# Solving Linear Ordinary And Partial Differential Equations by Factoring

Claude Michael Cassano

Abstract

Techniques and tools for exactly solving Ricatti and Linear Ordinary and Partial Differential equations are developed from factoring method. Therefrom, the one-space dimension the Wave and Helmholtz/Klein-Gordon equation may be factored; with example solution - leading to generalization of the Maxwell-Cassano equations of an electromagnetic-nuclear field for non-constant mass and what the general high energy Lagrangian equations really are (including Weak force, etc. equations) - guiding transformations between the general high energy Lagrangians equations in general coordinates and Cartesian coordinate PDEs.

A first order linear ordinary differential equation (LODE) may be written:

Since, for any HLODE, the middle coefficient may be transformed to any value; another second order HLODEs may be written (without loss of generality):

$$W = y' + Py = \left(ye^{\int Pdx}\right)' e^{-\int Pdx}$$

$$\Rightarrow y = e^{-\int Pdx}\int e^{\int Pdx} Wdx$$

**Theorem I.1**: For twice differentiable function $y$ and differentiable functions $g, h$ :

$$\left(\left(\left(ye^{\int gdx}\right)' e^{-\int gdx}\right)e^{\int hdx}\right)' e^{--\int hdx} = y'' + (g+h)y' + (g' + hg)y$$

*Proof:*

$$\left(\left(\left(ye^{\int gdx}\right)' e^{-\int gdx}\right)e^{\int hdx}\right)' e^{--\int hdx} = \left(\left(\left(ye^{\int gdx}\right)' e^{-\int gdx}\right)' e^{\int hdx} + \left(\left(ye^{\int gdx}\right)' e^{-\int gdx}\right)he^{\int hdx}\right)e^{--\int hdx}$$

$$= \left(\left(ye^{\int gdx}\right)' e^{-\int gdx}\right)' + \left(\left(ye^{\int gdx}\right)' e^{-\int gdx}\right)h$$

$$= \left(\left(y'e^{\int gdx} + yge^{\int gdx}\right)e^{-\int gdx}\right)' + \left(\left(y'e^{\int gdx} + yge^{\int gdx}\right)e^{-\int gdx}\right)h$$

$$= ((y' + yg))' + ((y' + yg))h$$

$$= y' + y'g + yg' + hy' + yhg$$

$$= y' + (g+h)y' + (g' + hg)y$$

□

**Corollary I.1**: For twice differentiable function $y$ and differentiable functions $g, h, P, Q$ :

$$P = g + h \quad \& \quad Q = g' + gh$$

$$\Rightarrow \left(\left(\left(ye^{\int gdx}\right)' e^{-\int gdx}\right)e^{\int hdx}\right)' e^{--\int hdx} = y'' + Py' + Qy$$

*Proof:*

So, from theorem I.1:

$$\left(\left(\left(ye^{\int gdx}\right)' e^{-\int gdx}\right)e^{\int hdx}\right)' e^{--\int hdx} = y'' + Py' + Qy$$

□

**Theorem I.2**: For twice differentiable function $y$ and differentiable functions $g, h$ :

$$\left(\left(\left(ye^{-\int gdx}\right)' e^{\int gdx}\right)e^{-\int hdx}\right)' e^{\int hdx} = y'' + (-g-h)y' + (-g' + hg)y$$

*Proof:*

$$\left(\left(\left(ye^{-\int gdx}\right)' e^{\int gdx}\right)e^{-\int hdx}\right)' e^{\int hdx} = \left(\left(\left(ye^{-\int gdx}\right)' e^{\int gdx}\right)' e^{-\int hdx} + \left(\left(ye^{-\int gdx}\right)' e^{\int gdx}\right)(-h)e^{-\int hdx}\right)e^{\int hdx}$$

$$= \left(\left(ye^{-\int gdx}\right)' e^{\int gdx}\right)' + \left(\left(ye^{-\int gdx}\right)' e^{\int gdx}\right)(-h)$$

$$= \left(\left(y'e^{-\int gdx} - gye^{-\int gdx}\right)e^{\int gdx}\right)' + \left(\left(y'e^{-\int gdx} - gye^{-\int gdx}\right)e^{\int gdx}\right)(-h)$$

$$= ((y' - gy))' + ((y' - gy))(-h)$$

$$= y'' - gy' - g'y - hy' + hgy$$

$$= y'' + (-g-h)y' + (-g' + hg)y$$

□

**Corollary I.2**: For twice differentiable function $y$ and differentiable functions $g, h, P, Q$ :

$$P = -g - h \quad \& \quad Q = -g' + gh$$

$$\Rightarrow \left(\left(\left(ye^{-\int gdx}\right)' e^{\int gdx}\right)e^{-\int hdx}\right)' e^{\int hdx} = y'' + Py' + Qy$$

*Proof:*

So, from theorem I.1:

$$\left(\left(\left(ye^{\int gdx}\right)' e^{-\int gdx}\right)e^{\int hdx}\right)' e^{--\int hdx} = y'' + Py' + Qy$$

□

**Corollary I.3**: For twice differentiable function $y$ and differentiable functions $g, h, P, Q, W$ :

$$y_1 = e^{-\int gdx}$$

is a solution to homogeneous ODE:

$$\left(\left(\left(ye^{\int gdx}\right)' e^{-\int gdx}\right) e^{\int hdx}\right)' e^{-\int hdx} = y'' + (g+h)y' + (g' + hg)y$$

$$= y'' + Py' + Qy \quad , \quad \left(P = g + h \quad \& \quad Q = g' + gh\right)$$

$$Q = -\left(-\frac{y_1'}{y_1}\right)' - \left(-\frac{y_1'}{y_1}\right)^2 + P\left(-\frac{y_1'}{y_1}\right)$$

$$\left(\left(\left(ye^{\int gdx}\right)' e^{-\int gdx}\right) e^{\int hdx}\right)' e^{-\int hdx} = W$$

$$\Rightarrow y = y_1 + c_1 y_1 \int y_1^{-2} e^{-\int Pdx} dx + y_1 \int y_1^{-2} e^{-\int Pdx}\left(\int y_1 W e^{\int Pdx} dx\right) dx$$

*Proof:*

$$\left(\left(\left(y_1 e^{\int gdx}\right)' e^{-\int gdx}\right) e^{\int hdx}\right)' e^{-\int hdx} = \left(\left(\left(e^{-\int gdx} e^{\int gdx}\right)' e^{-\int gdx}\right) e^{\int hdx}\right)' e^{-\int hdx}$$

$$= \left(\left((1)' e^{-\int gdx}\right) e^{\int hdx}\right)' e^{-\int hdx} = 0$$

$$y_1 = e^{-\int gdx} \Rightarrow y_1' = -ge^{-\int gdx} = -gy_1 \Rightarrow \frac{y_1'}{y_1} = -g$$

$$\Rightarrow \left(-\frac{y_1'}{y_1}\right)' - \left(-\frac{y_1'}{y_1}\right)^2 + P\left(-\frac{y_1'}{y_1}\right) = g' - g^2 + Pg = g' + g(P - g) = g' + gh = Q$$

$$\Rightarrow P = g + \left(\frac{Q - g'}{g}\right) \quad \& \quad -Q = g' + g\left(\frac{Q - g'}{g}\right)$$

i.e.:

$$h = \left(\frac{Q - g'}{g}\right) \Rightarrow P = g + h \quad \& \quad --Q = g' + gh$$

$$P = \left(-\frac{y_1'}{y_1}\right) + \left(\frac{Q - \left(-\frac{y_1'}{y_1}\right)'}{\left(-\frac{y_1'}{y_1}\right)}\right) \quad \& \quad Q = \left(-\frac{y_1'}{y_1}\right)' + \left(-\frac{y_1'}{y_1}\right)\left(\frac{Q - \left(-\frac{y_1'}{y_1}\right)'}{\left(-\frac{y_1'}{y_1}\right)}\right)$$

So, from theorem I.1:

$$\left(\left(\left(ye^{\int gdx}\right)' e^{-\int gdx}\right) e^{\int hdx}\right)' e^{-\int hdx} = W$$

$$\Rightarrow y = e^{-\int gdx} \int e^{\int (g-h)dx}\left(\int We^{\int hdx} dx\right) dx + c_1 e^{-\int gdx} \int e^{\int (g-h)dx} dx + c_2 e^{-\int gdx}$$

$$\Rightarrow y = c_2 y_1 + c_1 y_1 \int y_1^{-2} e^{-\int Pdx} dx + y_1 \int y_1^{-2} e^{-\int Pdx}\left(\int y_1 W e^{\int Pdx} dx\right) dx$$

□

**Corollary I.4**: If $y_1$ is a homogeneous solution to a linear homogeneous ordinary differential equation::

$$y_1'' + Py_1' + Qy_1 = 0 \qquad \left(P, Q \text{ differentiable functions}\right)$$

and if:

$$y'' + Py' + Qy = W \qquad \left(W \text{ differentiable function}\right)$$

$$\Rightarrow y = y_1 + c_1 y_1 \int y_1^{-2} e^{-\int Pdx} dx + y_1 \int y_1^{-2} e^{-\int Pdx}\left(\int y_1 W e^{\int Pdx} dx\right) dx$$

