Gneutrons Bound States

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Absrract

Using a generalizatuion of the Klein-Gordon equation in the framework of the Swarzschild-Whitehead solution I consider the gravitational bound states of two neutral, massive, point particles tentativily called Gneutrons

Introduction

A Gneutron pair is here by definition a system of two neutral massive bodies interacting gravitationally, assuming that we can deal with them as if they were point particles.
with masses mp and m.
A quantum gravitational state of a Gneutron pair is defined as a solution of two coupled Klein-Gordon equtations where the data of the two componemtss is exchanged. But in this paper, to start with , I assume that mp=m and therefore there is only one equation to solve.

The work presented in this paper is certainly incomplete because it relies heavily on a particular quantification conjecture and also on what some graphics may suggest. But what they suggest is so atractifI that I thought it was worthwhile to let it known.

> $restart:$ Maple 2020 program

> $with(tensor): with(plots):$

> **local** γ

$$\gamma \tag{1}$$

$Natural\ units: G := 1: \ \hbar := 1: \ c := 1:$

> $MU := 2.176434098\ 10^{-8}\ kg$

$$MU := 2.176434098\ 10^{-8}\ kg \tag{2}$$

> $LU := 1.616255205\ 10^{-35}\ m$ # meter

$$LU := 1.616255205\ 10^{-35}\ m \tag{3}$$

> $TU := 5.391247052\ 10^{-44}\ s$

$$TU := 5.391247052\ 10^{-44}\ s \tag{4}$$

Weight of a sand particle 0,67 - 23 mg

> $coord := [r, \theta, \phi, t]:$

> $g\_compts := array(symmetric, sparse, 1..4, 1..4):$

> $ginv := array(symmetric, sparse, 1..4, 1..4):$

---

dAlembertian definition

$\sqrt{|g|} = r^2 \cdot \sin(\theta)$

$$\# \,\Box\Psi = \frac{1}{\sqrt{|g|}}\left(\partial_{\mu}\sqrt{|g|}\, g^{\mu\nu}\partial_{\nu}\Psi\right), \mu, \nu = 1, 2, 3, 4$$

equivalent to

$$\# \,\Box\Psi = g^{\mu\nu}\nabla_{\mu}\left(\partial_{\nu}\Psi - \Gamma^{\alpha}_{\mu,\nu}\partial_{\alpha}\Psi\right)$$

Whitehead potentials :

> $g\_compts[4, 4] := -1 + \frac{2\cdot mp}{r}$: $g\_compts[1, 4] := \frac{2\cdot mp}{r}$: $g\_compts[1, 1] := \frac{(r + 2\cdot mp)}{r}$:
> $g\_compts[2, 2] := r^2$: $g\_compts[3, 3] := r^2\cdot\sin(\theta)^2$:

> $g := create([-1, -1], eval(g\_compts))$

$$g := \mathrm{table}\left(\left[compts = \begin{bmatrix} \frac{r+2\,mp}{r} & 0 & 0 & \frac{2\,mp}{r} \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin(\theta)^2 & 0 \\ \frac{2\,mp}{r} & 0 & 0 & -1 + \frac{2\,mp}{r} \end{bmatrix}, index\_char = [-1, -1]\right]\right) \quad \textbf{(5)}$$

> $ginv := invert(g, `detg`)$

$$ginv := \mathrm{table}\left(\left[compts = \begin{bmatrix} -\frac{-r+2\,mp}{r} & 0 & 0 & \frac{2\,mp}{r} \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2\sin(\theta)^2} & 0 \\ \frac{2\,mp}{r} & 0 & 0 & -\frac{r+2\,mp}{r} \end{bmatrix}, index\_char = [1, 1]\right]\right) \quad \textbf{(6)}$$

> $ginv[4, 4] := -\frac{r+2\,mp}{r}$: $ginv[4, 1] := \frac{2\,mp}{r}$: $ginv[1, 1] := -\frac{-r+2\,mp}{r}$: $ginv[2, 2] :=$

$\frac{1}{r^2}$: $ginv[3,3] := \frac{1}{r^2 \sin(\theta)^2}$: $ginv[1,4] := \frac{2\,mp}{r}$:

> $detg := detg$

$$detg := -r^4 \sin(\theta)^2 \quad \textbf{(7)}$$

> $sqrg := r^2 \cdot \sin(\theta)$

$$sqrg := r^2 \sin(\theta) \quad \textbf{(8)}$$

> $F[1] := simplify\left(\frac{1}{sqrg} \cdot diff\left(sqrg \cdot \left(ginv[1,1] \cdot diff(\psi(r,\theta,\phi,t), r) + ginv[1,4] \cdot diff(\psi(r,\theta,\phi,t), t)\right), r\right)\right)$

$$F_1 := \frac{1}{r^2}\left(r\,(r-2\,mp)\left(\frac{\partial^2}{\partial r^2}\psi(r,\theta,\phi,t)\right) + 2\,mp\left(\frac{\partial^2}{\partial r \partial t}\psi(r,\theta,\phi,t)\right) r + (-2\,mp + 2\,r)\left(\frac{\partial}{\partial r}\psi(r,\theta,\phi,t)\right) + 2\,mp\left(\frac{\partial}{\partial t}\psi(r,\theta,\phi,t)\right)\right) \quad \textbf{(9)}$$

> $F[2] := \frac{1}{sqrg} \cdot diff(sqrg \cdot ginv[2,2] \cdot diff(\psi(r,\theta,\phi,t), \theta), \theta)$

$$F_2 := \frac{\cos(\theta)\left(\frac{\partial}{\partial \theta}\psi(r,\theta,\phi,t)\right) + \sin(\theta)\left(\frac{\partial^2}{\partial \theta^2}\psi(r,\theta,\phi,t)\right)}{r^2 \sin(\theta)} \quad \textbf{(10)}$$

> $F[3] := \frac{1}{sqrg} \cdot diff(sqrg \cdot ginv[3,3] \cdot diff(\psi(r,\theta,\phi,t), \phi), \phi)$

$$F_3 := \frac{\frac{\partial^2}{\partial \phi^2}\psi(r,\theta,\phi,t)}{r^2 \sin(\theta)^2} \quad \textbf{(11)}$$

> $F[4] := \frac{1}{sqrg} \cdot diff\left(sqrg \cdot \left(ginv[4,4] \cdot diff(\psi(r,\theta,\phi,t), t) + ginv[4,1] \cdot diff(\psi(r,\theta,\phi,t), r)\right), t\right)$

$$F_4 := -\frac{(r+2\,mp)\left(\frac{\partial^2}{\partial t^2}\psi(r,\theta,\phi,t)\right)}{r} + \frac{2\,mp\left(\frac{\partial^2}{\partial r \partial t}\psi(r,\theta,\phi,t)\right)}{r} \quad \textbf{(12)}$$

