Goldbach Conjecture Solution in Every Even Numbers

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Abstract

This paper proposed proof of Goldbach Conjecture by using a function such that the "numbers solution of the conjecture" were bounded to the function. The function sketches after Eratoshenes Sieve under modulo term such that the function fullfilled prime condition in closed intervals.

keyword: ***Goldbach Conjecture, Goldbach Conjecture lower bound.***

Preface

Goldbach conjecture was the oldest open problem in mathematics. The problem stated in two parts one for any odd numbers and one for any even numbers. The conjecture for any odd numbers already been solved in 2013. As the other parts, had not been solved, even when I started wrote this paper in February 2021.

The parts that had not been solved stated that every even number can be stated as the sum of two primes. Its hard to proof because the solution of each even number rarely unique. Even more, the solution seem came in random pattern which correspond to randomness of primes itself as leftover in multiplication group.

The possibilities to proof the conjecture lies in generalization of conjecture such that the generalization model can sum up the conjecture throughly. In other hands, the summation/ generalization must approachable to at least one proof method that already accepted. In mathematics, the proof can be done by using direct proof, indirect proof, induction, and counter argument.

1 Initial Study

Every even numbers, was a product of some natural integers with 2. So every even number can be stated as $2m$ for every $m \in \mathbb{N}$. Let $k \in [0, m) \subseteq \mathbb{N}$ such that $2m = (m + k) + (m - k)$. As to made

$m \pm k$ (both $m+k$ and $m-k$) as the solution to Golbach Conjecture (GC) both of $m \pm k$ must be primes.

One way to show that both $m \pm k$ was prime, is to show that every $m \pm k$ must not be divisible to any number between $[2, m-k)$ for $m-k$ and any integers $[2, m+k)$ for $m+k$. But, as Eratosthenes sieve implemented, the numbers of integers can be reduced to some prime set. The prime set filled with every prime that had value less than $\sqrt{2m}$. In term of modulo, the statement $m \pm k$ was prime would satisfied:

$$(m \pm k) \mod x_i \not\equiv 0, \; \forall x_i \in X_m \tag{1}$$

for $m \in \mathbb{Z}^+$, $k \in [\,0, m\,) \subseteq \mathbb{Z}$ and $x_i \in X_m := \{x_i \text{ primes less than } \sqrt{2m}\}$.

Given theorem below:

Theorem 1. *for any given $a, b, k \in \mathbb{N}$ such that $a \mod b \equiv c$ for $c \in \mathbb{N}$, then $(a+k) \mod b \equiv c+k$*

Then, based of the theorem 1, congruation (1) can be simplified as:

$$|m \text{ mod } x| \not\equiv k, \forall x_i \in X_m \tag{2}$$

for $X_m := \{x_i | \, x \text{ primes less than } \sqrt{2m}\}$.

Notice that although by definition, (2) show the criterion of prime, there were cases, when $k = m-1$ would made $m-k$ not divisible by X_m and $m+k$ seem to be the same. It would made $m \pm k$ full filled condition (2). But, as 1 (one) was not prime (any more) then it wasn't a solution for GC. As implication, k must be restrict to $\big[\,0\,,\, m-2\big]$ instead, rather than $[\,0\,, m\,)$.

As other cases, when $m-k = x_i \in X$ it may made $m \pm k$ pairs as primes and made it into solutions of GC. But as definition any of $k = m - x_i$ won't satisfied condition (2). As the possibilities that $k = m - x_i$ may exists as primes, it would sum the condition (2) that restrict on $\big[\,0\,,\, m-2\big]$ be the lower bound of the numbers solution that should exists for GC in every m.

Let $|m \, \mathrm{mod} \, x_i|$ represent as both of "$m \, \mathrm{mod} \, x_i$" and "$(x_i - m) \, \mathrm{mod} \, x_i$" for any $x_i \in X_m$. Let $f(m,i)$ was numbers of solution that $|m \, \mathrm{mod} \, x_i|$ would had. As the function $f(m,i)$ can be summarized, the summarized, $f(m,i)$ would suffice function below:

$$f(m,i) = \begin{cases} 1 \quad for \quad |m \, \mathrm{mod} \, x_i| \equiv 0 \quad or \quad x_i = 2 \\ 2 \quad for \quad |m \, \mathrm{mod} \, x_i| \not\equiv 0 \quad and \quad x_i \neq 2 \end{cases}$$