*Proof:*

Let: $-g = -\frac{y_1'}{y_1} = -(\log(y_1))' \Rightarrow y_1 = e^{-\int gdx}$

$$\Rightarrow -\left(-\frac{y_1'}{y_1}\right)' - \left(-\frac{y_1'}{y_1}\right)^2 = -\left(-\frac{y_1(y_1')' - y_1'y_1'}{y_1^2}\right) - \left(-\frac{y_1'}{y_1}\right)^2 = -\frac{y_1''}{y_1}$$

$$\Rightarrow y_1\left[-\left(-\frac{y_1'}{y_1}\right)' - \left(-\frac{y_1'}{y_1}\right)^2\right] = -y_1'' = Py_1' + Qy_1$$

$$\Rightarrow Q = \left(-\frac{y_1'}{y_1}\right)' - \left(-\frac{y_1'}{y_1}\right)^2 + P\left(-\frac{y_1'}{y_1}\right) = g' - g^2 + Pg$$

$$\Rightarrow P = g + \left(\frac{Q - g'}{g}\right) \quad \& \quad -Q = g' + g\left(\frac{Q - g'}{g}\right)$$

$$\Rightarrow -\left(-\frac{y_1'}{y_1}\right)' - \left(-\frac{y_1'}{y_1}\right)^2 = -P\left(-\frac{y_1'}{y_1}\right) + Q$$

So, by corollary I.3:

$$\left(\left(\left(ye^{\int gdx}\right)' e^{-\int gdx}\right) e^{\int hdx}\right)' e^{-\int hdx} = W$$

$$\Rightarrow y = y_1 + c_1 y_1 \int y_1^{-2} e^{-\int Pdx} dx + y_1 \int y_1^{-2} e^{-\int Pdx}\left(\int y_1 W e^{\int Pdx} dx\right) dx$$

□

**Lemma I.5**: If $P, U \;\&\; V$ are differentiable functions, and:

$$u = e^{\int (U+V)dx}$$

$$\Rightarrow \begin{cases} u'' + Pu' + \left[-(U+V)' - (U+V)^2 - P(U+V)\right]u = 0 \\ u'' + Pu' + [-(U' + U^2) - (V' + V^2) - 2UV - PU - PV]u = 0 \end{cases}$$

*Proof:*

$$u = e^{\int (U+V)dx} \Rightarrow u' = (U+V)u \Rightarrow u'' = \left[(U+V)' + (U+V)^2\right]u$$

$$\Rightarrow u'' + Pu' = \left[(U+V)' + (U+V)^2 + P(U+V)\right]u = 0$$

$$\Rightarrow u'' + Pu' + \left[-(U+V)' - (U+V)^2 - P(U+V)\right]u = 0$$

$\Rightarrow u'' + Pu' + [-U' - V' - U^2 - 2UV - V^2 - PU - PV]u = 0$
$\Rightarrow u'' + Pu' + [-(U' + U^2) - (V' + V^2) - 2UV - PU - PV]u = 0$
□

**Corollary I.5**: If $P, U$ & $V$ are differentiable functions, and:

$$u = e^{\int (U-V)dx}$$
$$\Rightarrow \begin{cases} u'' + Pu' + \left[-(U-V)' - (U-V)^2 - P(U-V)\right]u = 0 \\ u'' + Pu' + [-(U' + U^2) + (V' - V^2) + 2UV - PU + PV]u = 0 \end{cases}$$

*Proof:*
immediate
□

**Corollary I.5a**: If $P, U$ & $V$ are differentiable functions, and:

$$u = e^{-\int (U+V)dx}$$
$$\Rightarrow \begin{cases} u'' + Pu' + \left[(U+V)' - (U+V)^2 + P(U+V)\right]u = 0 \\ u'' + Pu' + [[U' - U^2] + [V' - V^2] - 2UV + PU + PV]u = 0 \end{cases}$$

*Proof:*

$$u = e^{-\int (U+V)dx} = e^{\int ((-U)+(-V))dx}$$

By lemme I.5:

$$\Rightarrow \begin{cases} u'' + Pu' + \left[-((-U)+(-V))' - ((-U)+(-V))^2 - P((-U)+(-V))\right]u = 0 \\ u'' + Pu' + \left[-\left((-U)' + (-U)^2\right) - \left((-V)' + (-V)^2\right) - 2(-U)(-V) - P(-U) - P(-V)\right]u = 0 \end{cases}$$
$$\Rightarrow \begin{cases} u'' + Pu' + \left[(U+V)' - (U+V)^2 + P(U+V)\right]u = 0 \\ u'' + Pu' + [U' + V' - U^2 - 2UV - V^2 + PU + PV]u = 0 \end{cases}$$
$$\Rightarrow \begin{cases} u'' + Pu' + \left[(U+V)' - (U+V)^2 + P(U+V)\right]u = 0 \\ u'' + Pu' + [[U' - U^2] + [V' - V^2] - 2UV + PU + PV]u = 0 \end{cases}$$
□

**Corollary I.5b**: If $P, R$ & $T$ are differentiable functions, and:

$$u = e^{-\int \left(\frac{1}{2}R+T\right)dx}$$
$$\Rightarrow \begin{cases} u'' + Pu' + \left[\left[\left(\frac{1}{2}R\right)' - \left(\frac{1}{2}R\right)^2\right] + [T' - T^2] - RT + \frac{1}{2}PR + PT\right]u = 0 \\ u'' + Ru' + \left[\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] + [T' - T^2]\right]u = 0 \end{cases}$$
$$u = e^{\int \left(-\frac{1}{2}R+T\right)dx}$$
$$\Rightarrow \begin{cases} u'' + Pu' + \left[-\left(T - \left(\frac{1}{2}R\right)\right)' - \left(T - \left(\frac{1}{2}R\right)\right)^2 - P\left(T - \left(\frac{1}{2}R\right)\right)\right]u = 0 \\ u'' + Pu' + \left[-[T' + T^2] + \left[\left(\frac{1}{2}R\right)' - \left(\frac{1}{2}R\right)^2\right] + RT - PT + \frac{1}{2}PR\right]u = 0 \\ u'' + Ru' + \left[-[T' + T^2] + \left(\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right)\right]u = 0 \end{cases}$$
$$u = e^{\frac{1}{2}\int (P-R)dx}$$
$$\Rightarrow \begin{cases} u'' + Tu' + \left[-\left(\frac{1}{2}(P-R)\right)' - \left(\frac{1}{2}(P-R)\right)^2 - T\left(\frac{1}{2}(P-R)\right)\right]u = 0 \\ u'' + Tu' + \left[-\left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2\right] + \left[\left(\frac{1}{2}R\right)' - \left(\frac{1}{2}R\right)^2\right] + \frac{1}{2}PR - \frac{1}{2}PT + \frac{1}{2}RT\right]u = 0 \\ u'' + Ru' + \left[-\left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2\right] + \left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right]\right]u = 0 \end{cases}$$

*Proof:*

$$u = e^{-\int \left(\frac{1}{2}R+T\right)dx}$$
$$\Rightarrow \begin{cases} u'' + Pu' + \left[\left(\frac{1}{2}R\right)' + T' - \left(\left(\frac{1}{2}R\right) + T\right)^2 + P\left(\left(\frac{1}{2}R\right) + T\right)\right]u = 0 \\ u'' + Pu' + \left[\left[\left(\frac{1}{2}R\right)' - \left(\frac{1}{2}R\right)^2\right] + [T' - T^2] - 2\left(\frac{1}{2}R\right)T + P\left(\frac{1}{2}R\right) + PT\right]u = 0 \\ u'' + Ru' + \left[\left[\left(\frac{1}{2}R\right)' - \left(\frac{1}{2}R\right)^2\right] + [T' - T^2] - 2\left(\frac{1}{2}R\right)T + R\left(\frac{1}{2}R\right) + RT\right]u = 0 \end{cases}$$
$$\Rightarrow \begin{cases} u'' + Pu' + \left[\left[\left(\frac{1}{2}R\right)' - \left(\frac{1}{2}R\right)^2\right] + [T' - T^2] - RT + \frac{1}{2}PR + PT\right]u = 0 \\ u'' + Ru' + \left[\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] + [T' - T^2]\right]u = 0 \end{cases}$$
$$u = e^{\int \left(-\frac{1}{2}R+T\right)dx} = e^{\int \left(T-\frac{1}{2}R\right)dx}$$
$$\Rightarrow \begin{cases} u'' + Pu' + \left[-\left(T - \left(\frac{1}{2}R\right)\right)' - \left(T - \left(\frac{1}{2}R\right)\right)^2 - P\left(T - \left(\frac{1}{2}R\right)\right)\right]u = 0 \\ u'' + Pu' + \left[-[T' + T^2] + \left[\left(\frac{1}{2}R\right)' - \left(\frac{1}{2}R\right)^2\right] + RT - PT + \frac{1}{2}PR\right]u = 0 \\ u'' + Ru' + \left[-[T' + T^2] + \left(\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right)\right]u = 0 \end{cases}$$
$$u = e^{\frac{1}{2}\int (P-R)dx}$$

$$\Rightarrow \begin{cases} u'' + Tu' + \left[-\left(\tfrac{1}{2}(P-R)\right)' - \left(\tfrac{1}{2}(P-R)\right)^2 - T\left(\tfrac{1}{2}(P-R)\right)\right]u = 0 \\ u'' + Tu' + \left[-\left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right] + \left[\left(\tfrac{1}{2}R\right)' - \left(\tfrac{1}{2}R\right)^2\right] + \tfrac{1}{2}PR - \tfrac{1}{2}PT + \tfrac{1}{2}RT\right]u = 0 \\ u'' + Ru' + \left[-\left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right] + \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right]\right]u = 0 \end{cases}$$

□

**Corollary I.5c**: For twice differentiable function $y$ and differentiable functions $g, h, P, Q$ :

$$y'' + Py' + Qy = 0$$

$$\Rightarrow y = e^{-\int gdx}\left(c_1 \int e^{-\int (2g-P)dx} dx + c_2\right)$$

$$\Rightarrow y = e^{-\int gdx}\left(c_1 \int g e^{\int\left(g - \frac{Q}{g}\right)dx} dx + c_2\right)$$

$$\text{where: } P = g + \left(\frac{Q-g'}{g}\right) \;\; \& \;\; -Q = g' + g\left(\frac{Q-g'}{g}\right)$$

*Proof:*

From corolary I.3 & I.4 with $W = 0$.