> $dAlembert := F[1] + F[2] + F[3] + F[4];$

$$dAlembert := \frac{1}{r^2}\left(r\,(r-2\,mp)\left(\frac{\partial^2}{\partial r^2}\psi(r,\theta,\phi,t)\right) + 2\,mp\left(\frac{\partial^2}{\partial r \partial t}\psi(r,\theta,\phi,t)\right) r + (-2\,mp + 2\,r)\left(\frac{\partial}{\partial r}\psi(r,\theta,\phi,t)\right) + 2\,mp\left(\frac{\partial}{\partial t}\psi(r,\theta,\phi,t)\right)\right) + \frac{\cos(\theta)\left(\frac{\partial}{\partial \theta}\psi(r,\theta,\phi,t)\right) + \sin(\theta)\left(\frac{\partial^2}{\partial \theta^2}\psi(r,\theta,\phi,t)\right)}{r^2 \sin(\theta)} + \frac{\frac{\partial^2}{\partial \phi^2}\psi(r,\theta,\phi,t)}{r^2 \sin(\theta)^2} \quad \textbf{(13)}$$

$$-\frac{(r+2\,mp)\left(\frac{\partial^2}{\partial t^2}\psi(r,\theta,\phi,t)\right)}{r}+\frac{2\,mp\left(\frac{\partial^2}{\partial r\partial t}\psi(r,\theta,\phi,t)\right)}{r}$$

> $dA1 := \frac{1}{r^2}\Bigg(r\,(r-2\,mp)\left(\frac{\partial^2}{\partial r^2}\psi(r,\theta,\phi,t)\right)+2\,mp\left(\frac{\partial^2}{\partial r\partial t}\psi(r,\theta,\phi,t)\right)r+(-2\,mp+2\,r)\left(\frac{\partial}{\partial r}\psi(r,\theta,\phi,t)\right)+2\,mp\left(\frac{\partial}{\partial t}\psi(r,\theta,\phi,t)\right)\Bigg)+\left(-\frac{(r+2\,mp)\left(\frac{\partial^2}{\partial t^2}\psi(r,\theta,\phi,t)\right)}{r}+\frac{2\,mp\left(\frac{\partial^2}{\partial r\partial t}\psi(r,\theta,\phi,t)\right)}{r}\right):$

> $dA2 := +\frac{\cos(\theta)\left(\frac{\partial}{\partial\theta}\psi(r,\theta,\phi,t)\right)+\sin(\theta)\left(\frac{\partial^2}{\partial\theta^2}\psi(r,\theta,\phi,t)\right)}{r^2\sin(\theta)}+\frac{\frac{\partial^2}{\partial\phi^2}\psi(r,\theta,\phi,t)}{r^2\sin(\theta)^2}:$

Assuming that :

> $\psi(r,\theta,\phi,t) := \Phi(r,t)\cdot Y(\theta,\phi)$ **# Y being a spherical harmonic**

$$\psi := (r,\theta,\phi,t)\mapsto\Phi(r,t)\cdot Y(\theta,\phi) \quad (14)$$

> $dA1 := simplify(dA1)$

$$dA1 := \frac{1}{r^2}\Bigg(Y(\theta,\phi)\Bigg(r\,(r-2\,mp)\left(\frac{\partial^2}{\partial r^2}\Phi(r,t)\right)-r\,(r+2\,mp)\left(\frac{\partial^2}{\partial t^2}\Phi(r,t)\right)+4\,mp\left(\frac{\partial^2}{\partial r\partial t}\Phi(r,t)\right)r+(-2\,mp+2\,r)\left(\frac{\partial}{\partial r}\Phi(r,t)\right)+2\,mp\left(\frac{\partial}{\partial t}\Phi(r,t)\right)\Bigg)\Bigg) \quad (15)$$

>

> $dA2 := simplify(dA2);$

$$dA2 := \frac{1}{r^2\sin(\theta)^2}\Bigg(\Phi(r,t)\Bigg(\sin(\theta)\cos(\theta)\left(\frac{\partial}{\partial\theta}Y(\theta,\phi)\right)-\left(\frac{\partial^2}{\partial\theta^2}Y(\theta,\phi)\right)\cos(\theta)^2+\frac{\partial^2}{\partial\theta^2}Y(\theta,\phi)+\frac{\partial^2}{\partial\phi^2}Y(\theta,\phi)\Bigg)\Bigg) \quad (16)$$

> $dA2 := \frac{\Phi(r,t)}{r^2}\cdot\left(\frac{\partial^2}{\partial\theta^2}Y(\theta,\phi)+\frac{1}{\sin(\theta)^2}\frac{\partial^2}{\partial\phi^2}Y(\theta,\phi)+\frac{\cos(\theta)}{\sin(\theta)}\cdot\frac{\partial}{\partial\theta}Y(\theta,\phi)\right)$

$$dA2 := \frac{\Phi(r,t)\left(\frac{\partial^2}{\partial\theta^2}Y(\theta,\phi)+\frac{\frac{\partial^2}{\partial\phi^2}Y(\theta,\phi)}{\sin(\theta)^2}+\frac{\cos(\theta)\left(\frac{\partial}{\partial\theta}Y(\theta,\phi)\right)}{\sin(\theta)}\right)}{r^2} \quad (17)$$

But from the theory of Harmonic fuctions we know that

> $-\left(\frac{1}{\sin(\theta)}\cdot diff(\sin(\theta)\cdot diff(Y(\theta,\phi),\theta),\theta)+\frac{1}{\sin(\theta)^2}\cdot diff(Y(\theta,\phi),\phi,\phi)\right) = \ell\cdot(\ell+1)\cdot Y(\theta,\phi)$

$$-\frac{\cos(\theta)\left(\frac{\partial}{\partial\theta}Y(\theta,\phi)\right)+\sin(\theta)\left(\frac{\partial^2}{\partial\theta^2}Y(\theta,\phi)\right)}{\sin(\theta)}-\frac{\frac{\partial^2}{\partial\phi^2}Y(\theta,\phi)}{\sin(\theta)^2}=\ell(\ell+1)\,Y(\theta,\phi) \quad \mathbf{(18)}$$

Therefore:

> $dA2 := -\frac{\Phi(r,t)}{r^2}\cdot\ell\cdot(\ell+1)\cdot Y(\theta,\phi)$

$$dA2 := -\frac{\Phi(r,t)\,\ell(\ell+1)\,Y(\theta,\phi)}{r^2} \quad \mathbf{(19)}$$

