Gravitational Bound States

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Abstract
Using a generalization of the Klein-Gordon equation in the framework of the Schwarzschild-Whitehead solution I consider the gravitational bound states of two neutral, massive, point particles.

Introduction

An Axion is here by definition a system of two neutral massive bodies interacting gravitationally, assuming that we can deal with them as if they were point particles.
with masses mp and m.
A quantum gravitational state of an axion is defined as a solution of two coupled Klein-Gordon equtations where the data of the two componemtss is exchanged. But in this paper, to start with ,I assume that mp=m and therefore there is only one equation to solve.

The work presented in this paper is certainly incomplete because it relies heavily on a particular quantification conjecture and also on what some graphics may suggest. But what they suggest is so atractifI that I thought it was worthwhile to let it known.

> $restart:$ Maple 2020 program

> $with(tensor):with(plots):$

> **local** γ

$$\gamma \tag{1}$$

$Natural\ units: G := 1: \ \hbar := 1:$
$c := 1: \quad \#\ at\ the\ present\ cosmological\ era$

> $MU := 2.176434098\ 10^{-8}\ kg$

$$MU := 2.176434098\ 10^{-8}\ kg \tag{2}$$

> $LU := 1.616255205\ 10^{-35}\ m$ # meter

$$LU := 1.616255205\ 10^{-35}\ m \tag{3}$$

> $TU := 5.391247052\ 10^{-44}\ s$

$$TU := 5.391247052\ 10^{-44}\ s \tag{4}$$

Weight of a sand particle 0,67 - 23 mg

> $coord := [r, \theta, \phi, t]:$

> $g\_compts := array(symmetric, sparse, 1..4, 1..4):$

> $ginv := array(symmetric, sparse, 1..4, 1..4):$

----

dAlembertian definition

$\# \quad \sqrt{|g|} = r^2 \cdot \sin(\theta)$

$\# \ \Box\Psi = \frac{1}{\sqrt{|g|}} \left(\partial_{\mu} \sqrt{|g|}\, g^{\mu\nu} \partial_{\nu} \Psi \right) \ , \mu, \nu = 1, 2, 3, 4$

equivalent to

$\# \ \Box\Psi = g^{\mu\nu} \nabla_{\mu} \left(\partial_{\nu} \Psi - \Gamma^{\alpha}_{\mu,\nu}\, \partial_{\alpha} \Psi \right)$

Whitehead potentials :

$> g\_compts[4,4] := -1 + \frac{2 \cdot mp}{r} \ : \ g\_compts[1,4] := \frac{2 \cdot mp}{r} \ : g\_compts[1,1] := \frac{(r + 2 \cdot mp)}{r} \ :$
$g\_compts[2,2] := r^2 : \ g\_compts[3,3] := r^2 \cdot \sin(\theta)^2 :$

$> g := create([-1,-1], eval(g\_compts))$

$$g := \text{table}\left(\left[compts = \begin{bmatrix} \frac{r+2\,mp}{r} & 0 & 0 & \frac{2\,mp}{r} \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin(\theta)^2 & 0 \\ \frac{2\,mp}{r} & 0 & 0 & -1 + \frac{2\,mp}{r} \end{bmatrix}, index\_char = [-1, -1] \right]\right) \quad \textbf{(5)}$$

$> ginv := invert(g, \text{`detg`})$

$$ginv := \text{table}\left(\left[compts = \begin{bmatrix} -\frac{-r+2\,mp}{r} & 0 & 0 & \frac{2\,mp}{r} \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2 \sin(\theta)^2} & 0 \\ \frac{2\,mp}{r} & 0 & 0 & -\frac{r+2\,mp}{r} \end{bmatrix}, index\_char = [1, 1] \right]\right) \quad \textbf{(6)}$$

> $ginv[4,4] := -\frac{r+2\,mp}{r}$: $ginv[4,1] := \frac{2\,mp}{r}$: $ginv[1,1] := -\frac{-r+2\,mp}{r}$: $ginv[2,2] := \frac{1}{r^2}$: $ginv[3,3] := \frac{1}{r^2\sin(\theta)^2}$: $ginv[1,4] := \frac{2\,mp}{r}$:

> $detg := detg$

$$detg := -r^4 \sin(\theta)^2 \tag{7}$$

> $sqrg := r^2 \cdot \sin(\theta)$

$$sqrg := r^2 \sin(\theta) \tag{8}$$

> $F[1] := simplify\left(\frac{1}{sqrg} \cdot diff\left(sqrg \cdot \left(ginv[1,1] \cdot diff(\psi(r,\theta,\phi,t), r) + ginv[1,4] \cdot diff(\psi(r,\theta,\phi,t), t)\right), r\right)\right)$

$$F_1 := \frac{1}{r^2}\left(r\,(r-2\,mp)\left(\frac{\partial^2}{\partial r^2}\,\psi(r,\theta,\phi,t)\right) + 2\,mp\left(\frac{\partial^2}{\partial r\partial t}\,\psi(r,\theta,\phi,t)\right) r + (-2\,mp + 2\,r)\left(\frac{\partial}{\partial r}\,\psi(r,\theta,\phi,t)\right) + 2\,mp\left(\frac{\partial}{\partial t}\,\psi(r,\theta,\phi,t)\right)\right) \tag{9}$$

> $F[2] := \frac{1}{sqrg} \cdot diff(sqrg \cdot ginv[2,2] \cdot diff(\psi(r,\theta,\phi,t), \theta), \theta)$

$$F_2 := \frac{\cos(\theta)\left(\frac{\partial}{\partial \theta}\,\psi(r,\theta,\phi,t)\right) + \sin(\theta)\left(\frac{\partial^2}{\partial \theta^2}\,\psi(r,\theta,\phi,t)\right)}{r^2 \sin(\theta)} \tag{10}$$

> $F[3] := \frac{1}{sqrg} \cdot diff(sqrg \cdot ginv[3,3] \cdot diff(\psi(r,\theta,\phi,t), \phi), \phi)$

$$F_3 := \frac{\frac{\partial^2}{\partial \phi^2}\,\psi(r,\theta,\phi,t)}{r^2 \sin(\theta)^2} \tag{11}$$

> $F[4] := \frac{1}{sqrg} \cdot diff\left(sqrg \cdot \left(ginv[4,4] \cdot diff(\psi(r,\theta,\phi,t), t) + ginv[4,1] \cdot diff(\psi(r,\theta,\phi,t), r)\right), t\right)$

$$F_4 := -\frac{(r+2\,mp)\left(\frac{\partial^2}{\partial t^2}\,\psi(r,\theta,\phi,t)\right)}{r} + \frac{2\,mp\left(\frac{\partial^2}{\partial r\partial t}\,\psi(r,\theta,\phi,t)\right)}{r} \tag{12}$$