□

**Corollary I.5d**: For twice differentiable function $y$ and differentiable function $P$ :

$$\Rightarrow y'' + Py' = 0 \Rightarrow y = c_1 e^{\int\left(\frac{e^{-\int Pdx}}{\int e^{-\int Pdx}dx}\right)dx}\left[\int\left(-\frac{e^{-\int Pdx}}{\int e^{-\int Pdx}dx}\right)e^{\int\left(-\frac{e^{-\int Pdx}}{\int e^{-\int Pdx}dx} - \left[P - -\left(-\frac{e^{-\int Pdx}}{\int e^{-\int Pdx}dx}\right)\right]\right)dx} dx + c_2\right]$$

*Proof:*

$$y'' + Py' = 0 \qquad (Q = 0)$$

$$\Rightarrow (y')' + P(y') = 0 \Rightarrow \left(y' e^{\int Pdx}\right)' e^{-\int Pdx} = 0 \Rightarrow y' = c_1 e^{-\int Pdx}$$

$$\Rightarrow y = c_1 \int e^{-\int Pdx} dx + c_2 = y = e^{-\int gdx}$$

$$\Rightarrow \log\left(c_1 \int e^{-\int Pdx} dx + c_2\right) = -\int gdx$$

$Q = 0$ :

$$\Rightarrow y = c_1 e^{-\int gdx} \int e^{\int\left(g - \left(\frac{0-g'}{g}\right)\right)dx} dx + c_2 e^{-\int gdx}$$

$$\Rightarrow y = c_1 e^{-\int gdx} \int g e^{\int gdx} dx + c_2 e^{-\int gdx}$$

$$c_1 = 0 \Rightarrow g = 0 \Rightarrow h = \frac{0-g'}{g} = \frac{0}{0} \text{ is unacceptable}$$

$$c_2 = 0 \Rightarrow g = -\frac{e^{-\int Pdx}}{\int e^{-\int Pdx}dx} \Rightarrow \frac{g'}{g} = -P + g$$

$$\Rightarrow g - \left(\frac{0-g'}{g}\right) = g + \frac{g'}{g} = g - h \Rightarrow h = \left(\frac{Q-g'}{g}\right) = P - g \;\; ✓$$

$$\Rightarrow y'' + Py' = 0 \Rightarrow y = c_1 e^{\int\left(\frac{e^{-\int Pdx}}{\int e^{-\int Pdx}dx}\right)dx}\left[\int\left(-\frac{e^{-\int Pdx}}{\int e^{-\int Pdx}dx}\right)e^{\int\left(-\frac{e^{-\int Pdx}}{\int e^{-\int Pdx}dx} - \left[P - -\left(-\frac{e^{-\int Pdx}}{\int e^{-\int Pdx}dx}\right)\right]\right)dx} dx + c_2\right]$$

□

**Corollary I.5e**: For twice differentiable function $y$ and differentiable function $P$ :

$$\Rightarrow y'' + Py' + P'y = 0 \Rightarrow e^{-\int Pdx}\left(c_1 \int e^{\int\left(P - \left(\frac{P'-P'}{P}\right)\right)dx} dx + c_3\right)$$

*Proof:*

$$y'' + Py' + P'y = 0 \qquad (Q = P')$$

$$\Rightarrow (y' + Py)' = 0 \Rightarrow \left(\left(y e^{\int Pdx}\right)' e^{-\int Pdx}\right)' = 0$$

$$\Rightarrow \left(y e^{\int Pdx}\right)' = c e^{\int Pdx} \Rightarrow e^{-\int Pdx}\left(c_1 \int e^{\int Pdx} dx + c_2\right) = y = e^{-\int gdx}$$

$$\Rightarrow \log\left(e^{-\int Pdx}\left(c_1 \int e^{\int Pdx} dx + c_2\right)\right) = -\int gdx$$

$Q = P'$ :

$$\Rightarrow y = e^{-\int gdx} c_1 \int e^{\int\left(g - \left(\frac{P'-g'}{g}\right)\right)dx} dx + c_2 e^{-\int gdx}$$

$$c_1 = 0 \Rightarrow -\int gdx = \log\left(e^{-\int Pdx} c_2\right) = -\int Pdx + \log c_2 \Rightarrow g = P \Rightarrow h = 0 \Rightarrow Q = g' + gh = P' \;\; ✓$$

$$\Rightarrow y = e^{-\int g dx} \int c_1 e^{\int \left( g - \left( \frac{P' - P'}{g} \right) \right) dx} dx + c_2 e^{-\int g dx}$$
$$\Rightarrow y = e^{-\int P dx} \left( c_1 \int e^{\int \left( P - \left( \frac{P' - P'}{P} \right) \right) dx} dx + c_3 \right) = e^{-\int P dx} \left( c_1 \int e^{\int P dx} dx + c_3 \right)$$

□

In these, most elementary, $g$ was determined and plugged-into corolary I.3 for the complete HLODE solution. Thus, the key to fully understanding 2nd order LODEs is the $g$ .
Ricatti ODEs are expressed via the $g$'s:

Example:

$$g' + g^2 = \lambda^2 \Leftrightarrow g \in. \{\pm\lambda, -\lambda\cot(\lambda x)\}$$
$$\Rightarrow \left(g + \tfrac{1}{2}P\right)' + \left(g + \tfrac{1}{2}P\right)^2 + (-\lambda^2 - 0) = g' + g^2 + gP + \lambda^2 + \left[ \left( \left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2 + 0 \right) \right]$$
$$\Rightarrow y = e^{-\int \left( g + \frac{1}{2}P \right) dx} \Rightarrow y'' + Py' + \left( \lambda^2 + \left[ \left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2 \right] \right) = 0$$
$$\Rightarrow y = e^{-\int \left( g + \frac{1}{2}P \right) dx} \Rightarrow y'' + Py' + \left( \lambda^2 + \left[ \left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2 \right] \right) = 0$$
$$\Rightarrow y = e^{-\int g dx} \left[ c_1 + c_2 \int g e^{\int \left( g - \frac{Q}{g} \right) dx} dx \right]$$
$$\Rightarrow y = e^{-\int \left( g + \frac{1}{2}P \right) dx} \left[ c_1 + c_2 \int g e^{\int \left( \left( g + \frac{1}{2}P \right) - \left( \frac{\left( \lambda^2 + \left[ \left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2 \right] \right)}{\left( g + \frac{1}{2}P \right)} \right) \right) dx} dx \right]$$

(NOTE: The inhomogenous terms are being omitted merely to save space.)

**Theorem I.6**: If $y_1'' + P_1 y_1' + Q_1 y_1 = 0$ and $y_2'' + P_2 y_2' + Q_2 y_2 = 0$ and:

$$u = \frac{y_2}{y_1}$$

then

$$0 = u'' + \left( 2\frac{y_1'}{y_1} + P_2 \right) u' + \left[ (P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1 \right] u$$