> $dAlembert := dA1 + dA2;$

$$dAlembert := \frac{1}{r^2}\left(Y(\theta,\phi)\left(r\,(r-2\,mp)\left(\frac{\partial^2}{\partial r^2}\Phi(r,t)\right)-r\,(r+2\,mp)\left(\frac{\partial^2}{\partial t^2}\Phi(r,t)\right)+4\,mp\left(\frac{\partial^2}{\partial r\partial t}\Phi(r,t)\right)r+(-2\,mp+2\,r)\left(\frac{\partial}{\partial r}\Phi(r,t)\right)+2\,mp\left(\frac{\partial}{\partial t}\Phi(r,t)\right)\right)\right)-\frac{\Phi(r,t)\,\ell(\ell+1)\,Y(\theta,\phi)}{r^2} \quad \mathbf{(20)}$$

Assuming now that

> $\Phi(r,t) := A(r)\cdot\exp(I\cdot E\cdot t);$

$$\Phi := (r,t)\mapsto A(r)\cdot e^{I\cdot E\cdot t} \quad \mathbf{(21)}$$

We get:

> $EquA := coeff(dAlembert, Y(\theta,\phi))$

$$EquA := \frac{1}{r^2}\left(r\,(r-2\,mp)\left(\frac{d^2}{dr^2}A(r)\right)e^{IEt}+r\,(r+2\,mp)\,A(r)\,E^2\,e^{IEt}+4\,I\,mp\left(\frac{d}{dr}A(r)\right)E\,e^{IEt}\,r+(-2\,mp+2\,r)\left(\frac{d}{dr}A(r)\right)e^{IEt}+2\,I\,mp\,A(r)\,E\,e^{IEt}\right)-\frac{A(r)\,e^{IEt}\,\ell(\ell+1)}{r^2} \quad \mathbf{(22)}$$

and

> $EquA := coeff(EquA, e^{I\cdot E\cdot t})$

$$EquA := \frac{1}{r^2}\left(r\,(r-2\,mp)\left(\frac{d^2}{dr^2}A(r)\right)+r\,(r+2\,mp)\,A(r)\,E^2+4\,I\,mp\left(\frac{d}{dr}A(r)\right)E\,r+(-2\,mp+2\,r)\left(\frac{d}{dr}A(r)\right)+2\,I\,mp\,A(r)\,E\right)-\frac{A(r)\,\ell(\ell+1)}{r^2} \quad \mathbf{(23)}$$

And therefore

> $KGE := EquA-\left(\frac{c^3}{G\cdot\hbar}\right)^2\cdot m^2\cdot A(r)=0$

(24)

$$KGE := \frac{1}{r^2}\left(r\,(r-2\,mp)\left(\frac{d^2}{dr^2}A(r)\right)+r\,(r+2\,mp)\,A(r)\,E^2+4\,\mathrm{I}\,mp\left(\frac{d}{dr}A(r)\right)E\,r+\left(-2\,mp+2\,r\right)\left(\frac{d}{dr}A(r)\right)+2\,\mathrm{I}\,mp\,A(r)\,E\right)-\frac{A(r)\,\ell(\ell+1)}{r^2}-m^2A(r)=0 \tag{24}$$

Remark 1: The coefficient $\left(\frac{c^3}{G\cdot\hbar}\right)^2$:

is necessary when general units are used

Remark 2: This equation would not change if Droste´s coordinates were used. But it woud if Fock's or Brillouin's coordinates were used.

> $dsolve(KGE)$

$$A(r)=\_C1\,e^{\sqrt{-E^2+m^2}\,r}\,\mathrm{HeunC}\left(-4\,mp\sqrt{-E^2+m^2},4\,\mathrm{I}\,mp\,E,0,\left(-8\,E^2+4\,m^2\right)mp^2,-\ell^2-\ell+\left(8\,E^2-4\,m^2\right)mp^2,\frac{-r+2\,mp}{2\,mp}\right)+\_C2\,e^{\sqrt{-E^2+m^2}\,r}\,\mathrm{HeunC}\left(-4\,mp\sqrt{-E^2+m^2},-4\,\mathrm{I}\,mp\,E,0,\left(-8\,E^2+4\,m^2\right)mp^2,-\ell^2-\ell+\left(8\,E^2-4\,m^2\right)mp^2,\frac{-r+2\,mp}{2\,mp}\right)(r-2\,mp)^{-4\,\mathrm{I}\,mp\,E} \tag{25}$$

Below only the following solutions are considered

> $A1 := (\ell, r)\rightarrow e^{-\sqrt{-E(\ell)^2+m^2}\,r}\,\mathrm{HeunC}\left(-4\cdot mp\sqrt{-E(\ell)^2+m^2},4\,\mathrm{I}\cdot mp\cdot E(\ell),0,-8\cdot mp^2\cdot\left(E(\ell)^2-\frac{m^2}{2}\right),\left(-4\,m^2+8\,E(\ell)^2\right)mp^2-\ell^2-\ell,\frac{-r+2\,mp}{2\,mp}\right):$

Solutions with a factor $e^{+\sqrt{-E(\ell)^2+m^2}\,r}$ lead to functions without norm. And those with a coeffient _C2 different from 0 did not look satisfactory. They lead to solutions constrained in the intervals [0,2] or [2,∞]

$$e^{\sqrt{-E(\ell)^2+m^2}\,r} \tag{26}$$

> $\_C1 := 1;\ \_C2 := 0;$

$$\_C1 := 1$$
$$\_C2 := 0 \tag{27}$$

Mass selection--

> $\alpha := -4\,mp\sqrt{-E^2+m^2};\ \beta := +4\,\mathrm{I}\ mp\cdot E;\ \gamma := 0;\ \delta := -\left(8\,E^2-4\cdot m^2\right)mp^2;\ \eta := \left(-4\cdot m^2+8\cdot E^2\right)\cdot mp^2-\ell^2-\ell;$

$$\alpha := -4\,mp\sqrt{-E^2+m^2}$$
$$\beta := 4\,\mathrm{I}\,mp\,E$$
$$\gamma := 0$$
$$\delta := -\left(8\,E^2-4\,m^2\right)mp^2$$

$$\eta := -\ell^2 - \ell + \left(8\,E^2 - 4\,m^2\right) mp^2 \tag{28}$$

Maple 2020 Help page on HeunC functions suggests to use the inert quantization rule below, known to work in some familiar cases

