaction predicted by our particle model [1] is able to align the spins along the polarizers field axis, then we obtain exactly the same result in the classical and the quantum calculations. If it is not, the spin precess around the polarizer's axis and the projections of the spins on these axis are $\pm\,\hbar/2$. In this case the violation of the QM result is maximum, and it differs by a factor 3 from the classical result. Thus, the ability of the spin axis to align give a measure of the difference between QM and our classical model.

In the first case, we have been able to reproduce exactly the quantum correlations in the EPR-Bohm experiment with a hidden variable model which is local and realistic. This does not imply that Bell's theorem is incorrect because it should be remembered that we have been forced to modify Bell's basic assumption (3.2) in a non-trivial way to encompass non-linear classical theories. So, it simply means that Bell's theorem, as it currently stands, does not apply to this larger class of theories.

8 Bibliography

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