Numbers of Goldbach Conjecture Occurence in Every Even Numbers

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Abstract

This paper proposed proof of Goldbach Conjecture by using a function such that the numbers occurences of conjecture solution in any even numbers can be estimated. The function sketches after Eratoshenes Sieve under modulo term such that the function fulfilled prime sub-condition in closed intervals.

keyword: ***Goldbach Conjecture, Goldbach Conjecture lower bound.***

Preface

Goldbach conjecture was the oldest open problem in mathematics. The problem stated in two parts one for any odd numbers and one for any even numbers. The conjecture for any odd numbers already been solved in 2013. As the other parts of the conjecture , had not been solved in 2021.

The parts that had not been solved stated that every even number can be stated as the sum of two primes. Its hard to proof because the solution of each even number rarely unique. Even more, the solution seem came in random pattern which correspond to randomness of primes itself as leftover in multiplication group.

The possibilities to proof the conjecture lies in generalization of conjecture such that the generalization model can sum up the conjecture thoroughly. In other hands, the summation/ generalization must approachable to at least one proof method that publicly accepted by mathematics people. In mathematics, the proof can be done by using direct proof, indirect proof, induction, and counter argument. Every each of them correspond to specific approach that may be different for each other method's.

1 Initial Study

Every even numbers, was a product of some natural integers with 2. So every even number can be stated as $2m$ for every $m \in \mathbb{N}$. Let $r \in [\,0, m) \subseteq \mathbb{N}$ such that $2m = (m+r) + (m-r)$. As to made $m \pm r$ (both $m+r$ and $m-r$) as the solution to Golbach Conjecture (GC) both of $m \pm r$ must be primes.

One way to show that both $m \pm r$ was prime, is to show that every for $m - r$ not divisible to any integers between $[2, m-r)$; and $m+r$ was not divisible to any integers between $[2, m+r)$. But, as Eratosthenes sieve implemented, numbers of integers can be reduced to only some prime set. In summary, the prime set filled with every prime that had value less than $\sqrt{2m}$. In term of modulo, the statement $m \pm r$ was prime would satisfied:

$$(m \pm r) \mod x_i \not\equiv 0, \; \forall x_i \in X_m \tag{1}$$

for $m \in \mathbb{Z}^+$, $r \in [\,0, m\,) \subseteq \mathbb{Z}$ and $x_i \in X_m := \{x_i | \, x_i \text{ primes less than } \sqrt{2m}\}$ $i = 0, 1, 2, ...$ where $x_0 = 2$; $x_1 = 3$; $x_3 = 5$ and so on.

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Given theorem below:

Theorem 1. *for any given $a, b, c \in \mathbb{N}$, if a mod $b \equiv c$, then for any $r \in \mathbb{N}$, $(a + r)$ mod $b \equiv c + r$*

Based of the theorem 1, congruation (1) can be simplified as:

$$|m \text{ mod } x_i| \not\equiv r, \forall x_i \in X_m \tag{2}$$

Notice that although by definition, (2) show the criterion of prime, there were cases, when $r = m-1$ would made $m - r$ not divisible by X_m and $m + r$ seem to be the same. It would made $m \pm r$ fulfilled condition (2). But, as 1 (one) was not prime (any more) then it wasn't a solution for GC. As implication, r must be restrict to $\left[\,0\,,\,m-2\right]$ instead, rather than $[\,0\,, m\,)$.

As the other cases, when $m - r = x_i \in X_m$ it may made $m \pm r$ as pairs of primes which made $m \pm r$ be solutions of GC for given m. But, as for any of $r = m - x_i$ won't satisfied condition (2), it's just made condition (2) as criterion of most solution would have, but it not served as criterion that every solution of GC for m should have.

It's easy to tweak the condition (2), to described the criterion where $r = m - x_i$ made $m \pm r$ as primes, But it doesn't necessary needed. Even-though, it may led to more precise model, the condition (2) already enough to prove the conjecture.