> $dAlembert := F[1] + F[2] + F[3] + F[4];$

$$dAlembert := \frac{1}{r^2}\left(r\,(r-2\,mp)\left(\frac{\partial^2}{\partial r^2}\,\psi(r,\theta,\phi,t)\right) + 2\,mp\left(\frac{\partial^2}{\partial r\partial t}\,\psi(r,\theta,\phi,t)\right) r + (-2\,mp + 2\,r)\left(\frac{\partial}{\partial r}\,\psi(r,\theta,\phi,t)\right) + 2\,mp\left(\frac{\partial}{\partial t}\,\psi(r,\theta,\phi,t)\right)\right) + \frac{\cos(\theta)\left(\frac{\partial}{\partial \theta}\,\psi(r,\theta,\phi,t)\right) + \sin(\theta)\left(\frac{\partial^2}{\partial \theta^2}\,\psi(r,\theta,\phi,t)\right)}{r^2 \sin(\theta)} + \frac{\frac{\partial^2}{\partial \phi^2}\,\psi(r,\theta,\phi,t)}{r^2 \sin(\theta)^2} \tag{13}$$

$$-\frac{(r+2\,mp)\left(\frac{\partial^2}{\partial t^2}\,\psi(r,\theta,\phi,t)\right)}{r}+\frac{2\,mp\left(\frac{\partial^2}{\partial r\partial t}\,\psi(r,\theta,\phi,t)\right)}{r}$$

> $$dA1 := \frac{1}{r^2}\Bigg(r\,(r-2\,mp)\left(\frac{\partial^2}{\partial r^2}\,\psi(r,\theta,\phi,t)\right)+2\,mp\left(\frac{\partial^2}{\partial r\partial t}\,\psi(r,\theta,\phi,t)\right)r+(-2\,mp+2\,r)\left(\frac{\partial}{\partial r}\,\psi(r,\theta,\phi,t)\right)+2\,mp\left(\frac{\partial}{\partial t}\,\psi(r,\theta,\phi,t)\right)\Bigg)+\Bigg(-\frac{(r+2\,mp)\left(\frac{\partial^2}{\partial t^2}\,\psi(r,\theta,\phi,t)\right)}{r}+\frac{2\,mp\left(\frac{\partial^2}{\partial r\partial t}\,\psi(r,\theta,\phi,t)\right)}{r}\Bigg):$$

> $$dA2 := +\frac{\cos(\theta)\left(\frac{\partial}{\partial\theta}\,\psi(r,\theta,\phi,t)\right)+\sin(\theta)\left(\frac{\partial^2}{\partial\theta^2}\,\psi(r,\theta,\phi,t)\right)}{r^2\sin(\theta)}+\frac{\frac{\partial^2}{\partial\phi^2}\,\psi(r,\theta,\phi,t)}{r^2\sin(\theta)^2}:$$

Assuming that :

> $\psi(r,\theta,\phi,t) := \Phi(r,t)\cdot Y(\theta,\phi)$ `# Y being a spherical harmonic`

$$\psi := (r,\theta,\phi,t)\mapsto \Phi(r,t)\cdot Y(\theta,\phi) \tag{14}$$

> $dA1 := simplify(dA1)$

$$dA1 := \frac{1}{r^2}\Bigg(Y(\theta,\phi)\Bigg(r\,(r-2\,mp)\left(\frac{\partial^2}{\partial r^2}\,\Phi(r,t)\right)-r\,(r+2\,mp)\left(\frac{\partial^2}{\partial t^2}\,\Phi(r,t)\right)+4\,mp\left(\frac{\partial^2}{\partial r\partial t}\,\Phi(r,t)\right)r+(-2\,mp+2\,r)\left(\frac{\partial}{\partial r}\,\Phi(r,t)\right)+2\,mp\left(\frac{\partial}{\partial t}\,\Phi(r,t)\right)\Bigg)\Bigg) \tag{15}$$

>

> $dA2 := simplify(dA2);$

$$dA2 := \frac{1}{r^2\sin(\theta)^2}\Bigg(\Phi(r,t)\Bigg(\sin(\theta)\cos(\theta)\left(\frac{\partial}{\partial\theta}\,Y(\theta,\phi)\right)-\left(\frac{\partial^2}{\partial\theta^2}\,Y(\theta,\phi)\right)\cos(\theta)^2+\frac{\partial^2}{\partial\theta^2}\,Y(\theta,\phi)+\frac{\partial^2}{\partial\phi^2}\,Y(\theta,\phi)\Bigg)\Bigg) \tag{16}$$

> $$dA2 := \frac{\Phi(r,t)}{r^2}\cdot\left(\frac{\partial^2}{\partial\theta^2}\,Y(\theta,\phi)+\frac{1}{\sin(\theta)^2}\,\frac{\partial^2}{\partial\phi^2}\,Y(\theta,\phi)+\frac{\cos(\theta)}{\sin(\theta)}\cdot\frac{\partial}{\partial\theta}\,Y(\theta,\phi)\right)$$

$$dA2 := \frac{\Phi(r,t)\left(\frac{\partial^2}{\partial\theta^2}\,Y(\theta,\phi)+\frac{\frac{\partial^2}{\partial\phi^2}\,Y(\theta,\phi)}{\sin(\theta)^2}+\frac{\cos(\theta)\left(\frac{\partial}{\partial\theta}\,Y(\theta,\phi)\right)}{\sin(\theta)}\right)}{r^2} \tag{17}$$

But from the theory of Harmonic fuctions we know that

> $$-\left(\frac{1}{\sin(\theta)}\cdot diff(\sin(\theta)\cdot diff(Y(\theta,\phi),\theta),\theta)+\frac{1}{\sin(\theta)^2}\cdot diff(Y(\theta,\phi),\phi,\phi)\right)=\ell\cdot(\ell+1)\cdot Y(\theta,\phi)$$

$$-\frac{\cos(\theta)\left(\frac{\partial}{\partial\theta}Y(\theta,\phi)\right)+\sin(\theta)\left(\frac{\partial^2}{\partial\theta^2}Y(\theta,\phi)\right)}{\sin(\theta)}-\frac{\frac{\partial^2}{\partial\phi^2}Y(\theta,\phi)}{\sin(\theta)^2}=\ell(\ell+1)\,Y(\theta,\phi) \quad (18)$$

Therefore:

$$> dA2 := -\frac{\Phi(r,t)}{r^2}\cdot\ell\cdot(\ell+1)\cdot Y(\theta,\phi)$$

$$dA2 := -\frac{\Phi(r,t)\,\ell(\ell+1)\,Y(\theta,\phi)}{r^2} \quad (19)$$

$$> dAlembert := dA1 + dA2;$$

$$dAlembert := \frac{1}{r^2}\left(Y(\theta,\phi)\left(r\,(r-2\,mp)\left(\frac{\partial^2}{\partial r^2}\Phi(r,t)\right)-r\,(r+2\,mp)\left(\frac{\partial^2}{\partial t^2}\Phi(r,t)\right)+4\,mp\left(\frac{\partial^2}{\partial r\partial t}\Phi(r,t)\right)r+(-2\,mp+2\,r)\left(\frac{\partial}{\partial r}\Phi(r,t)\right)+2\,mp\left(\frac{\partial}{\partial t}\Phi(r,t)\right)\right)\right)-\frac{\Phi(r,t)\,\ell(\ell+1)\,Y(\theta,\phi)}{r^2} \quad (20)$$