Let $y(i,m) \in \big[\,0\,,\, m-2\,\big] \subseteq \mathbb{N}$ such that $|y(i,m) \, \mathrm{mod} \, x_i| \equiv |m \, \mathrm{mod} \, x_i|$. Let $Y(i,m)$ be the set that contains every $y(i,m)$. We can deduce that the number of element in $Y(i,m)$ (stated as $n(Y(i,m))$) suffice criterion below:

$$n(Y(i,m)) = \left\lfloor \frac{f(m,i) \cdot (m-1)}{x_i} \right\rfloor \tag{3}$$

Let $X_m := \{p_1, p_2, ..., p_k\}$ such that $pk \leq \sqrt{(2m)}$, but $p_{k+1} \geq \sqrt{2m}$. As we construct $\bigcup Y(i,m)$, for all $i \leq k \in \mathbb{N}$. Its obvious that every $y(i,m) \in Y(i,m)$ didn't suffice conditions (2). In contrast, all of its complement's would suffice (2) and would be counted as solution of GC. Let $K(i,m) = \bigcap Y^c(i,m)$. By definitions, $K(i,m)$ only contains k such that $|m \text{ mod } x_i| \not\equiv |k \, \mathrm{mod} \, x_i|$, $\forall \, x_i \in X_m$.

As we deduce that $m-1 = n(\,Y(i,m)\,) + n(\,Y^c(i,m)\,)$, we can approximate $n(\,Y^c(i,m)\,)$, and it's value would satisfied:

$$\begin{aligned} n(Y^c(i,m)) &\geq (m-1) - n(\,Y(i,m)\,) \\ &\geq (m-1) - \left\lfloor \frac{f(m,i)\cdot m}{2\cdot x_i} \right\rfloor \\ &\geq \left\lfloor (m-1)\cdot\left(1 - \frac{f(m,i)}{x_i}\right) \right\rfloor \end{aligned} \tag{4}$$

based on (4), we can construct $n(\,K(i,m)\,) = n(\bigcap Y^c(i,m)\,)$. It's value would be bounded by the product of $n(\,Y^c(i,m)\,)$ as shown:

$$\begin{aligned} n\Big(K(i,m)\Big) &\geq \left\lfloor m-1\cdot\left(\prod_{i=0}^{k} \frac{x_i - f(m,i)}{x_i}\right) \right\rfloor \\ &\geq \left\lfloor \frac{m-1}{2}\cdot\left(\prod_{i=1}^{k} \frac{x_i - 2}{x_i}\right) \right\rfloor \end{aligned} \tag{5}$$

let $g(m) := \lfloor \frac{m-1}{2} \cdot \prod_{i=1}^{k} \frac{x_i-2}{x_i} \rfloor$. As it is stated at (5), we conclude that $g(m)$ was the lower bound of $n(\,K(i,m)\,)$ for any $m \in \mathbb{N}$.

Let $\hat{g}(m) = \frac{m-1}{2} \cdot \prod_{i=1}^{k} \frac{x_i-2}{x_i}$. Notice that ratio $\frac{\hat{g}(k_2)}{\hat{g}(k_1)} \approx \frac{g(k_2)}{g(k_1)}$, would follows two criterion:

1. case for $k_1 + 1 = k_2$ where $X_{k_1} = X_{k_2} = \{x_0, x_1, ..., x_k\}$.

$$\begin{aligned} \frac{\hat{g}(k_2)}{\hat{g}(k_1)} &\geq \frac{\frac{k_2-1}{2}\cdot\left(\prod_{i=1}^{k} \frac{x_i-2}{x_i}\right)}{\frac{k_1-1}{2}\cdot\left(\prod_{i=1}^{k} \frac{x_i-2}{x_i}\right)} \\ &\geq \frac{k_1}{k_1 - 1} \geq 1 \end{aligned} \tag{6}$$