*Proof:*

The following are given:

$$y_1'' + P_1 y_1' + Q_1 y_1 = 0$$
$$y_2'' + P_2 y_2' + Q_2 y_2 = 0$$
$$u = \frac{y_2}{y_1}$$
$$\Rightarrow u' = \frac{y_1 y_2' - y_1' y_2}{y_1^2} = \frac{y_2'}{y_1} - \frac{y_1'}{y_1}\frac{y_2}{y_1}$$
$$\Rightarrow u'' = \frac{y_1 y_2'' - y_1' y_2'}{y_1^2} - \frac{y_1 y_1'' - y_1' y_1'}{y_1^2}\frac{y_2}{y_1} - \frac{y_1'}{y_1}\left(\frac{y_2}{y_1}\right)'$$
$$= \frac{y_2''}{y_1} - \frac{y_1'}{y_1^2} y_2' - \frac{y_1''}{y_1}\frac{y_2}{y_1} + \frac{y_1'}{y_1}\frac{y_1'}{y_1}\frac{y_2}{y_1} - \frac{y_1'}{y_1}\left( \frac{y_2'}{y_1} - \frac{y_1'}{y_1}\frac{y_2}{y_1} \right)$$
$$= \frac{1}{y_1}\left( y_2'' - \frac{y_1'}{y_1} y_2' - \frac{y_1''}{y_1} y_2 + \frac{y_1'}{y_1}\frac{y_1'}{y_1} y_2 - \frac{y_1'}{y_1} y_2' + \frac{y_1'}{y_1}\frac{y_1'}{y_1} y_2 \right)$$
$$= \frac{1}{y_1}\left[ y_2'' - 2\frac{y_1'}{y_1} y_2' + \frac{1}{y_1}\left( -y_1'' + 2\frac{y_1'}{y_1} y_1' \right) y_2 \right]$$
$$\Rightarrow u'' + \left( 2\frac{y_1'}{y_1} + P_2 \right) u' + \left[ (P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1 \right] u =$$
$$= \frac{1}{y_1}\left[ y_2'' - 2\frac{y_1'}{y_1} y_2' + \frac{1}{y_1}\left( -y_1'' + 2\frac{y_1'}{y_1} y_1' \right) y_2 \right] +$$
$$+ \left( 2\frac{y_1'}{y_1} + P_2 \right)\left[ \frac{y_2'}{y_1} - \frac{y_1'}{y_1}\frac{y_2}{y_1} \right] + \left[ (P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1 \right]\frac{y_2}{y_1}$$
$$= \frac{1}{y_1}\Bigg( y_2'' - 2\frac{y_1'}{y_1} y_2' + \frac{1}{y_1}\left( -y_1'' + 2\frac{y_1'}{y_1} y_1' \right) y_2 + \left( 2\frac{y_1'}{y_1} + P_2 \right)\left( y_2' - \frac{y_1'}{y_1} y_2 \right) +$$
$$+ \left[ (P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1 \right] y_2 \Bigg)$$
$$= \frac{1}{y_1}\Bigg( y_2'' - 2\frac{y_1'}{y_1} y_2' - \frac{1}{y_1} y_1'' y_2 + 2\frac{1}{y_1}\frac{y_1'}{y_1} y_1' y_2 + \left( 2\frac{y_1'}{y_1} + P_2 \right) y_2' - \left( 2\frac{y_1'}{y_1} + P_2 \right)\frac{y_1'}{y_1} y_2 +$$
$$+ \left[ (P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1 \right] y_2 \Bigg)$$
$$= \frac{1}{y_1}\Bigg( y_2'' - 2\frac{y_1'}{y_1} y_2' - \frac{1}{y_1} y_1'' y_2 + 2\frac{1}{y_1}\frac{y_1'}{y_1} y_1' y_2 + 2\frac{y_1'}{y_1} y_2' + P_2 y_2' - 2\frac{y_1'}{y_1}\frac{y_1'}{y_1} y_2 - P_2\frac{y_1'}{y_1} y_2 +$$
$$+ \left[ (P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1 \right] y_2 \Bigg)$$
$$= \frac{1}{y_1}\Bigg( y_2'' - 2\frac{y_1'}{y_1} y_2' + 2\frac{y_1'}{y_1} y_2' - \frac{1}{y_1} y_1'' y_2 + 2\frac{y_1'}{y_1}\frac{y_1'}{y_1} y_2 - 2\frac{y_1'}{y_1}\frac{y_1'}{y_1} y_2 + P_2 y_2' - P_2\frac{y_1'}{y_1} y_2 +$$
$$+ \left[ (P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1 \right] y_2 \Bigg)$$
$$= \frac{1}{y_1}\left( y_2'' - \frac{1}{y_1} y_1'' y_2 + P_2 y_2' - P_2\frac{y_1'}{y_1} y_2 + \left[ (P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1 \right] y_2 \right)$$

$$= \frac{1}{y_1}\left[ y_2'' + P_2y_2' + Q_2y_2 - \frac{1}{y_1}y_1''y_2 - P_2\frac{y_1'}{y_1}y_2 + P_2\frac{y_1'}{y_1}y_2 - P_1\frac{y_1'}{y_1}y_2 - Q_1y_2 \right]$$
$$= \frac{1}{y_1}\left[ y_2'' + P_2y_2' + Q_2y_2 - \frac{1}{y_1}y_1''y_2 - P_1\frac{y_1'}{y_1}y_2 - Q_1y_2 \right]$$
$$= \frac{1}{y_1}\left[ (y_2'' + P_2y_2' + Q_2y_2) - \frac{y_2}{y_1}(y_1'' + P_1y_1' + Q_1y_1) \right] = \frac{1}{y_1}\left[ 0 - \frac{y_2}{y_1}0 \right] = 0$$

□

**Theorem I.7**: For arbitrary diferentiable function $R$ and diferentiable functions $P_1$, $P_2$, $Q_2$, $y_1$, $y_2$ such that:

$$y_1'' + P_1y_1' + \left(-\left[\left(\tfrac{1}{2}(R-P_2)\right)' + \left(\tfrac{1}{2}(R-P_2)\right)^2 + P_1\left(\tfrac{1}{2}(R-P_2)\right)\right]\right)y_1 = 0 \quad ,$$

$$\left(y_1'' + P_1y_1' + \left(-\left[\left(\tfrac{1}{2}(R-P_2) + \left(\tfrac{1}{2}P_1\right)\right)' + \left(\tfrac{1}{2}(R-P_2) + \left(\tfrac{1}{2}P_1\right)\right)^2 - \left[\left(\tfrac{1}{2}P_1\right)' + \left(\tfrac{1}{2}P_1\right)^2\right]\right]\right)y_1 = 0 \right)$$, and:

$$y_2'' + P_2y_2' + Q_2y_2 = 0 \quad \text{then:}$$

$$u = y_2e^{-\frac{1}{2}\int(R-P_2)dx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P_2\right)' + \left(\tfrac{1}{2}P_2\right)^2\right] + Q_2\right]u = 0$$

*Proof:*

Let
$$y_1 = e^{\frac{1}{2}\int(R-P_2)dx} \Rightarrow y_1' = \tfrac{1}{2}(R-P_2)y_1 \Rightarrow y_1'' + P_1y_1' = \left[\left[\tfrac{1}{2}(R-P_2)' + \tfrac{1}{4}(R-P_2)^2\right] + \tfrac{1}{2}P_1(R-P_2)\right]y_1$$
$$\Rightarrow y_1'' + P_1y_1' - \left[\left[\tfrac{1}{2}(R-P_2)' + \tfrac{1}{4}(R-P_2)^2\right] + \tfrac{1}{2}P_1(R-P_2)\right]y_1 = 0$$
$$Q_1 = -\tfrac{1}{2}(R-P_2)' - \tfrac{1}{4}(R-P_2)^2 - \tfrac{1}{2}P_1(R-P_2)$$
$$= -\left[\left(\tfrac{1}{2}(R-P_2) + \left(\tfrac{1}{2}P_1\right)\right)' + \left(\tfrac{1}{2}(R-P_2) + \left(\tfrac{1}{2}P_1\right)\right)^2 - \left[\left(\tfrac{1}{2}P_1\right)' + \left(\tfrac{1}{2}P_1\right)^2\right]\right]$$
$$\Rightarrow y_1'' + P_1y_1' + Q_1y_1 = 0$$

So, by theorem I.1:
$$u = \frac{y_2}{y_1} = y_2e^{-\frac{1}{2}\int(R-P_2)dx}$$

then
$$0 = u'' + Ru' + \left[ (P_2 - P_1)\frac{y_1'}{y_1} + Q_2 - Q_1 \right]u$$
$$\Rightarrow 0 = u'' + Ru' + \left[ \tfrac{1}{2}(P_2-P_1)(R-P_2) + Q_2 + \left[\left[\tfrac{1}{2}(R-P_2)' + \tfrac{1}{4}(R-P_2)^2\right] + \tfrac{1}{2}P_1(R-P_2)\right]\right]u$$
$$\Rightarrow 0 = u'' + Ru' +$$
$$+ \left[ \tfrac{1}{2}(P_2R - P_2^2 - P_1R + P_1P_2) + Q_2 + \tfrac{1}{2}(R-P_2)' + \tfrac{1}{4}(R^2 - 2RP_2 + P_2^2) + \tfrac{1}{2}P_1R - \tfrac{1}{2}P_1P_2 \right]u$$
$$\Rightarrow 0 = u'' + Ru' +$$
$$+ \left[ \tfrac{1}{2}P_2R - \tfrac{1}{2}P_2^2 - \tfrac{1}{2}P_1R + \tfrac{1}{2}P_1P_2 + Q_2 + \tfrac{1}{2}(R-P_2)' + \tfrac{1}{4}R^2 - \tfrac{1}{2}RP_2 + \tfrac{1}{4}P_2^2 + \tfrac{1}{2}P_1R - \tfrac{1}{2}P_1P_2 \right]u$$
$$\Rightarrow 0 = u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P_2\right)' + \left(\tfrac{1}{2}P_2\right)^2\right] + Q_2\right]u$$

□

The following lemma I.8 verifies corollary I.5b

**Lemma I.8**: If $P, U$ & $V$ are differentiable functions, and:

$$w'' + Pw' + Qw = 0 \ \ \& \ \ u = we^{\frac{1}{2}\int(P-R)dx}$$
$$\Rightarrow u = \left(c_1\int e^{2\int Tdx}dx + c_2\right)e^{-\int\left(\frac{1}{2}R\right)dx}e^{-\int Tdx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] + [T' - T^2]\right]u = 0$$