Assuming now that

$$> \Phi(r,t) := A(r)\cdot\exp(I\cdot E\cdot t);$$

$$\Phi := (r,t)\mapsto A(r)\cdot e^{I\cdot E\cdot t} \quad (21)$$

We get:

$$> EquA := coeff(dAlembert, Y(\theta,\phi))$$

$$EquA := \frac{1}{r^2}\left(r\,(r-2\,mp)\left(\frac{d^2}{dr^2}A(r)\right)e^{IEt}+r\,(r+2\,mp)\,A(r)\,E^2\,e^{IEt}+4\,I\,mp\left(\frac{d}{dr}A(r)\right)E\,e^{IEt}\,r+(-2\,mp+2\,r)\left(\frac{d}{dr}A(r)\right)e^{IEt}+2\,I\,mp\,A(r)\,E\,e^{IEt}\right)-\frac{A(r)\,e^{IEt}\,\ell(\ell+1)}{r^2} \quad (22)$$

and

$$> EquA := coeff(EquA, e^{I\cdot E\cdot t})$$

$$EquA := \frac{1}{r^2}\left(r\,(r-2\,mp)\left(\frac{d^2}{dr^2}A(r)\right)+r\,(r+2\,mp)\,A(r)\,E^2+4\,I\,mp\left(\frac{d}{dr}A(r)\right)E\,r+(-2\,mp+2\,r)\left(\frac{d}{dr}A(r)\right)+2\,I\,mp\,A(r)\,E\right)-\frac{A(r)\,\ell(\ell+1)}{r^2} \quad (23)$$

And therefore

$$> KGE := EquA-\left(\frac{c^3}{G\cdot\hbar}\right)^2\cdot m^2\cdot A(r)=0$$

(24)

$$KGE := \frac{1}{r^2}\left(r\,(r-2\,mp)\left(\frac{d^2}{dr^2}A(r)\right)+r\,(r+2\,mp)\,A(r)\,E^2+4\,\mathrm{I}\,mp\left(\frac{d}{dr}A(r)\right)E\,r+(-2\,mp+2\,r)\left(\frac{d}{dr}A(r)\right)+2\,\mathrm{I}\,mp\,A(r)\,E\right)-\frac{A(r)\,\ell(\ell+1)}{r^2}-m^2A(r)=0 \tag{24}$$

This equation would not change if Droste´s coordinates were used. But it woud if Fock's or Brillouin's coordinates were used.

> $dsolve(KGE)$

$$A(r)=\_C1\,e^{\sqrt{-E^2+m^2}\,r}\,\mathrm{HeunC}\left(-4\,mp\sqrt{-E^2+m^2},4\,\mathrm{I}\,mp\,E,0,(-8\,E^2+4\,m^2)\,mp^2,-\ell^2-\ell+(8\,E^2-4\,m^2)\,mp^2,\frac{-r+2\,mp}{2\,mp}\right)+\_C2\,e^{\sqrt{-E^2+m^2}\,r}\,\mathrm{HeunC}\left(-4\,mp\sqrt{-E^2+m^2},-4\,\mathrm{I}\,mp\,E,0,(-8\,E^2+4\,m^2)\,mp^2,-\ell^2-\ell+(8\,E^2-4\,m^2)\,mp^2,\frac{-r+2\,mp}{2\,mp}\right)(r-2\,mp)^{-4\,\mathrm{I}\,mp\,E} \tag{25}$$

Below only the following solutions are considered

> $A1 := (\ell,r)\to e^{-\sqrt{-E(\ell)^2+m^2}\,r}\,\mathrm{HeunC}\left(-4\cdot mp\sqrt{-E(\ell)^2+m^2},4\,\mathrm{I}\cdot mp\cdot E(\ell),0,-8\cdot mp^2\cdot\left(E(\ell)^2-\frac{m^2}{2}\right),(-4\,m^2+8\,E(\ell)^2)\,mp^2-\ell^2-\ell,\frac{-r+2\,mp}{2\,mp}\right):$

Solutions with a factor $e^{+\sqrt{-E(\ell)^2+m^2}\,r}$ lead to functions without norm. And those with a coeffient _C2 different from 0 did not look satisfactory. They lead to solutions constrained in the intervals [0,2] or [2,∞]

$$e^{\sqrt{-E(\ell)^2+m^2}\,r} \tag{26}$$

> $\_C1 := 1;\ \_C2 := 0;$

$$\_C1 := 1$$
$$\_C2 := 0 \tag{27}$$

Mass selection---

> $\alpha := -4\,mp\sqrt{-E^2+m^2};\ \beta := +4\,\mathrm{I}\ mp\cdot E;\ \gamma := 0;\ \delta := -(8\,E^2-4\cdot m^2)\,mp^2;\ \eta := (-4\cdot m^2+8\cdot E^2)\cdot mp^2-\ell^2-\ell;$

$$\alpha := -4\,mp\sqrt{-E^2+m^2}$$
$$\beta := 4\,\mathrm{I}\,mp\,E$$
$$\gamma := 0$$
$$\delta := -(8\,E^2-4\,m^2)\,mp^2$$
$$\eta := -\ell^2-\ell+(8\,E^2-4\,m^2)\,mp^2 \tag{28}$$

Maple 2020 Help page on HeunC functions suggests to use the inert quantization rule below, known to work in some familiar cases

> $Equu := \delta = -\left(\ell + \frac{(\gamma+\beta+2)}{2}\right)\cdot\alpha$ 3# a + front sign is OK also

$$Equ := -(8E^2 - 4m^2)\, mp^2 = 4\,(\ell + 1 + 2\,\mathrm{I}\, mp\, E)\, mp\sqrt{-E^2 + m^2} \quad \textbf{(29)}$$

> $solve(Equ)$

$$\{\ell = \ell, E = E, m = m, mp = 0\}, \left\{\ell = -\frac{-2\,\mathrm{I}\,E\sqrt{-E^2+m^2}\;mp + 2E^2 mp - m^2 mp + \sqrt{-E^2+m^2}}{\sqrt{-E^2+m^2}}, E = E, m = m, mp = mp\right\}, \left\{\ell = -\frac{2\,\mathrm{I}\,E\sqrt{-E^2+m^2}\;mp + 2E^2 mp - m^2 mp + \sqrt{-E^2+m^2}}{\sqrt{-E^2+m^2}}, E = E, m = m, mp = mp\right\}, \{\ell = \ell, E = 0, m = 0, mp = mp\} \quad \textbf{(30)}$$

> $Equ(\ell) := \ell = -\frac{2\,\mathrm{I}\,E\sqrt{-E^2+m^2}\;mp + 2E^2 mp - m^2 mp + \sqrt{-E^2+m^2}}{\sqrt{-E^2+m^2}}$:

> $mp := 1;\ \ m := mp$

$$mp := 1$$
$$m := 1 \quad \textbf{(31)}$$

> $simplify(Equ(\ell));$

$$\ell = \frac{-2\,\mathrm{I}\,E\sqrt{-E^2+1} - 1 + 2E^2 - \sqrt{-E^2+1}}{\sqrt{-E^2+1}} \quad \textbf{(32)}$$

> $E(0) := solve(Equ(0), E)$:

> $E(0) := evalf(E(0));$

$$E(0) := -0.9921567416 + 0.1250000000\,\mathrm{I},\ 0.9921567416 + 0.1250000000\,\mathrm{I} \quad \textbf{(33)}$$

> $E(1) := solve(Equ(1), E)$:

> $E(1) := evalf(E(1));$

$$E(1) := 0.9770466965 + 0.0541891430\,\mathrm{I},\ -0.9770466964 + 0.0541891431\,\mathrm{I} \quad \textbf{(34)}$$

> $E(2) := solve(Equ(2), E)$:

> $E(2) := evalf(E(2));$

$$E(2) := 0.9798879590 + 0.0281288567\,\mathrm{I},\ -0.9798879593 + 0.0281288568\,\mathrm{I} \quad \textbf{(35)}$$