Let $|m \bmod x_i|$ represent as both of "$m \bmod x_i$" and "$(x_i - m) \bmod x_i$" for any $x_i \in X_m$. Let $f(m, i)$ was numbers of solution that $|m \bmod x_i|$ would had. As the function $f(m, i)$ can be summarized, the summarized, $f(m, i)$ would suffice function below:

$$f(m,i) = \begin{cases} 1 & for \quad |m \bmod x_i| \equiv 0 \quad or \quad x_i = 2 \\ 2 & for \quad |m \bmod x_i| \not\equiv 0 \quad and \quad x_i \neq 2 \end{cases}$$

Let $y(i,m) \in \left[\,0\,,\,m-2\,\right] \subseteq \mathbb{N}$ such that $|y(i,m) \bmod x_i| \equiv |m \bmod x_i|$. Let $Y(i,m)$ be the set that contains every $y(i,m)$. We can deduce that the number of element in $Y(i,m)$ (stated as $n(Y(i,m))$) suffice criterion below:

$$n(Y(i,m)) \quad \leq \quad \left\lceil \frac{f(m,i)\cdot(m-1)}{x_i} \right\rceil \tag{3}$$

As we deduce that $m-1 = n(\,Y(i,m)\,) + n(\,Y^c(i,m)\,)$, we can approximate $n(\,Y^c(i,m)\,)$, and it's value would satisfied:

$$\begin{aligned} n(Y^c(i,m)) \quad &\geq \quad (m-1) - n(\,Y(i,m)\,) \\ &\geq \quad (m-1) - \left\lceil \frac{f(m,i)\cdot(m-1)}{x_i} \right\rceil \\ &\geq \quad \left\lfloor (m-1)\cdot\left(1 - \frac{f(m,i)}{x_i}\right) \right\rfloor \end{aligned} \tag{4}$$

Let X_m such that $X_m := \{p_1, p_2, ..., p_k\}$ which $pk \leq \sqrt{(2m)}$, but $p_{k+1} \geq \sqrt{2m}$. Let $\bigcup Y(i,m)$ be the union of $Y(i,m)$, Notice that:

$$\bigcup_{i=0}^{k} Y(i,m) = Y(0,m) \bigcup \left(Y(1,m) \bigcap Y^c(0,m) \right) \bigcup ... \bigcup \left(Y(k,m) \bigcap_{i=0}^{k-1} Y^c(i,m) \right)$$

As we construct $\bigcup Y(i,m)$, for all $i \leq k \in \mathbb{N}$. Its obvious that every $y(i,m) \in Y(i,m)$ didn't suffice conditions (2). In contrast, any members of it's complement $(\bigcap Y^c(i,m))$ would suffice (2) and also counted as solution of GC for given m. Let $R(m) = \bigcap Y^c(i,m)$, by definitions, $R(m)$ only contains r such that $|m \bmod x_i| \not\equiv |r \bmod x_i|$, $\forall\, x_i \in X_m$. based on (4) and (5), we can construct $n(\,R(m)\,) = n\big(\,\bigcap Y^c(i,m)\big)$. Therefore:

$$\begin{aligned} n\big(\,R(m)\,\big) \quad &\geq \quad \left\lfloor \left\lfloor \left\lfloor (m-1)\cdot\frac{x_0 - f(m,0)}{x_0} \right\rfloor \cdot \frac{x_1 - f(m,1)}{x_1} \right\rfloor \cdot ... \right\rfloor \cdot \frac{x_k - f(m,k)}{x_k} \right\rfloor \\ &\geq \quad \left\lfloor \left(\left(\frac{m-1}{2} \cdot \frac{x_0 - f(m,0)}{x_0} \right) \cdot \frac{x_1 - f(m,1)}{x_1} \right) \cdot ... \right) \cdot \frac{x_k - f(m,k)}{x_k} \right\rfloor \\ &\geq \quad \left\lfloor \frac{m-1}{2} \prod_{i=0}^{k} \frac{x_i - f(m,i)}{x_i} \right\rfloor \end{aligned} \tag{5}$$