> $Eqm := \delta = -\left(\ell + \frac{(\gamma+\beta+2)}{2}\right)\cdot\alpha$

$$Eqm := -\left(8\,E^2 - 4\,m^2\right) mp^2 = 4\,\left(\ell + 1 + 2\,\mathrm{I}\,mp\,E\right) mp\,\sqrt{-E^2 + m^2} \tag{29}$$

But I found convenient to consider also the variant:

> $Eqp := \delta = +\left(\ell + \frac{(\gamma+\beta+2)}{2}\right)\cdot\alpha$

$$Eqp := -\left(8\,E^2 - 4\,m^2\right) mp^2 = -4\,\left(\ell + 1 + 2\,\mathrm{I}\,mp\,E\right) mp\,\sqrt{-E^2 + m^2} \tag{30}$$

> $solve(\ Eqm)$

$$\{\ell = \ell, E = E, m = m, mp = 0\}, \left\{\ell = -\frac{2\,\mathrm{I}\,E\,\sqrt{-E^2+m^2}\,mp + 2\,E^2\,mp - m^2\,mp + \sqrt{-E^2+m^2}}{\sqrt{-E^2+m^2}}, E = E, m = m, mp = mp\right\}, \left\{\ell = -\frac{-2\,\mathrm{I}\,E\,\sqrt{-E^2+m^2}\,mp + 2\,E^2\,mp - m^2\,mp + \sqrt{-E^2+m^2}}{\sqrt{-E^2+m^2}}, E = E, m = m, mp = mp\right\}, \{\ell = \ell, E = 0, m = 0, mp = mp\} \tag{31}$$

> $solve(Eqp)$

$$\{\ell = \ell, E = E, m = m, mp = 0\}, \left\{\ell = \frac{-2\,\mathrm{I}\,E\,\sqrt{-E^2+m^2}\,mp + 2\,E^2\,mp - m^2\,mp - \sqrt{-E^2+m^2}}{\sqrt{-E^2+m^2}}, E = E, m = m, mp = mp\right\}, \left\{\ell = \frac{2\,\mathrm{I}\,E\,\sqrt{-E^2+m^2}\,mp + 2\,E^2\,mp - m^2\,mp - \sqrt{-E^2+m^2}}{\sqrt{-E^2+m^2}}, E = E, m = m, mp = mp\right\}, \{\ell = \ell, E = 0, m = 0, mp = mp\} \tag{32}$$

> $Eqm(\ell) := \ell = -\frac{2\,\mathrm{I}\,E\,\sqrt{-E^2+m^2}\,mp + 2\,E^2\,mp - m^2\,mp + \sqrt{-E^2+m^2}}{\sqrt{-E^2+m^2}}$

$$Eqm := \ell \mapsto \ell = -\frac{2\cdot\mathrm{I}\cdot E\cdot\sqrt{-E^2+m^2}\cdot mp + 2\cdot E^2\cdot mp - m^2\cdot mp + \sqrt{-E^2+m^2}}{\sqrt{-E^2+m^2}} \tag{33}$$

> $Eqp(\ell) := \ell = \frac{-2\,\mathrm{I}\,E\,\sqrt{-E^2+m^2}\,mp + 2\,E^2\,mp - m^2\,mp - \sqrt{-E^2+m^2}}{\sqrt{-E^2+m^2}}$

$$Eqp := \ell \mapsto \ell = \frac{-2\cdot\mathrm{I}\cdot E\cdot\sqrt{-E^2+m^2}\cdot mp + 2\cdot E^2\cdot mp - m^2\cdot mp - \sqrt{-E^2+m^2}}{\sqrt{-E^2+m^2}} \tag{34}$$

> $mp := 1;\ \ m := mp$

$$mp := 1$$
$$m := 1 \tag{35}$$

> simplify(Eqm(ℓ));

$$\ell = \frac{-2\,\mathrm{I}\,E\sqrt{-E^2+1} - 2E^2 + 1 - \sqrt{-E^2+1}}{\sqrt{-E^2+1}} \tag{36}$$

> simplify(Eqp(ℓ));

$$\ell = \frac{-2\,\mathrm{I}\,E\sqrt{-E^2+1} + 2E^2 - 1 - \sqrt{-E^2+1}}{\sqrt{-E^2+1}} \tag{37}$$

> Eqm(0) := solve(Eqm(0), E) :

> Eqm(0) := evalf(Eqm(0));

$$Eqm(0) := 0. \tag{38}$$

>

> Eqp(0) := solve(Eqp(0), E) :

> Eqp(0) := evalf(Eqp(0));

$$Eqp(0) := -0.9921567416 + 0.1250000000\,\mathrm{I},\ 0.9921567416 + 0.1250000000\,\mathrm{I} \tag{39}$$

>

> Eqm(1) := solve(Eqm(1), E) :

> Eqm(1) := evalf(Eqm(1));

$$Eqm(1) := 0.3916217141\,\mathrm{I} \tag{40}$$

>

> Eqp(1) := solve(Eqp(1), E) :

> Eqp(1) := evalf(Eqp(1));

$$Eqp(1) := 0.9770466965 + 0.0541891430\,\mathrm{I},\ -0.9770466964 + 0.0541891431\,\mathrm{I} \tag{41}$$

>

> Eqm(2) := solve(Eqm(2), E) :

> Eqm(2) := evalf(Eqm(2));

$$Eqm(2) := 3.\ 10^{-10} + 0.6937422864\,\mathrm{I} \tag{42}$$

>

> Eqp(2) := solve(Eqp(2), E) :

> Eqp(2) := evalf(Eqp(2));

$$Eqp(2) := 0.9798879590 + 0.0281288567\,\mathrm{I},\ -0.9798879593 + 0.0281288568\,\mathrm{I} \tag{43}$$

>

> Eqm(3) := solve(Eqm(3), E) :

> Eqm(3) := evalf(Eqm(3));

$$Eqm(3) := 4.\ 10^{-10} + 0.9677251442\,\mathrm{I} \tag{44}$$

>

> Eqp(3) := solve(Eqp(3), E) :

> Eqp(3) := evalf(Eqp(3));

(45)

$$Eqp(3) := 0.9841272246 + 0.0161374277\, I,\ -0.9841272246 + 0.0161374278\, I \quad \textbf{(45)}$$

> $Eqm(4) := solve(Eqm(4), E):$

> $Eqm(4) := evalf(Eqm(4));$

$$Eqm(4) := 1.\ 10^{-10} + 1.230104699\, I \quad \textbf{(46)}$$

>

> $Eqp(4) := solve(Eqp(4), E):$

> $Eqp(4) := evalf(Eqp(4));$

$$Eqp(4) := 0.9876374546 + 0.0099476503\, I,\ -0.9876374547 + 0.0099476504\, I \quad \textbf{(47)}$$

>

> $Eqm(5) := solve(Eqm(5), E):$

> $Eqm(5) := evalf(Eqm(5));$

$$Eqm(5) := 1.\ 10^{-10} + 1.487024960\, I \quad \textbf{(48)}$$

>

> $Eqp(5) := solve(Eqp(5), E):$

> $Eqp(5) := evalf(Eqp(5));$

$$Eqp(5) := 0.9902846581 + 0.0064875200\, I,\ -0.9902846583 + 0.0064875194\, I \quad \textbf{(49)}$$

They are all time decaying states.. But while there is a frquency associated with Eqp modes this is not the case with the Eqm ones

The six Equp lowest mode solutions are listed below. The coma separates positive and negative modes.