> $E(3) := solve(Equ(3), E)$:

> $E(3) := evalf(E(3));$

$$E(3) := 0.9841272246 + 0.0161374277\,\mathrm{I},\ -0.9841272246 + 0.0161374278\,\mathrm{I} \quad \textbf{(36)}$$

> $E(4) := solve(Equ(4), E)$:

> $E(4) := evalf(E(4));$

$$E(4) := 0.9876374546 + 0.0099476503\,\mathrm{I},\ -0.9876374547 + 0.0099476504\,\mathrm{I} \quad \textbf{(37)}$$

> $E(5) := solve(Equ(5), E)$:

> $E(5) := evalf(E(5));$

$$E(5) := 0.9902846581 + 0.0064875200\,\mathrm{I},\ -0.9902846583 + 0.0064875194\,\mathrm{I} \quad \textbf{(38)}$$

> $E(6) := solve(Equ(6), E)$:

> $E(6) := evalf(E(6));$

$$E(6) := 0.9922743134 + 0.0044143267\,I,\ -0.9922743132 + 0.0044143275\,I \quad \textbf{(39)}$$

The six lowest mode solutions are listed below. The coma separates positive and negative modes. Notice the always positive imaginary coefficient of I meaning that the modes are always decaying in time (TU is the time unit)

The unverse of the real part might be interpreted as the frequency ν of the transition from the excited state to the stationary one, while the inverse of the maginary part τ ,that it is always positive , can be interpreted as the mean live of the excited state.

> $A1R := (\ell, r) \to \mathrm{Re}(A1(\ell, r));\ A1I := (\ell, r) \to \mathrm{Im}(A1(\ell, r));$

$$A1R := (\ell, r) \mapsto \Re(A1(\ell, r))$$

$$A1I := (\ell, r) \mapsto \Im(A1(\ell, r)) \quad \textbf{(40)}$$

> $F1 := (\ell, r) \to r^2 \cdot A1(\ell, r) \cdot conjugate(A1(\ell, r))$

$$F1 := (\ell, r) \mapsto r^2 \cdot A1(\ell, r) \cdot \overline{A1(\ell, r)} \quad \textbf{(41)}$$

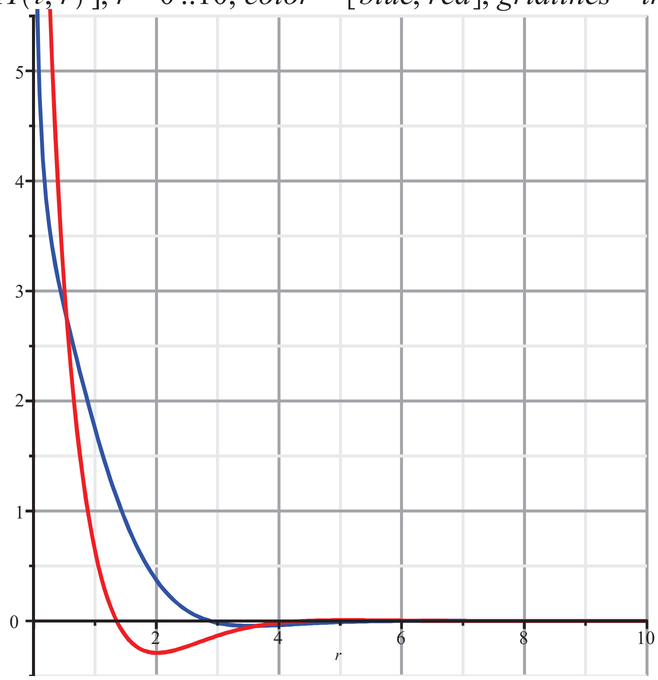
>

> $\ell := 0;\ E(0) := E(0)[1];$

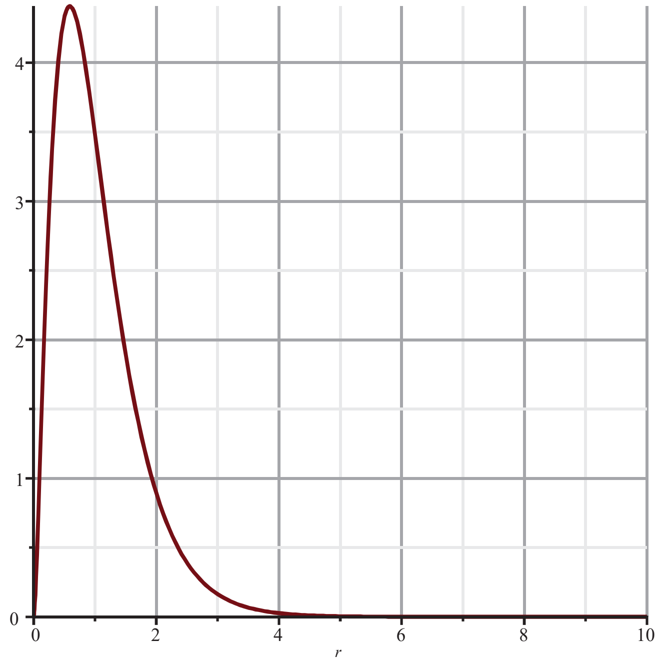
$$\ell := 0$$

$$E(0) := -0.9921567416 + 0.1250000000\,I \quad \textbf{(42)}$$

> $plot([A1R(\ell, r), A1I(\ell, r)], r = 0..10, color = [blue, red], gridlines = true)$



> $plot(F1(\ell, r), r = 0..10, gridlines = true)$

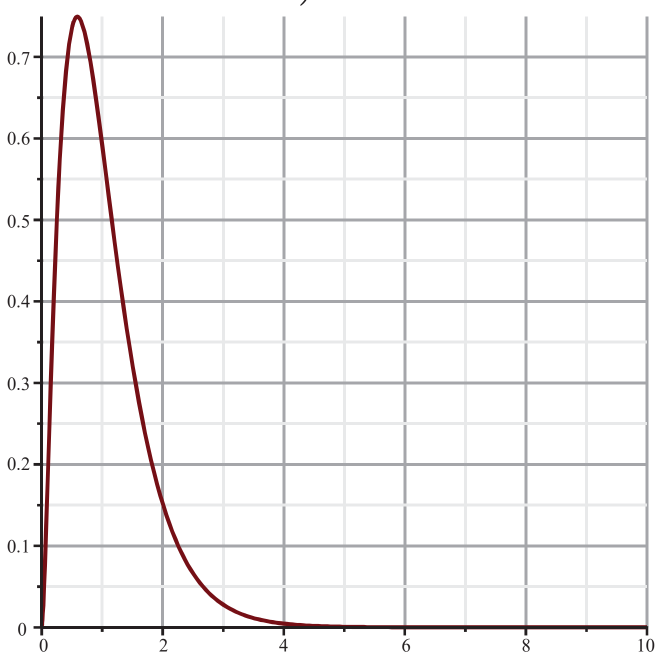


> #IntF1(0) := evalf(int(F1(0, r), r = 0 ..10))

> IntF1(0) := 5.881599832

$$IntF1(0) := 5.881599832 \quad (43)$$

> $plot\left(\frac{F1(\ell, r)}{IntF1(\ell)}, r = 0 ..10, gridlines = true\right)$