*Proof:*

From theorem I.7:
$$w'' + Pw' + Qw = 0 \ \ \& \ \ u = we^{\frac{1}{2}\int(P-R)dx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' - \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' - \left(\tfrac{1}{2}P\right)^2\right] + Q\right]u = 0$$

So, for $Q = 0$ :
$$w'' + Pw' = 0 \ \ \& \ \ u = we^{\frac{1}{2}\int(P-R)dx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' - \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' - \left(\tfrac{1}{2}P\right)^2\right]\right]u = 0$$
$$= (w')' + P(w') = \left(w'e^{\int Pdx}\right)'e^{-\int Pdx} \Rightarrow w'e^{\int Pdx} = c_1$$
$$\Rightarrow w = c_1\int e^{-\int Pdx}dx + c_2$$
$$\Rightarrow u = \left(c_1\int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int(P-R)dx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right]u = 0$$
$$\Rightarrow u = \left(c_1\int e^{2\int\left(-\frac{1}{2}P\right)dx}dx + c_2\right)e^{-\int\left(\frac{1}{2}R\right)dx}e^{-\int\left(-\frac{1}{2}P\right)dx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] + \left[\left(-\tfrac{1}{2}P\right)' - \left(-\tfrac{1}{2}P\right)^2\right]\right]u = 0$$
$$\Rightarrow u = \left(c_1\int e^{2\int\left(\frac{1}{2}P\right)dx}dx + c_2\right)e^{-\int\left(\frac{1}{2}R\right)dx}e^{-\int\left(\frac{1}{2}P\right)dx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right]u = 0$$

So, for $Q = P'$ :
$$w'' + Pw' + P' = 0 \ \ \& \ \ u = we^{\frac{1}{2}\int(P-R)dx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' - \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' - \left(\tfrac{1}{2}P\right)^2\right] + P'\right]u = 0$$
$$= (w')' + (Pw)' = (w' + Pw)' \Rightarrow w' + Pw = c_1$$
$$\Rightarrow c_1 = \left(we^{\int Pdx}\right)'e^{-\int Pdx} \Rightarrow \left(we^{\int Pdx}\right)' = c_1e^{\int Pdx} \Rightarrow w = e^{-\int Pdx}\left(c_1\int e^{\int Pdx}dx + c_2\right)$$
$$\Rightarrow u = e^{-\int Pdx}\left(c_1\int e^{\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int(P-R)dx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right] + P'\right]u = 0$$
$$\Rightarrow u = \left(c_1\int e^{\int Pdx}dx + c_2\right)e^{-\frac{1}{2}\int(P+R)dx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] + \left[\left(\tfrac{1}{2}P\right)' - \left(\tfrac{1}{2}P\right)^2\right]\right]u = 0$$
$$\Rightarrow u = \left(c_1\int e^{2\int\left(\frac{1}{2}P\right)dx}dx + c_2\right)e^{-\int\left(\frac{1}{2}R\right)dx}e^{-\int\left(\frac{1}{2}P\right)dx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] + \left[\left(\tfrac{1}{2}P\right)' - \left(\tfrac{1}{2}P\right)^2\right]\right]u = 0$$
$$\Rightarrow u = \left(c_1\int e^{2\int\left(-\frac{1}{2}P\right)dx}dx + c_2\right)e^{-\int\left(\frac{1}{2}R\right)dx}e^{-\int\left(-\frac{1}{2}P\right)dx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(-\tfrac{1}{2}P\right)' + \left(-\tfrac{1}{2}P\right)^2\right]\right]u = 0$$

So, more generally:
$$u = \left(c_1\int e^{2\int Tdx}dx + c_2\right)e^{-\int\left(\frac{1}{2}R\right)dx}e^{-\int Tdx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] + [T' - T^2]\right]u = 0$$

$$u=\left(c_1\int e^{-2\int(-T)dx}dx+c_2\right)e^{-\int\left(\frac{1}{2}R\right)dx}e^{\int(-T)dx}\Rightarrow u''+Ru'+\left[\left[\left(\tfrac{1}{2}R\right)'+\left(\tfrac{1}{2}R\right)^2\right]-\left[(-T)'+(-T)^2\right]\right]u=0$$

□

Comparing to corollary I.5b:

$$u=e^{\frac{1}{2}\int(P-R)dx}\Leftrightarrow\left(-T\Leftrightarrow\tfrac{1}{2}P\right)$$

$$\Rightarrow\begin{cases}u''+Tu'+\left[-\left(\tfrac{1}{2}(P-R)\right)'-\left(\tfrac{1}{2}(P-R)\right)^2-T\left(\tfrac{1}{2}(P-R)\right)\right]u=0\\ u''+Tu'+\left[-\left[\left(\tfrac{1}{2}P\right)'+\left(\tfrac{1}{2}P\right)^2\right]+\left[\left(\tfrac{1}{2}R\right)'-\left(\tfrac{1}{2}R\right)^2\right]+\tfrac{1}{2}PR-\tfrac{1}{2}PT+\tfrac{1}{2}RT\right]u=0\\ u''+Ru'+\left[-\left[\left(\tfrac{1}{2}P\right)'+\left(\tfrac{1}{2}P\right)^2\right]+\left[\left(\tfrac{1}{2}R\right)'+\left(\tfrac{1}{2}R\right)^2\right]\right]u=0\end{cases}$$

$$u=e^{-\int\left(\frac{1}{2}R+T\right)dx}$$

$$\Rightarrow\begin{cases}u''+Pu'+\left[\left(\tfrac{1}{2}R\right)'+T'-\left(\left(\tfrac{1}{2}R\right)+T\right)^2+P\left(\left(\tfrac{1}{2}R\right)+T\right)\right]u=0\\ u''+Pu'+\left[\left[\left(\tfrac{1}{2}R\right)'-\left(\tfrac{1}{2}R\right)^2\right]+[T'-T^2]-2\left(\tfrac{1}{2}R\right)T+P\left(\tfrac{1}{2}R\right)+PT\right]u=0\\ u''+Pu'+\left[\left[\left(\tfrac{1}{2}R\right)'-\left(\tfrac{1}{2}R\right)^2\right]+[T'-T^2]-RT+\tfrac{1}{2}PR+PT\right]u=0\\ u''+Ru'+\left[\left[\left(\tfrac{1}{2}R\right)'+\left(\tfrac{1}{2}R\right)^2\right]+[T'-T^2]\right]u=0\\ u''+Ru'+\left[\left[\left(\tfrac{1}{2}R\right)'+\left(\tfrac{1}{2}R\right)^2\right]-\left[(-T)'+(-T)^2\right]\right]u=0\end{cases}$$

The following sequences allow further use of $g$'s for 2nd order LODE solutions.

**Theorem II.1**: Given the sequence on differentiable functions $s_i, P_i$ :

$$s_{n+1}-s_n=-\frac{1}{2}P_{n+1}\quad,\quad s_n=s_1-\frac{1}{2}\left(\sum_{i=2}^{n}P_i\right)\qquad,\ (\forall n\in\mathbb{N});$$

these sequence expressions follow:

$$\begin{aligned}\left[s_n+\frac{1}{2}\left(\sum_{i=1}^{n}P_i\right)\right]'+\left[s_n+\frac{1}{2}\left(\sum_{i=1}^{n}P_i\right)\right]^2&=\\ &=s_n'+s_n^2+s_n\left(\sum_{i=1}^{n}P_i\right)+\left[\left[\frac{1}{2}\left(\sum_{i=1}^{n}P_i\right)\right]'+\left[\frac{1}{2}\left(\sum_{i=1}^{n}P_i\right)\right]^2\right]=\\ &=s_m'+s_m^2+s_m\left(\sum_{i=1}^{m}P_i\right)+\left[\left[\frac{1}{2}\left(\sum_{i=1}^{m}P_i\right)\right]'+\left[\frac{1}{2}\left(\sum_{i=1}^{m}P_i\right)\right]^2\right]=\\ &=\left[s_m+\frac{1}{2}\left(\sum_{i=1}^{m}P_i\right)\right]'+\left[s_m+\frac{1}{2}\left(\sum_{i=1}^{m}P_i\right)\right]^2\quad,\ \left(\forall m,n\in\mathbb{N}\ ,\ m\ge n\right);\end{aligned}$$

*Proof:*

$$\begin{aligned}s_1'+s_1^2+s_1P_1&=s_1'+\frac{1}{2}P_1'-\frac{1}{2}P_1'+s_1^2+s_1P_1+\frac{1}{4}P_1^2-\frac{1}{4}P_1^2\\ &=\left(s_1+\frac{1}{2}P_1\right)'+\left(s_1+\frac{1}{2}P_1\right)^2-\left[\left(\frac{1}{2}P_1\right)'+\left(\frac{1}{2}P_1\right)^2\right]\end{aligned}$$

$$\Rightarrow\left(s_1+\frac{1}{2}P_1\right)'+\left(s_1+\frac{1}{2}P_1\right)^2=s_1'+s_1^2+s_1P_1+\left[\left(\frac{1}{2}P_1\right)'+\left(\frac{1}{2}P_1\right)^2\right]$$