The inverse of the real part might be interpreted as the frequency ν of the transition from the excited state to the stationary one, while the inverse of the maginary part τ ,that it is always positive , can be interpreted as the mean live of the excited state.

> $A1R := (\ell, r) \rightarrow \mathrm{Re}(A1(\ell, r));\ \ A1I := (\ell, r) \rightarrow \mathrm{Im}(A1(\ell, r));$

$$A1R := (\ell, r) \mapsto \Re(A1(\ell, r))$$

$$A1I := (\ell, r) \mapsto \Im(A1(\ell, r)) \quad \textbf{(50)}$$

> $F1 := (\ell, r) \rightarrow r^2 \cdot A1(\ell, r) \cdot conjugate(A1(\ell, r))$

$$F1 := (\ell, r) \mapsto r^2 \cdot A1(\ell, r) \cdot \overline{A1(\ell, r)} \quad \textbf{(51)}$$

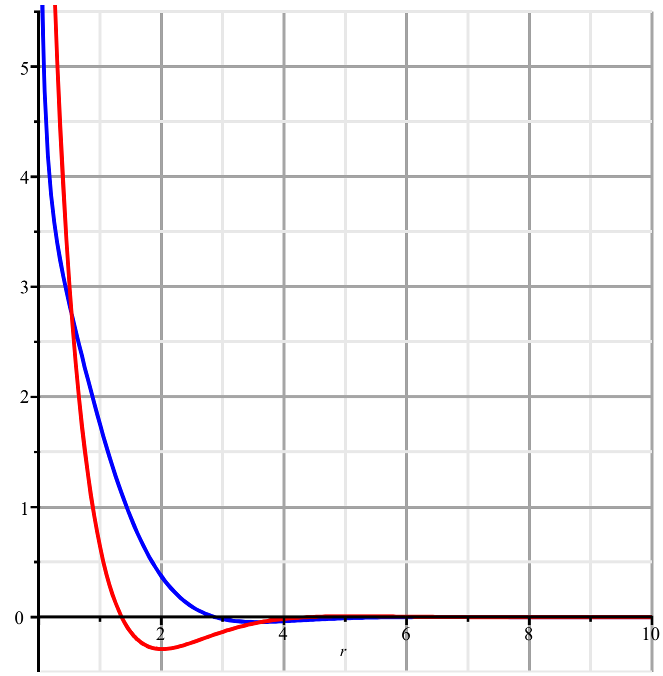
>

> $\ell := 0;\quad E(0) := Eqp(0)[1];$

$$\ell := 0$$

$$E(0) := -0.9921567416 + 0.1250000000\, I \quad \textbf{(52)}$$

> $plot([A1R(\ell, r), A1I(\ell, r)], r = 0..10, color = [blue, red], gridlines = true)$



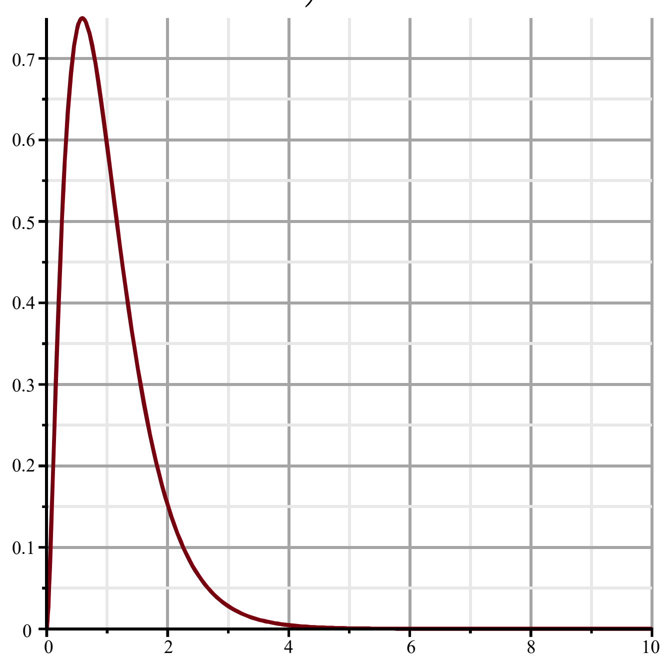
> #plot(F1(ℓ, r), r = 0 ..10, gridlines = true, color = yellow)

> #IntF1(0) := evalf(int(F1(0, r), r = 0 ..10))

> IntF1(0) := 5.881599832

$$IntF1(0) := 5.881599832 \quad \textbf{(53)}$$

> $plot\left(\dfrac{F1(\ell, r)}{IntF1(\ell)}, r = 0\,..10, gridlines = true\right)$

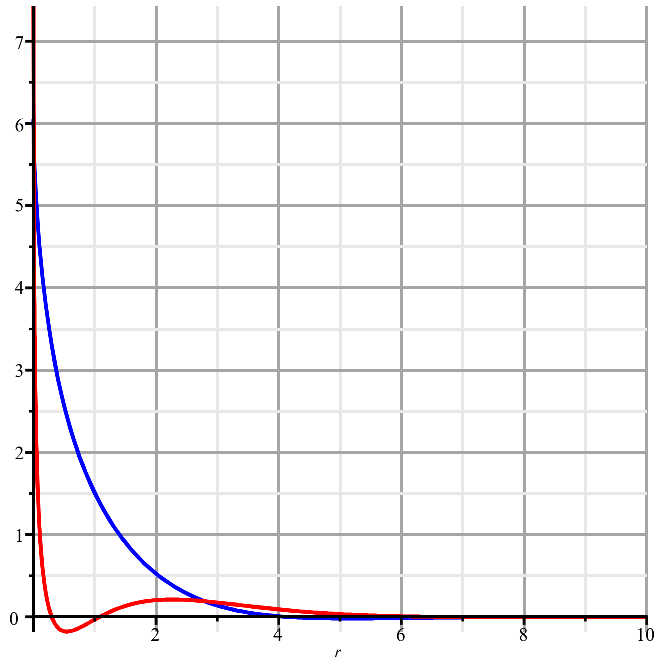


> ℓ := 1; E(1) := Eqp(1)[1];

$$\ell := 1$$

$$E(1) := 0.9770466965 + 0.0541891430\,I \quad \textbf{(54)}$$

> plot([A1R(ℓ, r), A1I(ℓ, r)], r = 0 ..10, color = [blue, red], gridlines = true);