Let $g(m) := \frac{m-1}{4} \prod_{i=1}^{k} \frac{x_i - 2}{x_i}$. Since $f(m,i) \leq 2$ especially for $i \geq 1$ where $x_i \geq 3$, we conclude that $\lfloor g(m) \rfloor$ was the lower bound of $n(\,R(m)\,)$ for any $m > 4$; $m \in \mathbb{N}$. Notice that, for any $m_{1,2} \in \mathbb{N}$ such that $m_2 > m_1$, ratio $\frac{g(m_2)}{g(m_1)}$, would follows two criterion:

1. case for $m_1 + 1 = m_2$ where $X_{m_1} = X_{m_2} = \{x_0, x_1, ..., x_k\}$.

$$\begin{aligned} \frac{g(m_2)}{g(m_1)} \quad &\geq \quad \frac{\frac{m_2-1}{4}\prod_{i=1}^{k}\frac{x_i-2}{x_i}}{\frac{m_1-1}{4}\prod_{i=1}^{k}\frac{x_i-2}{x_i}} \\ &\geq \quad \frac{m_1}{m_1-1} \\ &\geq \quad 1 \end{aligned} \tag{6}$$

2. case for $m_1 = \frac{x_k^2+1}{2}$ and $m_2 = \frac{x_{k+1}^2+1}{2}$
such that $X_{(k_1-1)} = X_{k_1} - \{x_k\}$ and $X_{(k_2-1)} = X_{k_2} - \{x_{k+1}\}$

$$\begin{aligned}
\frac{g(m_2)}{g(m_1)} &\geq \frac{\frac{m_2-1}{4}\prod_{i=1}^{k+1}\frac{x_i-2}{x_i}}{\frac{m_1-1}{4}\prod_{i=1}^{k}\frac{x_i-2}{x_i}} \\
&\geq \frac{\frac{x_{k+1}^2-1}{8}\cdot\frac{x_{k+1}-2}{x_{k+1}}\cdot\prod_{i=1}^{k}\frac{x_i-2}{x_i}}{\frac{x_k^2-1}{8}\cdot\prod_{i=1}^{k}\frac{x_i-2}{x_i}} \\
&\geq \frac{(x_{k+1}^2-1)\cdot\frac{x_{k+1}-2}{x_{k+1}}}{x_k^2-1}
\end{aligned}$$

Since minimum gap of two prime with index $k \geq 2$ was two, then $x_{k+1} \geq x_k + 2$. Therefore:

$$\begin{aligned}
\frac{g(m_2)}{g(m_1)} &\geq \frac{(x_k+2)^2-1}{x_k^2-1}\cdot\frac{x_k}{x_k+2} \\
&\geq \frac{x_k^3+4x_k^2+3x_k}{x_k^3+2x_k^2-x_k-2}
\end{aligned} \tag{7}$$

Notice that $4x_k^2 + 3x_k > 2x_k^2 - x_k - 2$ for every $k \in \mathbb{N}$ and (7) was well defined in $\mathbb{N}$. As implication, case for $m_1 = \frac{x_k^2+1}{2}$ and $m_2 = \frac{x_{k+1}^2+1}{2}$ gave results $\frac{g(m_2)}{g(m_1)} \geq 1$.

Since both cases that shown in (6) and (7) gave results that $\frac{g(m_2)}{g(m_1)} \geq 1$. It easily shown that $g(m) \geq 1$ for every $m \in [\,45\,,\,61\,]$. As $61 = \frac{x_5^2+1}{2}$ was the lower bound of $\frac{x_i^2+1}{2}$ for $i \geq 5 \in \mathbb{N}$, we can conclude that $g(m) \geq 1$ for every $m \in [\,45\,,\,\infty\,)$. it made $\lfloor\, g(m) \rfloor \geq 1$ for every $m \in [\,45\,,\,\infty\,)$.