Let: $s_1=s_2+\frac{1}{2}P_2$ :

$$\begin{aligned}\Rightarrow\left(s_1+\frac{1}{2}P_1\right)'+\left(s_1+\frac{1}{2}P_1\right)^2&=\left(s_2+\frac{1}{2}P_2+\frac{1}{2}P_1\right)'+\left(s_2+\frac{1}{2}P_2+\frac{1}{2}P_1\right)^2\\ &=\left[s_2+\frac{1}{2}(P_2+P_1)\right]'+\left[s_2+\frac{1}{2}(P_2+P_1)\right]^2\\ &=s_2'+s_2^2+s_2(P_2+P_1)+\left[\left[\frac{1}{2}(P_2+P_1)\right]'+\left[\frac{1}{2}(P_2+P_1)\right]^2\right]\end{aligned}$$

$$\begin{aligned}\Rightarrow s_1'+s_1^2+s_1P_1+\left[\left(\frac{1}{2}P_1\right)'+\left(\frac{1}{2}P_1\right)^2\right]&=\\ &=s_2'+s_2^2+s_2(P_2+P_1)+\left[\left[\frac{1}{2}(P_2+P_1)\right]'+\left[\frac{1}{2}(P_2+P_1)\right]^2\right]\end{aligned}$$

Let: $s_2=s_3+\frac{1}{2}P_3$ :

$$\begin{aligned}\Rightarrow\left[s_2+\frac{1}{2}(P_2+P_1)\right]'+\left[s_2+\frac{1}{2}(P_2+P_1)\right]^2&=\left[s_3+\frac{1}{2}(P_3+P_2+P_1)\right]'+\left[s_3+\frac{1}{2}(P_3+P_2+P_1)\right]^2\\ &=s_3'+s_3^2+s_3(P_3+P_2+P_1)+\left[\left[\frac{1}{2}(P_3+P_2+P_1)\right]'+\left[\frac{1}{2}(P_3+P_2+P_1)\right]^2\right]\end{aligned}$$

$$\begin{aligned}\Rightarrow s_1'+s_1^2+s_1P_1+\left[\left(\frac{1}{2}P_1\right)'+\left(\frac{1}{2}P_1\right)^2\right]&=\\ &=s_3'+s_3^2+s_3(P_3+P_2+P_1)+\left[\left[\frac{1}{2}(P_3+P_2+P_1)\right]'+\left[\frac{1}{2}(P_3+P_2+P_1)\right]^2\right]\end{aligned}$$

if: 

$$\begin{aligned}s_n'+s_n^2+s_n\left(\sum_{i=1}^{n}P_i\right)+\left[\left[\frac{1}{2}\left(\sum_{i=1}^{n}P_i\right)\right]'+\left[\frac{1}{2}\left(\sum_{i=1}^{n}P_i\right)\right]^2\right]&=\\ &=s_m'+s_m^2+s_m\left(\sum_{i=1}^{m}P_i\right)+\left[\left[\frac{1}{2}\left(\sum_{i=1}^{m}P_i\right)\right]'+\left[\frac{1}{2}\left(\sum_{i=1}^{m}P_i\right)\right]^2\right]\end{aligned}$$

Let: $s_m=s_{m+1}+\frac{1}{2}P_{m+1}$ :

$$\begin{aligned}s_m'+s_m^2+s_m\left(\sum_{i=1}^{m}P_i\right)+\left[\left[\frac{1}{2}\left(\sum_{i=1}^{m}P_i\right)\right]'+\left[\frac{1}{2}\left(\sum_{i=1}^{m}P_i\right)\right]^2\right]&=\\ &=\left[s_m+\frac{1}{2}\left(\sum_{i=1}^{m}P_i\right)\right]'+\left[s_m+\frac{1}{2}\left(\sum_{i=1}^{m}P_i\right)\right]^2\\ &=\left[s_{m+1}+\frac{1}{2}\left(\sum_{i=1}^{m+1}P_i\right)\right]'+\left[s_{m+1}+\frac{1}{2}\left(\sum_{i=1}^{m+1}P_i\right)\right]^2\end{aligned}$$

$$= s'_{m+1} + s^2_{m+1} + s_{m+1}\left(\sum_{i=1}^{m+1} P_i\right) + \left[\left[\frac{1}{2}\left(\sum_{i=1}^{m+1} P_i\right)\right]' + \left[\frac{1}{2}\left(\sum_{i=1}^{m+1} P_i\right)\right]^2\right]$$

$$\Rightarrow s'_n + s_n^2 + s_n\left(\sum_{i=1}^{n} P_i\right) + \left[\left[\frac{1}{2}\left(\sum_{i=1}^{n} P_i\right)\right]' + \left[\frac{1}{2}\left(\sum_{i=1}^{n} P_i\right)\right]^2\right] =$$

$$= s'_{m+1} + s^2_{m+1} + s_{m+1}\left(\sum_{i=1}^{m+1} P_i\right) + \left[\left[\frac{1}{2}\left(\sum_{i=1}^{m+1} P_i\right)\right]' + \left[\frac{1}{2}\left(\sum_{i=1}^{m+1} P_i\right)\right]^2\right]$$

□

(reproduced from my "Solving Ricatti Ordinary Differential Equations")

**Theorem II.2**: $\forall s, P_n \in \mathbb{R}$, $\forall n \in \mathbb{N}$ :

$$\left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)^2 + \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)P_{n+1} = \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)^2 - \left[\left(\frac{1}{2}P_{n+1}\right)' + \left(\frac{1}{2}P_{n+1}\right)^2\right]$$

*Proof:*

$$s' + s^2 + P_1 s = \left(s + \frac{1}{2}P_1\right)' + \left(s + \frac{1}{2}P_1\right)^2 - \left[\left(\frac{1}{2}P_1\right)' + \left(\frac{1}{2}P_1\right)^2\right]$$

$$\Rightarrow \left(s + \frac{1}{2}P_1\right)' + \left(s + \frac{1}{2}P_1\right)^2 + \left(s + \frac{1}{2}P_1\right)P_2 = \left(s + \frac{1}{2}P_1 + \frac{1}{2}P_2\right)' + \left(s + \frac{1}{2}P_1 + \frac{1}{2}P_2\right)^2 - \left[\left(\frac{1}{2}P_2\right)' + \left(\frac{1}{2}P_2\right)^2\right]$$

So, if for $n$ :

$$\left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)^2 + \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)P_{n+1} = \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)^2 - \left[\left(\frac{1}{2}P_{n+1}\right)' + \left(\frac{1}{2}P\right.$$

then:

$$\left(s + \sum_{i=1}^{n+2}\left(\frac{1}{2}P_i\right)\right)' = \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)' + \left(\frac{1}{2}P_{n+2}\right)'$$

$$\left(s + \sum_{i=1}^{n+2}\left(\frac{1}{2}P_i\right)\right)^2 = \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)^2 + 2\left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)\left(\frac{1}{2}P_{n+2}\right) + \left(\frac{1}{2}P_{n+2}\right)^2$$

$$= \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)^2 + \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)(P_{n+2}) + \left(\frac{1}{2}P_{n+2}\right)^2$$

$$= \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)^2 + \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)(P_{n+2}) + \left(\frac{1}{2}P_{n+2}\right)^2$$

$$\Rightarrow \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)^2 + \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)P_{n+2} = \left(s + \sum_{i=1}^{n+2}\left(\frac{1}{2}P_i\right)\right)' - \left(\frac{1}{2}P_{n+2}\right)' +$$

$$+ \left(s + \sum_{i=1}^{n+2}\left(\frac{1}{2}P_i\right)\right)^2 - \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)(P_{n+2}) - \left(\frac{1}{2}P_{n+2}\right)^2 +$$

$$+ \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)P_{n+2}$$

$$= \left(s + \sum_{i=1}^{n+2}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n+2}\left(\frac{1}{2}P_i\right)\right)^2 - \left[\left(\frac{1}{2}P_{n+2}\right)' + \left(\frac{1}{2}P_{n+2}\right)^2\right]$$

So true for all $n \in \mathbb{N} \geq 1$, by Induction.

□

**Corollary II.2**: $\forall s, P_n \in \mathbb{R}$, $\forall n \in \mathbb{N}$ :

$$\left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)^2 + \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)P_{n+1} = -Q$$

$$\Rightarrow y = e^{-\int\left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)dx} \Rightarrow y'' + P_{n+1}y' + Qy = 0$$

*Proof:*

immediate.

□

**Corollary II.2.1**: $\forall s, P_n \in \mathbb{R}$, $\forall n \in \mathbb{N}$ :

$$\left(s + \sum_{i=1}^{n+2}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n+2}\left(\frac{1}{2}P_i\right)\right)^2 - \left[\left(\frac{1}{2}P_{n+2}\right)' + \left(\frac{1}{2}P_{n+2}\right)^2\right] = -Q$$

$$\Rightarrow y = e^{-\int\left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)dx} \Rightarrow y'' + Qy = 0$$

*Proof:*

immediate.