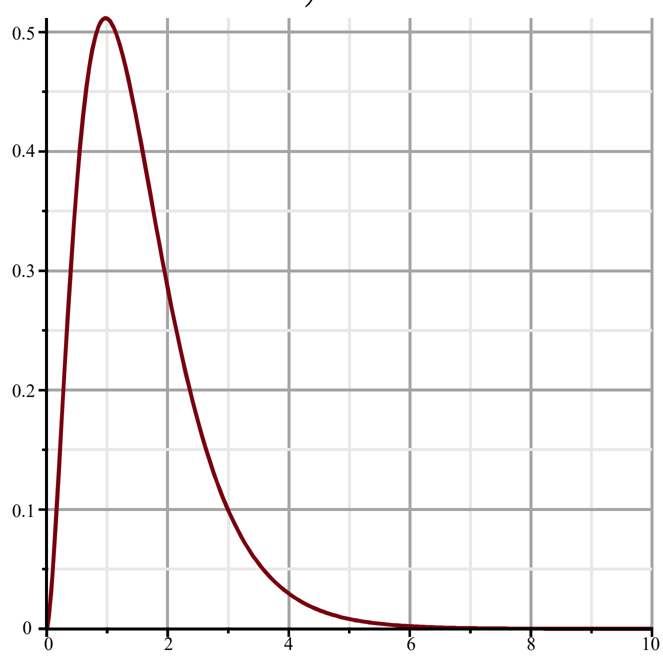
> #plot(F1(ℓ, r), r=0..10, gridlines=true, color=yellow)

> #IntF1(1) := evalf(int(F1(ℓ, r), r=0..10))

> IntF1(1) := 4.443555244

$$IntF1(1) := 4.443555244 \quad (55)$$

> $plot\left(\frac{F1(\ell, r)}{IntF1(\ell)}, r=0..10, gridlines=true\right)$

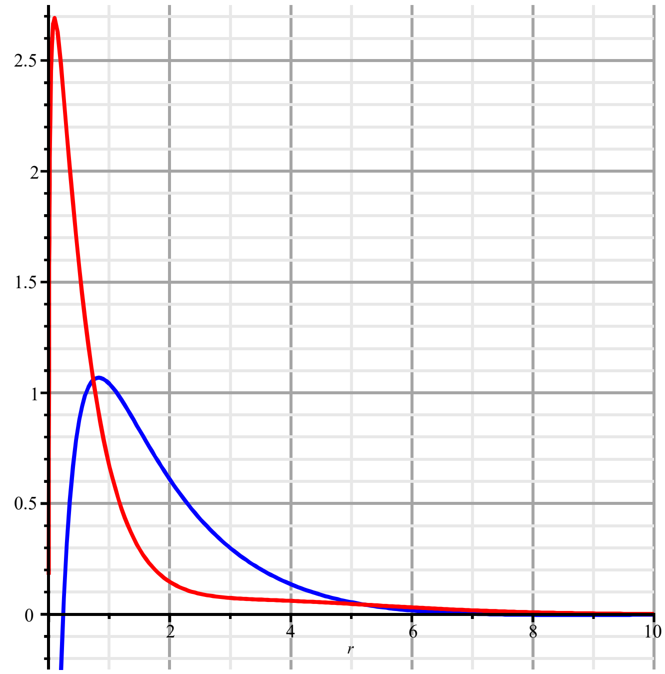


> ℓ := 2;   E(2) := Eqp(2)[1];

$$\ell := 2$$

$$E(2) := 0.9798879590 + 0.0281288567\,I \quad (56)$$

> plot([A1R(ℓ, r), A1I(ℓ, r)], r=0..10, color=[blue, red], gridlines=true);



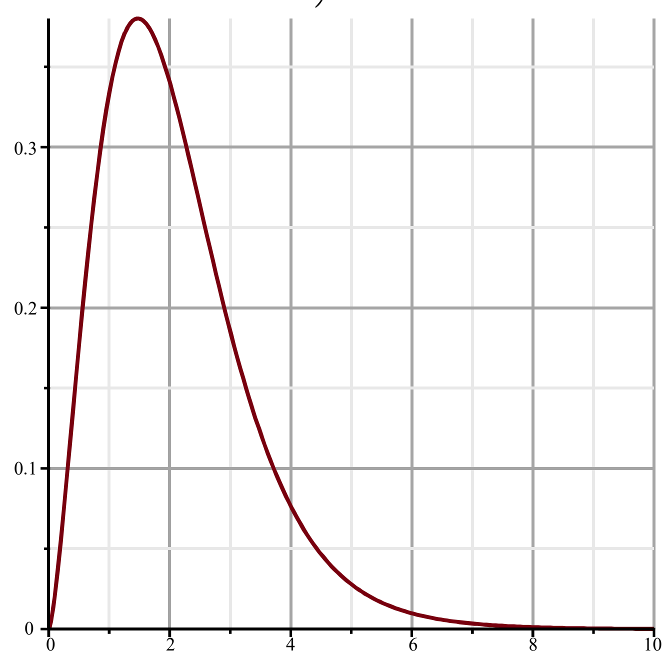
> #plot(F1(2, r), r = 0 ..10, gridlines = true, color = yellow)

> #IntF1(2) := evalf(int(F1(2, r), r = 0 ..10))