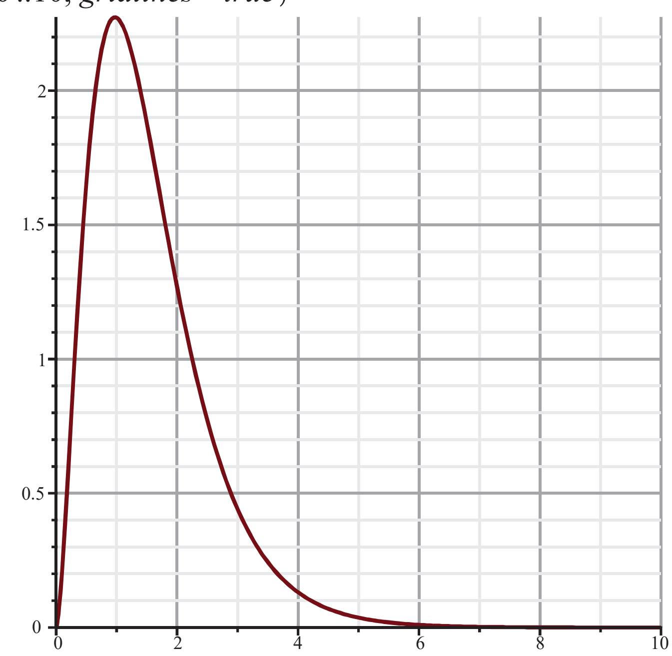
> ℓ := 1; E(1) := E(1)[1];

$$\ell := 1$$

$$E(1) := 0.9770466965 + 0.0541891430\, I \quad (44)$$

> plot([A1R(ℓ, r), A1I(ℓ, r)], r = 0 ..10, color = [blue, red], gridlines = true);

> $plot(F1(\ell, r), r = 0..10, gridlines = true)$

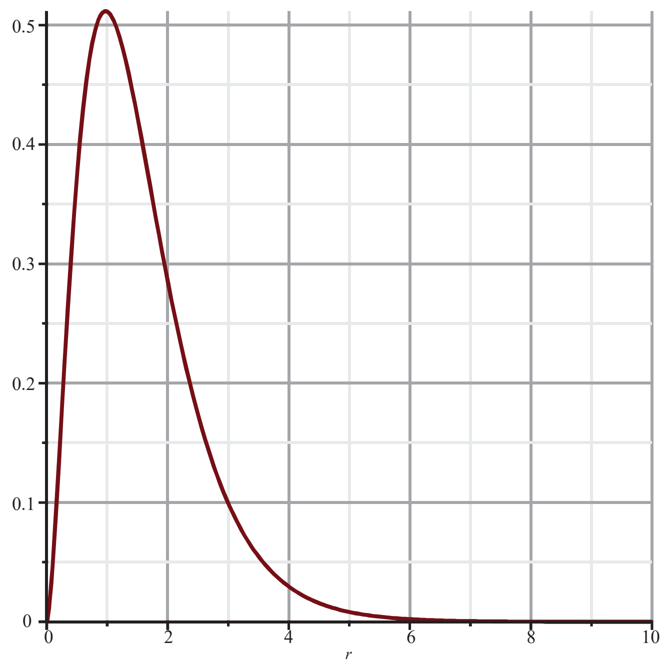


> $\#IntF1(1) := evalf(int(F1(\ell, r), r = 0..10))$

> $IntF1(1) := 4.443555244$

$$IntF1(1) := 4.443555244 \tag{45}$$

> $plot\left(\frac{F1(\ell, r)}{IntF1(\ell)}, r = 0..10, gridlines = true\right)$

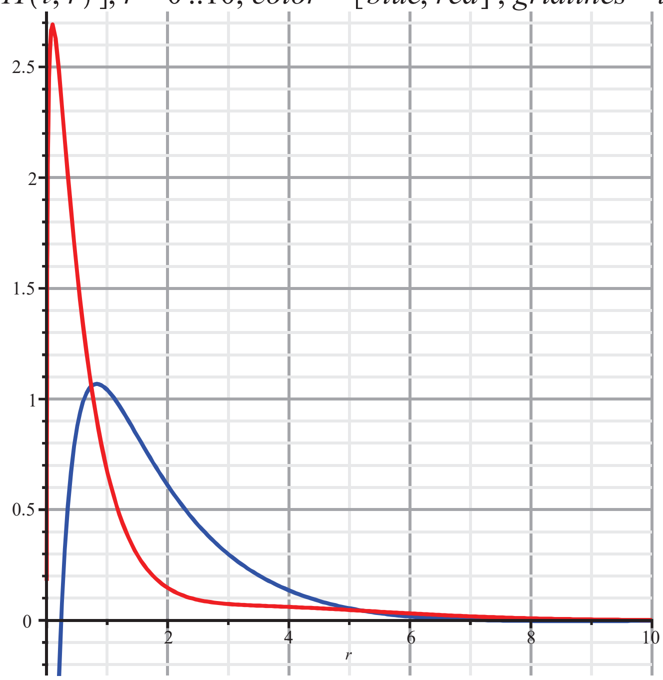


> $\ell := 2;\quad E(2) := E(2)[1];$

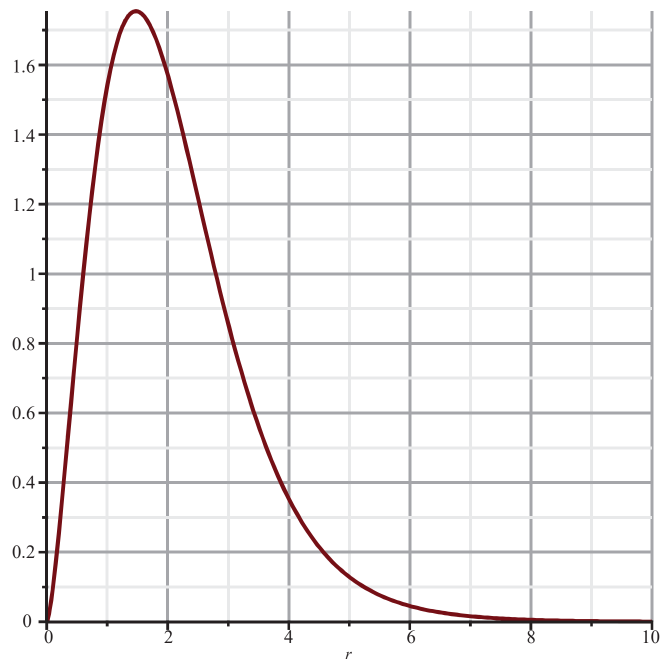
$$\ell := 2$$

$$E(2) := 0.9798879590 + 0.0281288567\,\mathrm{I} \tag{46}$$

> $plot([A1R(\ell, r), A1I(\ell, r)], r = 0 ..10, color = [blue, red], gridlines = true);$