2. case for $k_1 = \frac{x_k^2+1}{2}$ and $k_2 = \frac{x_{k+1}^2+1}{2}$
   such that $X_{(k_1-1)} = X_{k_1} - \{x_k\}$ and $X_{(k_2-1)} = X_{k_2} - \{x_{k+1}\}$

$$\begin{aligned} \frac{\hat{g}(k_2)}{\hat{g}(k_1)} &\geq \frac{\left(\frac{k_2}{2} - 1\right)\cdot\left(\prod_{i=1}^{k+1} \frac{x_i-2}{x_i}\right)}{\left(\frac{k_1}{2} - 1\right)\cdot\left(\prod_{i=1}^{k} \frac{x_i-2}{x_i}\right)} \\ &\geq \frac{\frac{x_{k+1}^2-1}{4}\cdot\left(\prod_{i=1}^{k} \frac{x_i-2}{x_i}\right)\cdot\left(\frac{x_{k+1}-2}{x_{k+1}}\right)}{\frac{x_k^2-1}{4}\cdot\left(\prod_{i=1}^{k} \frac{x_i-2}{x_i}\right)} \\ &\geq \frac{\left(x_{(k+1)}^2 - 1\right)\cdot\left(\frac{x_{k+1}-2}{x_{k+1}}\right)}{x_k^2 - 1} \end{aligned} \tag{7}$$

Since minimum gap of x_i and x_{i+1} for $i \geq 2$ was two, then:

$$\begin{aligned} \frac{\hat{g}(k_2)}{\hat{g}(k_1)} &\geq \frac{x_{k+1}^2 - 1}{x_k^2 - 1} \cdot \frac{x_{k+1} - 2}{x_{k+1}} \\ &\geq \frac{(x_k + 2)^2 - 1}{x_k^2 - 1} \cdot \frac{x_k + 2 - 2}{x_k + 2} \\ &\geq \frac{x_k^3 + 4x_k^2 + 3x_k}{x_k^3 + 2x_k^2 - x_k - 2} \end{aligned} \tag{8}$$

Note that $4x_k^3 + 3x_k > 2x_k^2 - x_k - 2$ for every $k \in \mathbb{N}$ and (8) was well defined in $\mathbb{N}$. As implication, case for $k_1 = \frac{x_k^2+1}{2}$ and $k_2 = \frac{x_{k+1}^2+1}{2}$ gave results $\frac{\hat{g}(k_2)}{\hat{g}(k_1)} \geq \frac{g(k_2)}{g(k_1)} \geq 1$.

As both cases that shown in (6) and (8) gave results that $\frac{g(k_2)}{g(k_1)} \geq 1$. It easily shown that $g(m) \geq 1$ for every $m \in [\,7\,,\,13\,]$. As $13 = \frac{x_2^2+1}{2}$ was the lower bound of $\frac{x_i^2+1}{2}$ for $i \geq 3 \in \mathbb{N}$, we can conclude that $g(m) \geq 1$ for every $m \in [\,7\,,\,\infty\,)$.

2 Proof

Let $h(m)$ be a function that mapped every $m \geq 2 \in \mathbb{N}$ to total numbers of $k \in [\,0\,,\,m-2\,]$ such that both $m \pm k$ was primes. Notice that sum of $(m + k) + (m - k) = 2m$ construct every even numbers that greater than 2 as $m \in \mathbb{N}$ went up. Its obvious that $h(m)$ mapped m to the numbers of solutions that Goldbach Conjecture had described for $2m$.

Exists lower bound function $g(m) := \left\lfloor \frac{m-1}{2} \cdot \prod_{i=1}^{k} \frac{x_i - 2}{x_i} \right\rfloor$ for $x_i \in X_m := \{x_i \text{ primes}, x_i \leq \sqrt{2m}\,\}$ such that $h(m) \geq g(m)$ for every $m \in \mathbb{N}$. Since $g(m) \geq 1$ for $m \geq 7$, then $h(m)$ would satisfied $h(m) \geq 1$ for $m \geq 7$. Since its already known that $h(m) \geq 1$ for every $m \in [\,2\,,\,6\,]$, then $h(m) \geq 1$ for every $m \in [\,2\,,\,\infty\,)$.

As every m had $m \pm k$ that were primes, and sum of $(m + k) + (m - k) = 2m$ construct every even numbers that greater than 2, then its true that every even number that greater than 2 can be represent as sum of two primes. (**Q.E.D**)

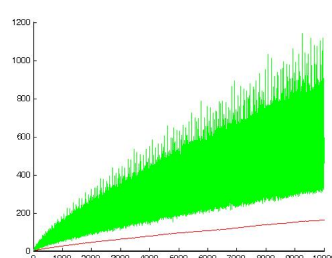

Figure 1: $h(m)$(green) bounded by $g(m)$ (red) for $m \in [\,2, 10000\,]$