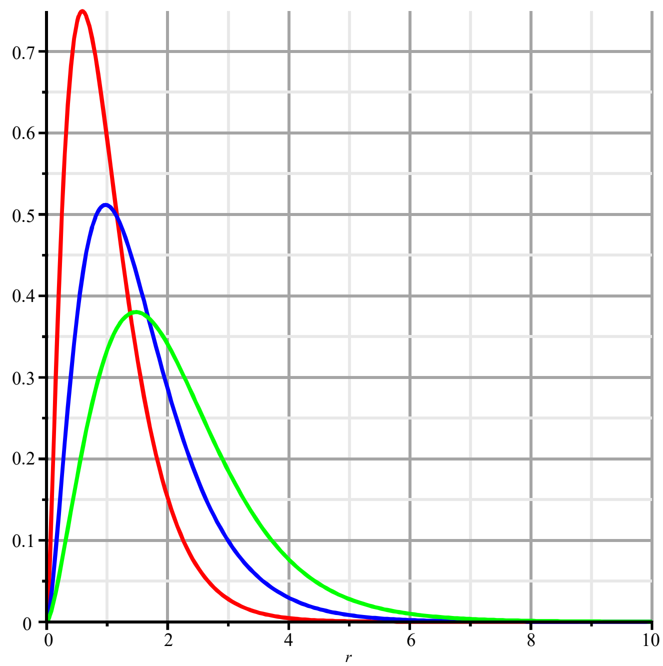
> IntF1(2) := 4.615589828

$$IntF1(2) := 4.615589828 \quad (57)$$

> $plot\left(\frac{F1(2, r)}{IntF1(2)}, r = 0 ..10, gridlines = true\right)$



> $plot\left(\left[\frac{F1(0, r)}{IntF1(0)}, \frac{F1(1, r)}{IntF1(1)}, \frac{F1(2, r)}{IntF1(2)}\right], r = 0 ..10, color = [red, blue, green], gridlines = true\right)$



> $\#plot\left(\left[\frac{F1(0,r)}{IntF1(0)}, \frac{F1(1,r)}{IntF1(1)}, \frac{F1(2,r)}{IntF1(2)}\right], r=0..2, color=[red, blue, green], gridlines=true\right)$

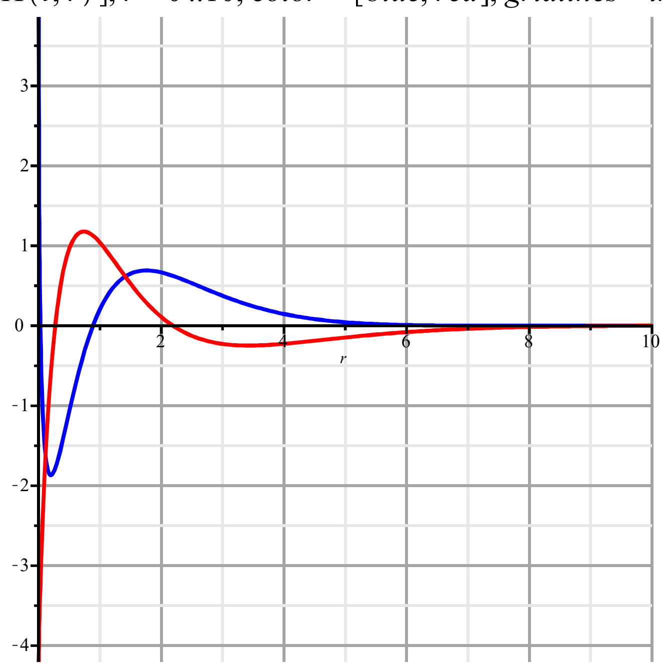
Notice that these three modes reach their maximum value inside the horizon interval[0,2]. There are no equivalent modes at the Newtonian approximation (L.Bel Cf. C. I. M.E. Relatività Generale. Edizioni Cremonese. Roma 1965)

> $\ell := 3;\quad E(3) := Eqp(3)[1];$

$$\ell := 3$$

$$E(3) := 0.9841272246 + 0.0161374277\,\mathrm{I} \tag{58}$$

> $plot([A1R(\ell, r), A1I(\ell, r)], r=0..10, color=[blue, red], gridlines=true);$



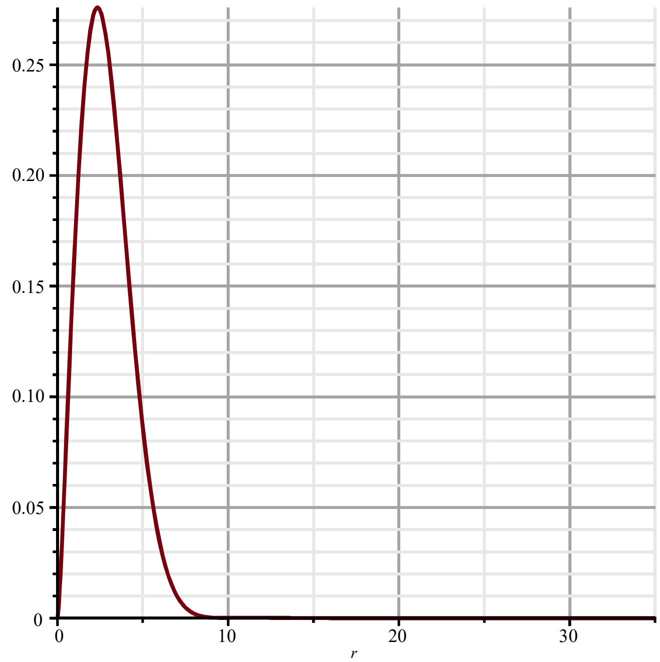
> $\#\ plot(F1(3,r), r=0..10, gridlines=true, color=yellow)$

> $\#IntF1(3) := evalf(int(F1(3, r), r=0..35))$

> $IntF1(3) := 6.802439516$

$$IntF1(3) := 6.802439516 \quad (59)$$

> $plot\left(\frac{F1(3, r)}{IntF1(3)}, r = 0..35, gridlines = true\right)$



> $\ell := 4; \quad E(4) := Eqp(4)[1];$

$$\ell := 4$$
$$E(4) := 0.9876374546 + 0.0099476503\,I \quad (60)$$

> $plot([A1R(\ell, r), A1I(\ell, r)], r = 0..20, color = [blue, red], gridlines = true);$

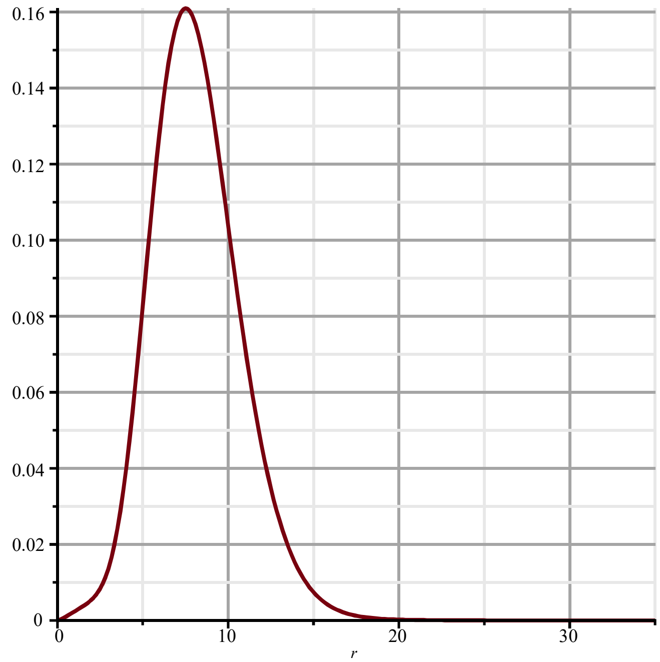
> $\#plot(F1(\ell, r), r = 0..20, gridlines = true, color = yellow)$