> $plot(F1(2, r), r = 0 ..10, gridlines = true)$

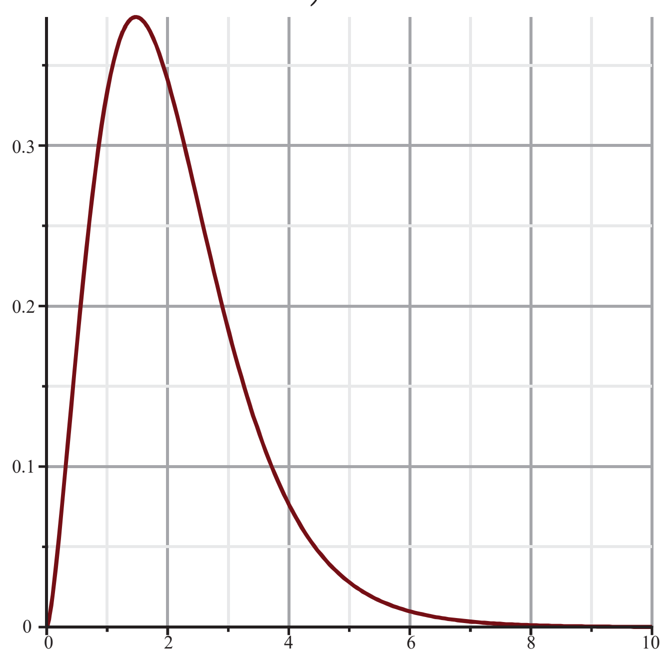


> $\#IntF1(2) := evalf(int(F1(2, r), r = 0 .. 10))$

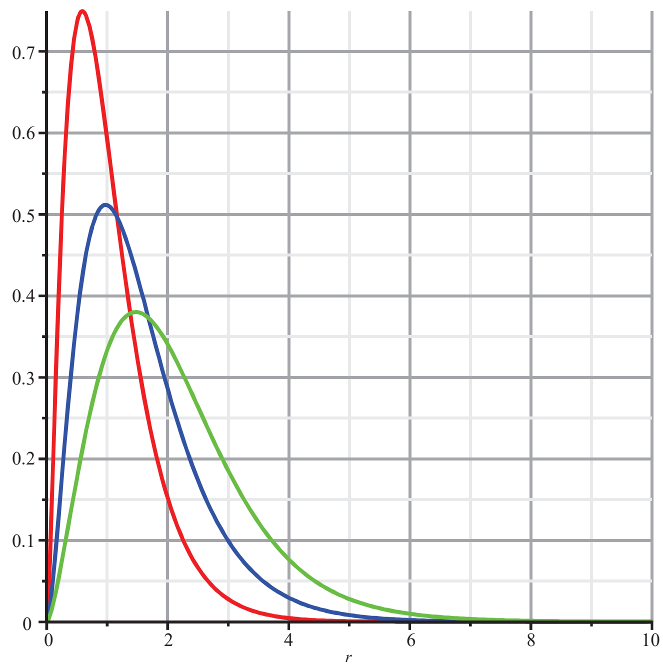
> $IntF1(2) := 4.615589828$

$$IntF1(2) := 4.615589828 \quad (47)$$

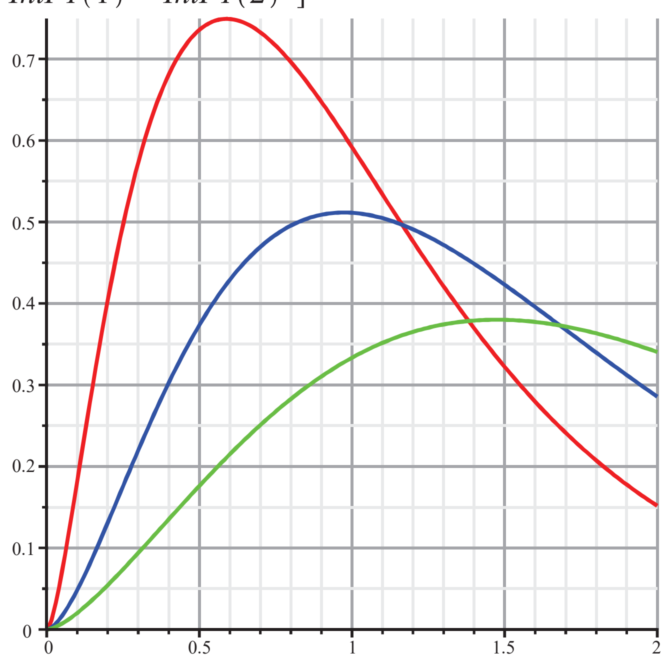
> $plot\left(\frac{F1(2, r)}{IntF1(2)}, r = 0 .. 10, gridlines = true\right)$



> $plot\left(\left[\frac{F1(0, r)}{IntF1(0)}, \frac{F1(1, r)}{IntF1(1)}, \frac{F1(2, r)}{IntF1(2)}\right], r = 0 .. 10, color = [red, blue, green], gridlines = true\right)$



> $plot\left(\left[\frac{F1(0,r)}{IntF1(0)}, \frac{F1(1,r)}{IntF1(1)}, \frac{F1(2,r)}{IntF1(2)}\right], r=0..2, color=[red, blue, green], gridlines=true\right)$



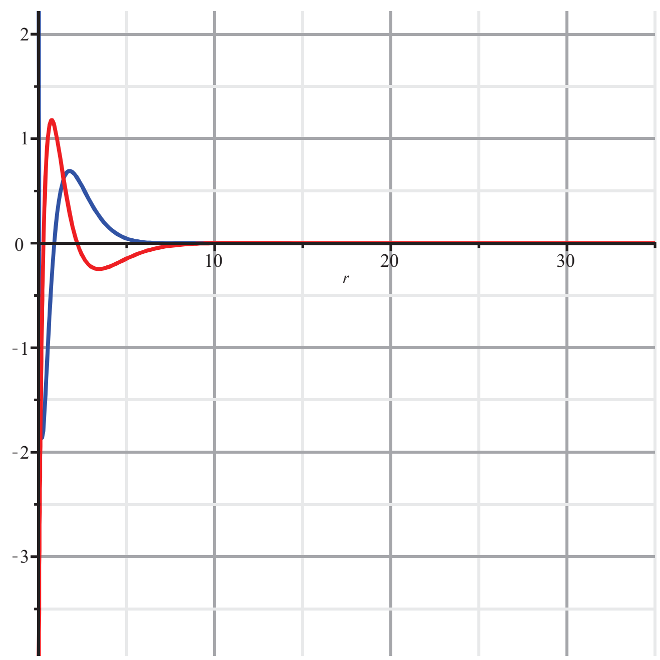
Notice that these three modes reach their maximum value inside the horizon interval[0,2]. There are no equivalent modes at the Newtonian approximation (L.Bel Cf. C. I. M.E. Relatività Generale. Edizioni Cremonese. Roma 1965)

> $\ell := 3;\quad E(3) := E(3)[1];$

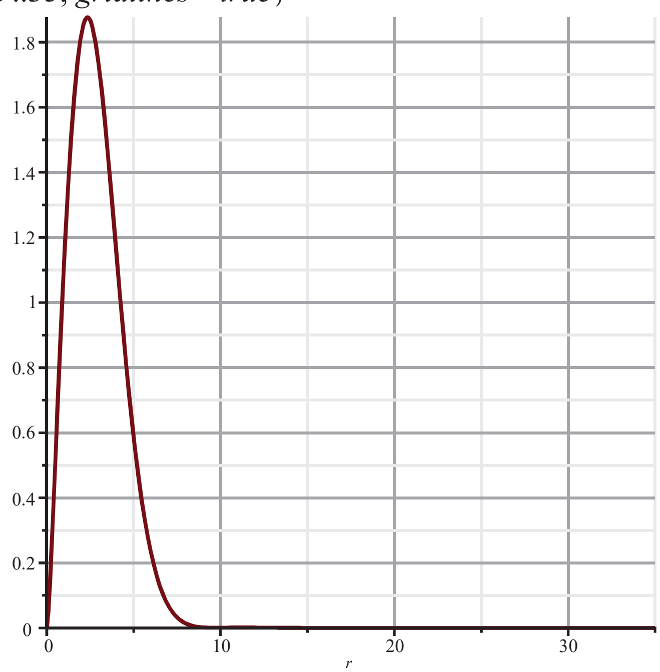
$$\ell := 3$$

$$E(3) := 0.9841272246 + 0.0161374277\ I \qquad \textbf{(48)}$$

> $plot([A1R(\ell, r), A1I(\ell, r)], r=0..35, color=[blue, red], gridlines=true);$