2 Proof

Let $h(m)$ be a function that mapped every $m \geq 2 \in \mathbb{N}$ to total numbers of $k \in [\,0\,,\,m-2\,]$ such that both $m \pm k$ was primes. For example, for $m = 5$ exists $R = \{0, 2\}$ such that each of 5 ± 0; $5 - 2$; and $5 + 2$ was primes. That's made $h(5) = 2$. Notice that sum of $(m+r) + (m-r) = 2m$ construct every even numbers that greater than 4 as $m \in \mathbb{N}$ went up. Its obvious that $h(m)$ mapped m to the numbers of solutions that Goldbach Conjecture had described for $2m$.

Exists lower bound function

$$g(m) := \frac{m-1}{4}\cdot\prod_{i=1}^{k}\frac{x_i-2}{x_i}$$

for $m > 4$; $x_i \in X_m := \{x_i |\, x_i \text{ primes},\, x_i \leq \sqrt{2m}\,\}$.

or

$$g(m) := \frac{m-1}{4} \cdot \prod_{\substack{p \in \mathbb{P} \\ 2 < p \leq \sqrt{2m}}}^{k} 1 - \frac{2}{p}$$

Note that $R(m)$ filled with any r that fulfilled condition (2). Also, note that every members of $R(m)$ would also included to determine the value of $h(m)$ for given m. In addition $h(m)$ value may also determined by the exception that already mentioned but not described in condition (2). because of that, we can said that $h(m) \geq n\big(R(m)\big)$.

As it already shown that $\lfloor g(m) \rfloor$ was the lower bound of $n\big(R(m)\big)$. Based on syllogism of previous two statements before, we can conclude that $h(m) \geq \lfloor g(m) \rfloor$ for every $m > 4 \in \mathbb{N}$.

Since $g(m) \geq 1$ for $m \in [45, \infty)$ and $h(m) \geq \lfloor g(m) \rfloor$, it's obvious that for every $m \in [45, \infty)$, made $h(m) \geq 1$. As it was already known that $h(m) \geq 1$ for every $m \in [2, 44]$, then it's obvious that $h(m) \geq 1$ for every $m \in [2, \infty)$.

As $h(m) \geq 1$, it show that every m had $m \pm r$ that were primes. As $h(m)$ also represent as function that mapped $2m$ (for $m > 2$) to numbers of possibilities that made $m \pm r$ as primes, then its true that every even number that greater than 2 can be represent as sum of two primes. (**Q**.**E**.**D**)

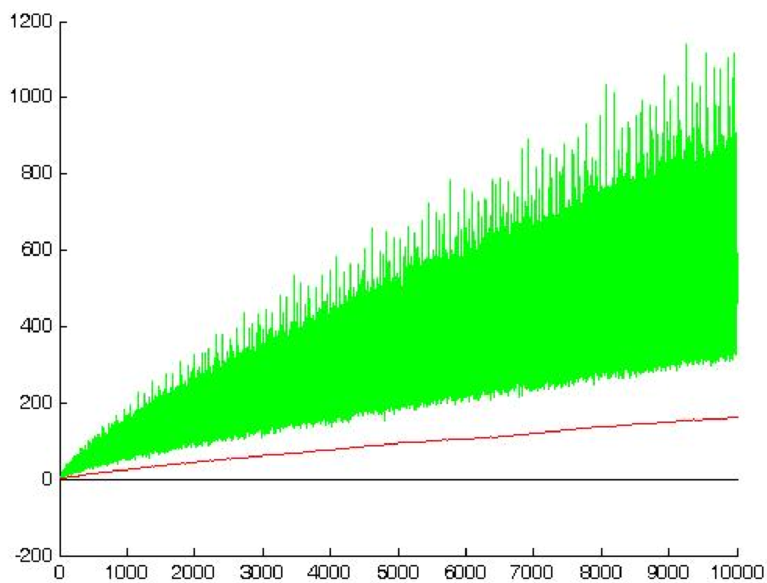

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Figure 1: $h(m)$(green) bounded by $g(m)$ (red) for $m \in [2, 10^5]$