> $\#IntF1(4) := evalf(int(F1(4, r), r = 0..35))$

> $IntF1(4) := 374.6385090$

$$IntF1(4) := 374.6385090 \quad (61)$$

> $plot\left(\dfrac{F1(4, r)}{IntF1(4)}, r = 0..35, gridlines = true\right)$



> $\ell := 5;\quad E(5) := Eqp(5)[1];$

$$\ell := 5$$
$$E(5) := 0.9902846581 + 0.0064875200\,I \quad (62)$$

> $plot([A1R(\ell, r), A1I(\ell, r)], r = 0..30, color = [blue, red], gridlines = true);$

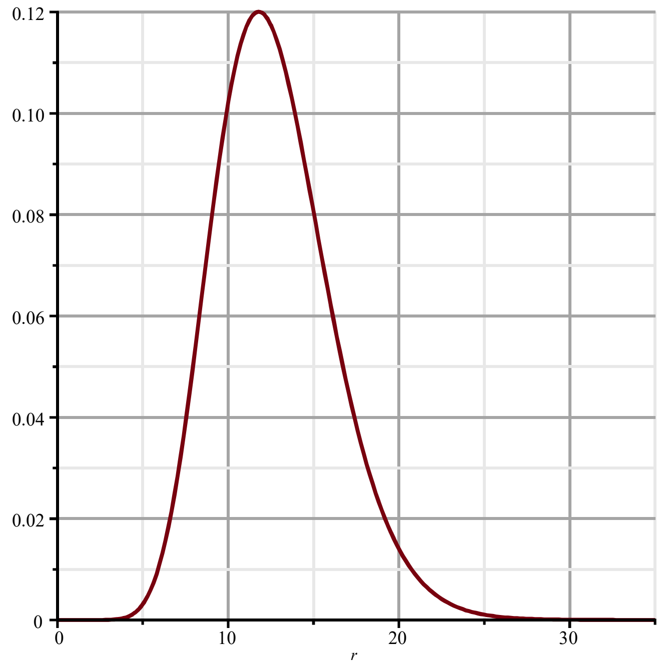
> $\#plot(F1(\ell, r), r = 0..30, gridlines = true, color = yellow)$

> $\#IntF1(\ell) := evalf(int(F1(\ell, r), r = 0..35))$

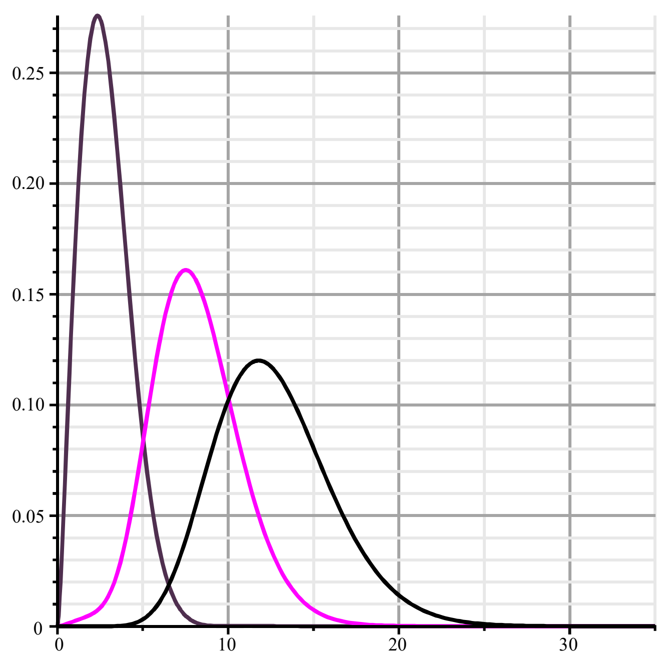
> $IntF1(5) := 925708.6231$

$$IntF1(5) := 925708.6231 \quad (63)$$

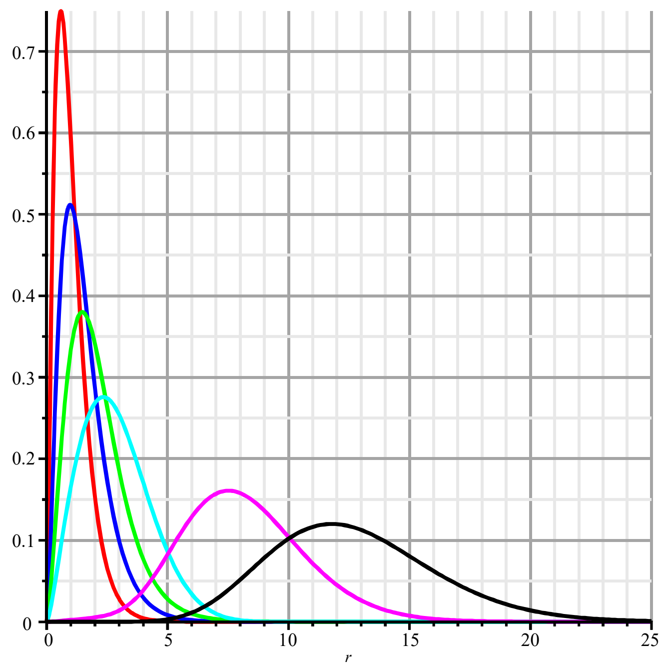
> $plot\left(\frac{F1(5, r)}{IntF1(5)}, r = 0 ..35, gridlines = true\right)$



> $plot\left(\left[\frac{F1(3, r)}{IntF1(3)}, \frac{F1(4, r)}{IntF1(4)}, \frac{F1(5, r)}{IntF1(5)}\right], r = 0 ..35, color = [violet, magenta, black], gridlines = true\right)$



> $plot\left(\left[\frac{F1(0, r)}{IntF1(0)}, \frac{F1(1, r)}{IntF1(1)}, \frac{F1(2, r)}{IntF1(2)}, \frac{F1(3, r)}{IntF1(3)}, \frac{F1(4, r)}{IntF1(4)}, \frac{F1(5, r)}{IntF1(5)}\right], r = 0 ..25, color = [red, blue, green, cyan, magenta, black], gridlines = true\right)$


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
20
25
r

>