□

**Theorem II.3**: $\forall s, P_n \in \mathbb{R}$, $\forall n \in \mathbb{N}$ :

$$s' + s^2 + s\left(\sum_{i=1}^{n} P_i\right) + \frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n} P_i P_j\right) = \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)^2 - \sum_{i=1}^{n}\left[\left(\frac{1}{2}P_i\right)' + \left(\frac{1}{2}P_i\right)^2\right]$$

*Proof:*

$$s' + s^2 + P_1 s = \left(s + \frac{1}{2}P_1\right)' + \left(s + \frac{1}{2}P_1\right)^2 - \left[\left(\frac{1}{2}P_1\right)' + \left(\frac{1}{2}P_1\right)^2\right]$$

$$\Rightarrow s' + s^2 + P_1 s + P_2\left(s + \frac{1}{2}P_1\right) = s' + s^2 + (P_1 + P_2)s + \frac{1}{2}(P_1 P_2)$$

$$= \left(s + \frac{1}{2}(P_1 + P_2)\right)' + \left(s + \frac{1}{2}(P_1 + P_2)\right)^2 - \left[\left(\frac{1}{2}(P_1 + P_2)\right)' + \left(\frac{1}{2}(P_1 + P_2)\right)^2\right] + \frac{1}{2}P_1 P_2$$

$$= \left(s + \frac{1}{2}(P_1 + P_2)\right)' + \left(s + \frac{1}{2}(P_1 + P_2)\right)^2 +$$
$$-\left[\left(\frac{1}{2}(P_1)\right)' + \left(\frac{1}{2}(P_1)\right)^2\right] - \left[\left(\frac{1}{2}(P_2)\right)' + \left(\frac{1}{2}(P_2)\right)^2\right] - \frac{1}{2}P_1P_2 + \frac{1}{2}P_1P_2$$
$$= \left(s + \frac{1}{2}(P_1 + P_2)\right)' + \left(s + \frac{1}{2}(P_1 + P_2)\right)^2 +$$
$$-\left[\left(\frac{1}{2}(P_1)\right)' + \left(\frac{1}{2}(P_1)\right)^2\right] - \left[\left(\frac{1}{2}(P_2)\right)' + \left(\frac{1}{2}(P_2)\right)^2\right]$$
$$\left(s + \frac{1}{2}P_1 + \frac{1}{2}P_2\right)' + \left(s + \frac{1}{2}P_1 + \frac{1}{2}P_2\right)^2 +$$
$$-\left[\left(\frac{1}{2}P_1\right)' + \left(\frac{1}{2}P_1\right)^2\right] - \left[\left(\frac{1}{2}P_2\right)' + \left(\frac{1}{2}P_2\right)^2\right]$$
$$= s' + s^2 + s\left(\sum_{i=1}^{n} P_i\right) + \frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n} P_iP_j\right) + sP_{n+1} + \frac{1}{2}\left(\sum_{i=1}^{n} P_iP_{n+1}\right)$$

So, if for $n$ :

$$s' + s^2 + s\left(\sum_{i=1}^{n} P_i\right) + \frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n} P_iP_j\right) = \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)^2 - \sum_{i=1}^{n}\left[\left(\frac{1}{2}P_i\right)' + \left(\frac{1}{2}P_i\right)^2\right]$$

then:

$$s' + s^2 + s\left(\sum_{i=1}^{n+1} P_i\right) + \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n+1} P_iP_j\right) = s' + s^2 + s\left(\sum_{i=1}^{n} P_i\right) + sP_{n+1} + \frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n} P_iP_j\right) + \frac{1}{2}\left(\sum_{i=1}^{n} P_iP_{n+1}\right)$$
$$= s' + s^2 + s\left(\sum_{i=1}^{n} P_i\right) + \frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n} P_iP_j\right) + sP_{n+1} + \frac{1}{2}\left(\sum_{i=1}^{n} P_iP_{n+1}\right)$$
$$= \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)^2 - \sum_{i=1}^{n}\left[\left(\frac{1}{2}P_i\right)' + \left(\frac{1}{2}P_i\right)^2\right] +$$
$$+ sP_{n+1} + \frac{1}{2}\left(\sum_{i=1}^{n} P_iP_{n+1}\right)$$
$$= \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)' - \left(\frac{1}{2}P_{n+1}\right)' +$$
$$+ \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right) - \left(\frac{1}{2}P_{n+1}\right)\right)^2 +$$
$$-\sum_{i=1}^{n+1}\left[\left(\frac{1}{2}P_i\right)' + \left(\frac{1}{2}P_i\right)^2\right] + \left[\left(\frac{1}{2}P_{n+1}\right)' + \left(\frac{1}{2}P_{n+1}\right)^2\right] +$$
$$+ sP_{n+1} + \frac{1}{2}\left(\sum_{i=1}^{n} P_iP_{n+1}\right)$$
$$= \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)' - \left(\frac{1}{2}P_{n+1}\right)' +$$
$$+ \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)^2 - \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)(P_{n+1}) + \left(\frac{1}{2}P_{n+1}\right)^2 +$$
$$-\sum_{i=1}^{n+1}\left[\left(\frac{1}{2}P_i\right)' + \left(\frac{1}{2}P_i\right)^2\right] + \left[\left(\frac{1}{2}P_{n+1}\right)' + \left(\frac{1}{2}P_{n+1}\right)^2\right] +$$
$$+ sP_{n+1} + \frac{1}{2}\left(\sum_{i=1}^{n} P_iP_{n+1}\right)$$
$$= \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)^2 - \left(\frac{1}{2}P_{n+1}\right)' +$$
$$- sP_{n+1} - (P_{n+1})\sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right) + \left(\frac{1}{2}P_{n+1}\right)^2 +$$
$$-\sum_{i=1}^{n+1}\left[\left(\frac{1}{2}P_i\right)' + \left(\frac{1}{2}P_i\right)^2\right] + \left[\left(\frac{1}{2}P_{n+1}\right)' + \left(\frac{1}{2}P_{n+1}\right)^2\right] +$$
$$+ sP_{n+1} + \frac{1}{2}\left(\sum_{i=1}^{n} P_iP_{n+1}\right)$$
$$= \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)^2 +$$
$$- (P_{n+1})\sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right) + \left(\frac{1}{2}P_{n+1}\right)^2 +$$
$$-\sum_{i=1}^{n+1}\left[\left(\frac{1}{2}P_i\right)' + \left(\frac{1}{2}P_i\right)^2\right] + \left[\left(\frac{1}{2}P_{n+1}\right)^2\right] +$$
$$+ \frac{1}{2}\left(\sum_{i=1}^{n} P_iP_{n+1}\right)$$
$$= \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)^2 +$$
$$- (P_{n+1})\sum_{i=1}^{n}\left(\frac{1}{2}P_i\right) + (P_{n+1})\left(\frac{1}{2}P_{n+1}\right) + 2\left(\frac{1}{2}P_{n+1}\right)^2 +$$
$$-\sum_{i=1}^{n+1}\left[\left(\frac{1}{2}P_i\right)' + \left(\frac{1}{2}P_i\right)^2\right] +$$
$$+ \frac{1}{2}\left(\sum_{i=1}^{n} P_iP_{n+1}\right)$$
$$= \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n+1}\left(\frac{1}{2}P_i\right)\right)^2 +$$
$$-\sum_{i=1}^{n+1}\left[\left(\frac{1}{2}P_i\right)' + \left(\frac{1}{2}P_i\right)^2\right]$$

So true for all $n \in \mathbb{N} \geq 1$ , by Induction.

□

**Corollary II.3**: $\forall s, P_n \in \mathbb{R}$ , $\forall n \in \mathbb{N}$ :

$$s' + s^2 + s\left(\sum_{i=1}^{n} P_i\right) = -\frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n} P_i P_j\right)$$
$$\Rightarrow y = e^{-\int s dx} \Rightarrow y'' + \left(\sum_{i=1}^{n} P_i\right)y' + \left[\frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n} P_i P_j\right)\right]y = 0$$

*Proof:*
immediate.
□

**Corollary II.3.1**: $\forall s, P_n \in \mathbb{R}$ , $\forall n \in \mathbb{N}$ :

$$\left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)^2 = \sum_{i=1}^{n}\left[\left(\frac{1}{2}P_i\right)' + \left(\frac{1}{2}P_i\right)^2\right]$$
$$\Rightarrow y = e^{-\int\left(s+\sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)dx} \Rightarrow y'' + \left(\sum_{i=1}^{n}\left[\left(\frac{1}{2}P_i\right)' + \left(\frac{1}{2}P_i\right)^2\right]\right)y = 0$$

*Proof:*
immediate.
□

Since: $-g = s$ these can lead to further Ricatti and LODE solutions.

$$s' + s^2 + s\left(\sum_{i=1}^{n} P_i\right) + \frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n} P_i P_j\right) = \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)' + \left(s + \sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)\right)^2 - \sum_{i=1}^{n}\left[\left(\frac{1}{2}P_i\right)' + \left(\frac{1}{2}P_i\right)^2\right]$$
$$P = \left(\sum_{i=1}^{n} P_i\right) \quad , \quad Q = \frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n} P_i P_j\right) \Rightarrow s' + s^2 + sP + Q = 0 = \left(s + \frac{1}{2}P\right)' + \left(s + \frac{1}{2}P\right)^2 - \left(\frac{1}{2}P' + \frac{1}{4}\sum_{i=1}^{n} P_i^2\right)$$
$$0 = \left(\sum_{i=1}^{n} P_i\right) \quad , \quad Q = \frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n} P_i P_j\right) \Rightarrow s' + s^2 + \frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n} P_i P_j\right) = 0 = s' + s^2 - \sum_{i=1}^{n}\left[\left(\frac{1}{2}P_i\right)^2\right]$$
$$\Rightarrow \sum_{i=1}^{n} P_i = 0 \Rightarrow \frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n} P_i P_j\right) = -\sum_{i=1}^{n}\left(\frac{1}{2}P_i\right)^2$$