> $plot(F1(3, r), r = 0..35, gridlines = true)$

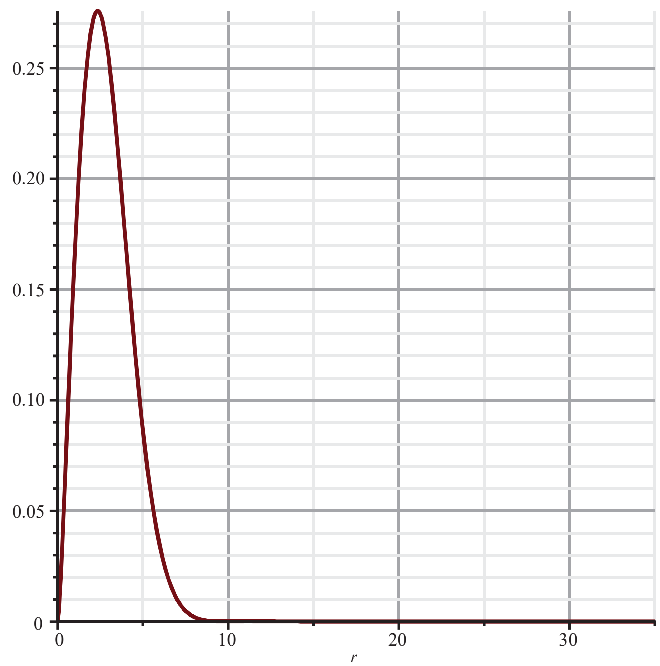


> $\#IntF1(3) := evalf(int(F1(3, r), r = 0..35))$

> $IntF1(3) := 6.802439516$

$$IntF1(3) := 6.802439516 \quad (49)$$

> $plot\left(\frac{F1(3, r)}{IntF1(3)}, r = 0..35, gridlines = true\right)$



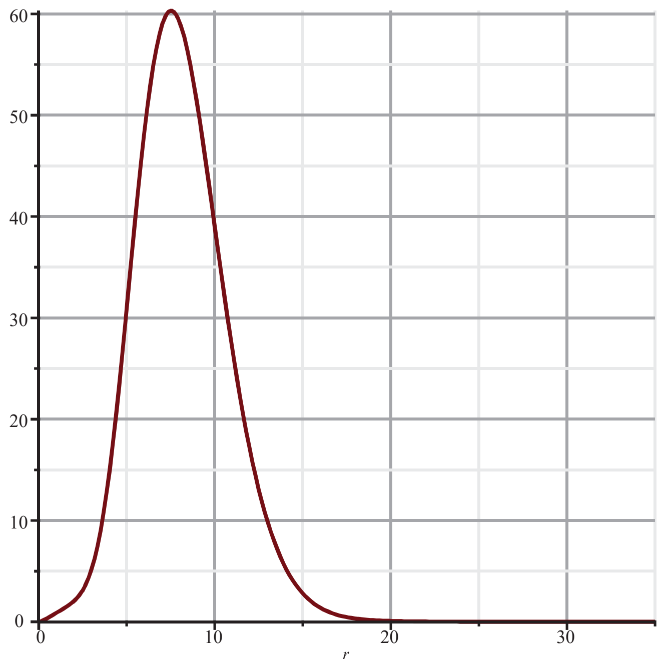
> $\ell := 4;\quad E(4) := E(4)[1];$

$$\ell := 4$$

$$E(4) := 0.9876374546 + 0.0099476503\,\mathrm{I} \tag{50}$$

> $plot([A1R(\ell, r), A1I(\ell, r)], r = 0..35, color = [blue, red], gridlines = true);$

> $plot(F1(\ell, r), r = 0..35, gridlines = true)$

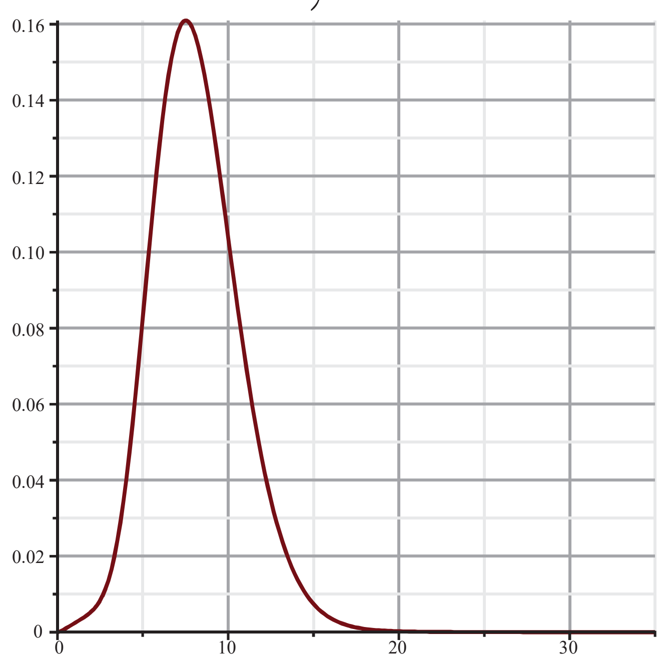


> #IntF1(4) := evalf(int(F1(4, r), r = 0 ..35))

> IntF1(4) := 374.6385090

$$IntF1(4) := 374.6385090 \quad \textbf{(51)}$$

> $plot\left(\frac{F1(4, r)}{IntF1(4)}, r = 0 ..35, gridlines = true\right)$

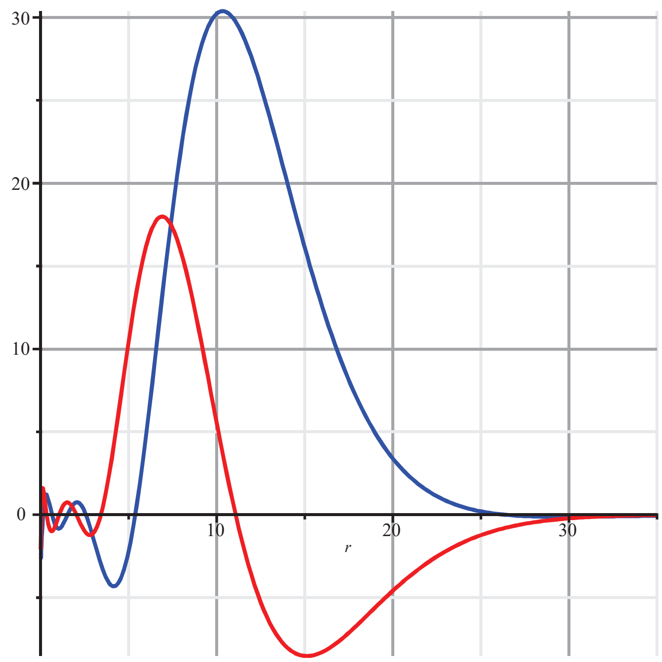


> ℓ := 5; E(5) := E(5)[1];

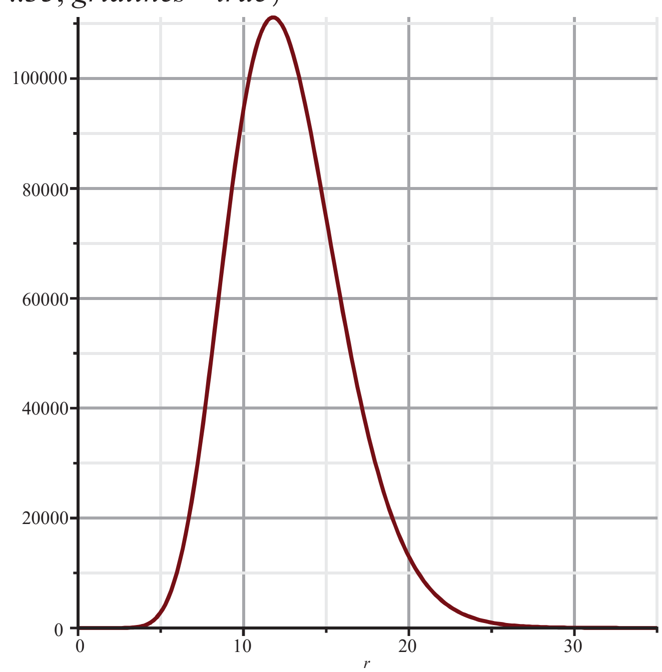
$$\ell := 5$$

$$E(5) := 0.9902846581 + 0.0064875200\ \mathrm{I} \quad \textbf{(52)}$$

> plot([A1R(ℓ, r), A1I(ℓ, r)], r = 0 ..35, color = [blue, red], gridlines = true);



> $plot(F1(\ell, r), r = 0\,..35, gridlines = true)$

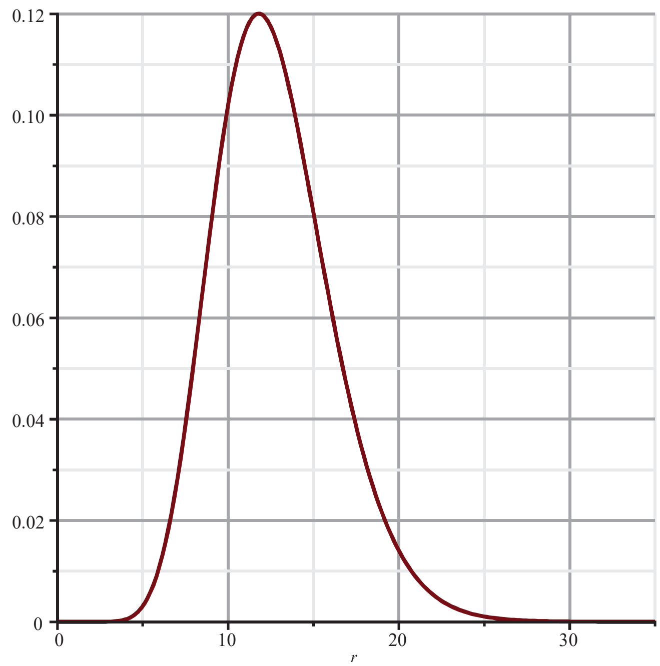


> $\#IntF1(\ell) := evalf(int(F1(\ell, r), r = 0\,..35))$

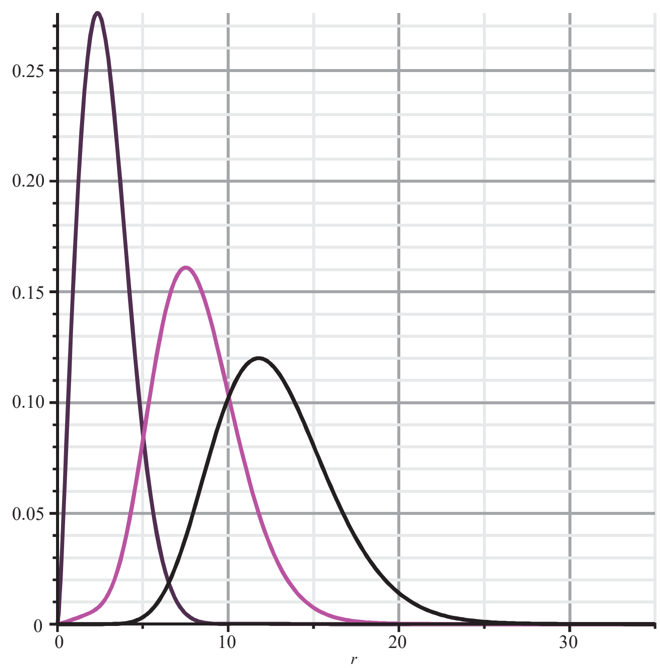
> $IntF1(5) := 925708.6231$

$$IntF1(5) := 925708.6231 \tag{53}$$

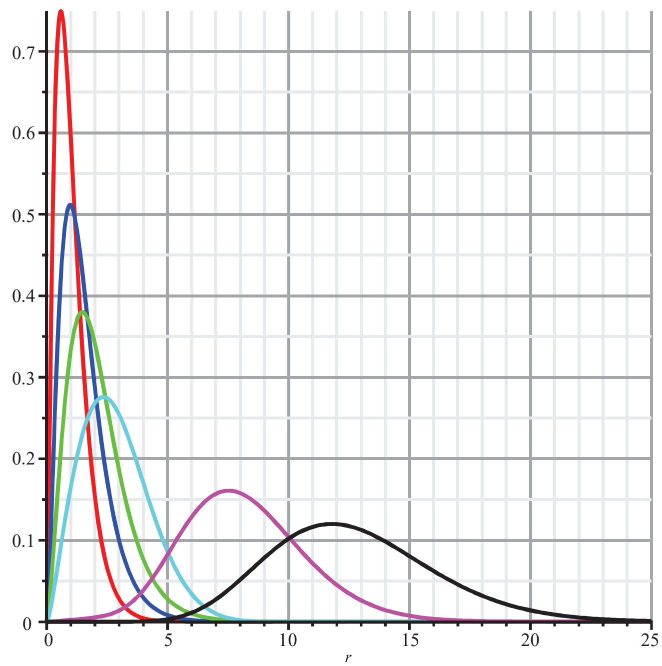
> $plot\left(\dfrac{F1(5, r)}{IntF1(5)}, r = 0\,..35, gridlines = true\right)$



> $plot\left(\left[\frac{F1(3,r)}{IntF1(3)}, \frac{F1(4,r)}{IntF1(4)}, \frac{F1(5,r)}{IntF1(5)}\right], r=0..35, color=[violet, magenta, black], gridlines=true\right)$



> $plot\left(\left[\frac{F1(0,r)}{IntF1(0)}, \frac{F1(1,r)}{IntF1(1)}, \frac{F1(2,r)}{IntF1(2)}, \frac{F1(3,r)}{IntF1(3)}, \frac{F1(4,r)}{IntF1(4)}, \frac{F1(5,r)}{IntF1(5)}\right], r=0..25, color=[red, blue, green, cyan, magenta, black], gridlines=true\right)$


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
20
25
r

>