And, 2nd order linear Partial differential equations may be factored, just as LODE's; thus solved as first order partials

**Theorem III.1**: For thrice differentiable function $y$ , twice differentiable function $g_1$ and differentiable functions $g_2, g_3$ .

$$\left(\left(\left(\left(\left(ye^{\int g_1 dx}\right)' e^{-\int g_1 dx}\right)e^{\int g_2 dx}\right)' e^{-\int g_2 dx}\right)e^{\int g_3 dx}\right)' e^{-\int g_3 dx} =$$
$$= y''' + (g_1 + g_2 + g_3)y'' + [(g_1' + g_2 g_1) + (g_1' + g_3 g_1) + (g_2' + g_3 g_2)]y' + $$
$$+ [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3 g_2)g_1]y$$

*Proof:*

$$\left(\left(\left(\left(\left(ye^{\int g_1 dx}\right)' e^{-\int g_1 dx}\right)e^{\int g_2 dx}\right)' e^{-\int g_2 dx}\right)e^{\int g_3 dx}\right)' e^{-\int g_3 dx} =$$
$$= \left(\left(\left(\left((y' + yg_1)e^{\int g_1 dx}e^{-\int g_1 dx}\right)e^{\int g_2 dx}\right)' e^{-\int g_2 dx}\right)e^{\int g_3 dx}\right)' e^{-\int g_3 dx}$$
$$= \left(\left(\left(((y' + yg_1))e^{\int g_2 dx}\right)' e^{-\int g_2 dx}\right)e^{\int g_3 dx}\right)' e^{-\int g_3 dx}$$
$$= \left(\left(\left((y' + yg_1)' e^{\int g_2 dx} + (y' + yg_1)g_2 e^{\int g_2 dx}\right)e^{-\int g_2 dx}\right)e^{\int g_3 dx}\right)' e^{-\int g_3 dx}$$
$$= \left(\left(\left((y' + yg_1)' + (y' + yg_1)g_2\right)\right)e^{\int g_3 dx}\right)' e^{-\int g_3 dx}$$
$$= \left(\left(\left((y' + yg_1)' + (y' + yg_1)g_2\right)\right)' e^{\int g_3 dx} + \left(\left((y' + yg_1)' + (y' + yg_1)g_2\right)\right)g_3 e^{\int g_3 dx}\right)e^{-\int g_3 dx}$$
$$= \left(\left((y' + yg_1)' + (y' + yg_1)g_2\right)\right)' + \left(\left((y' + yg_1)' + (y' + yg_1)g_2\right)\right)g_3$$
$$= \left((y' + yg_1)'' + (y' + yg_1)'g_2 + (y' + yg_1)g_2'\right) + (y' + yg_1)'g_3 + (y' + yg_1)g_2 g_3$$
$$= (y' + yg_1)'' + (y' + yg_1)'g_2 + (y' + yg_1)'g_3 + (y' + yg_1)g_2' + (y' + yg_1)g_2 g_3$$
$$= (y' + yg_1)'' + (y' + yg_1)'(g_2 + g_3) + (y' + yg_1)(g_2' + g_2 g_3)$$
$$= (y'' + y'g_1 + yg_1')' + (y'' + y'g_1 + yg_1')(g_2 + g_3) + (y' + yg_1)(g_2' + g_2 g_3)$$
$$= y''' + y''g_1 + y'g_1' + y'g_1' + yg_1'' + y''(g_2 + g_3) +$$
$$+ y'g_1(g_2 + g_3) + yg_1'(g_2 + g_3) + y'(g_2' + g_2 g_3) + yg_1(g_2' + g_2 g_3)$$
$$= y''' + y''g_1 + y''(g_2 + g_3) + y'g_1' + y'g_1' + y'g_1(g_2 + g_3) + y'(g_2' + g_2 g_3) +$$
$$+ yg_1'' + yg_1'(g_2 + g_3) + yg_1(g_2' + g_2 g_3)$$
$$= y''' + (g_1 + g_2 + g_3)y'' + (g_1' + g_1' + g_1(g_2 + g_3) + (g_2' + g_2 g_3))y' +$$
$$+ (g_1'' + g_1'(g_2 + g_3) + g_1(g_2' + g_2 g_3))y$$
$$= y''' + (g_1 + g_2 + g_3)y'' + [2g_1' + g_1(g_2 + g_3) + g_2' + g_2 g_3]y' +$$
$$+ [g_1'' + g_1'(g_2 + g_3) + g_1(g_2' + g_2 g_3)]y$$
$$= y''' + (g_1 + g_2 + g_3)y'' + [(g_1' + g_2 g_1) + (g_1' + g_3 g_1) + (g_2' + g_3 g_2)]y' +$$
$$+ [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3 g_2)g_1]y$$

□

**Theorem III.2**: For thrice differentiable function $y$ , twice differentiable function $g_1$ and differentiable functions $g_2, g_3$ and $P, Q, R$ :

$$P \equiv (g_1 + g_2 + g_3) \ , \ Q \equiv (g_1' + g_2 g_1) + (g_1' + g_3 g_1) + (g_2' + g_3 g_2) \ ,$$
$$R \equiv g_1'' + (g_2 + g_3) g_1' + (g_2' + g_3 g_2) g_1$$
$$\Rightarrow y''' + P y'' + Q y' + R y = W$$
$$\Rightarrow y = e^{-\int g_1 dx} \left( \int e^{\int (g_1 - g_2) dx} \left( \int e^{\int (g_2 - g_3) dx} \left( \int W e^{\int g_3 dx} dx + c_1 \right) dx + c_2 \right) dx + c_3 \right)$$
$$\Rightarrow y = e^{-\int g_1 dx} \left( \int e^{\int (g_1 - g_2) dx} \left( \int e^{\int (g_2 - g_3) dx} \left( \int W e^{\int g_3 dx} dx + c_1 \right) dx + c_2 \right) dx + c_3 \right)$$

*Proof:*

$$P \equiv (g_1 + g_2 + g_3) \ , \ Q \equiv (g_1' + g_2 g_1) + (g_1' + g_3 g_1) + (g_2' + g_3 g_2) \ ,$$
$$R \equiv g_1'' + (g_2 + g_3) g_1' + (g_2' + g_3 g_2) g_1$$
$$y''' + P y'' + Q y' + R y = W$$
$$\Rightarrow y''' + (g_1 + g_2 + g_3) y'' + [(g_1' + g_2 g_1) + (g_1' + g_3 g_1) + (g_2' + g_3 g_2)] y' +$$
$$+ [g_1'' + (g_2 + g_3) g_1' + (g_2' + g_3 g_2) g_1] y$$
$$\Rightarrow \left( \left( \left( \left( \left( y e^{\int g_1 dx} \right)' e^{-\int g_1 dx} \right) e^{\int g_2 dx} \right)' e^{-\int g_2 dx} \right) e^{\int g_3 dx} \right)' e^{-\int g_3 dx} = W$$
$$\Rightarrow \left( \left( \left( \left( y e^{\int g_1 dx} \right)' e^{-\int g_1 dx} \right) e^{\int g_2 dx} \right)' e^{-\int g_2 dx} \right) e^{\int g_3 dx} = \int W e^{\int g_3 dx} dx + c_1$$
$$\Rightarrow \left( \left( \left( y e^{\int g_1 dx} \right)' e^{-\int g_1 dx} \right) e^{\int g_2 dx} \right)' e^{-\int g_2 dx} = e^{-\int g_3 dx} \left( \int W e^{\int g_3 dx} dx + c_1 \right)$$
$$\Rightarrow \left( y e^{\int g_1 dx} \right)' e^{-\int g_1 dx} = e^{-\int g_2 dx} \int e^{\int (g_2 - g_3) dx} \left( \int W e^{\int g_3 dx} dx + c_1 \right) dx + c_2$$
$$\Rightarrow \left( y e^{\int g_1 dx} \right)' = e^{\int (g_1 - g_2) dx} \left( \int e^{\int (g_2 - g_3) dx} \left( \int W e^{\int g_3 dx} dx + c_1 \right) dx + c_2 \right)$$
$$\Rightarrow y = e^{-\int g_1 dx} \left( \int e^{\int (g_1 - g_2) dx} \left( \int e^{\int (g_2 - g_3) dx} \left( \int W e^{\int g_3 dx} dx + c_1 \right) dx + c_2 \right) dx + c_3 \right)$$

□

**Lemma III.3**: For thrice differentiable functions $g_1, g_2, g_3, P, Q, R$ :

$$\left. \begin{array}{|l|} \hline P \equiv (g_1 + g_2 + g_3) \\ \hline Q \equiv (g_1' + g_2 g_1) + (g_1' + g_3 g_1) + (g_2' + g_3 g_2) \\ \hline R \equiv g_1'' + (g_2 + g_3) g_1' + (g_2' + g_3 g_2) g_1 \\ \hline \end{array} \right\}$$
$$\Rightarrow \left\{ \begin{array}{|l|} \hline g_1'' - \frac{3}{2} (g_1^2)' + P g_1' + g_1^3 - P g_1^2 + Q g_1 = R \\ \hline g_2' - g_2^2 + (P - g_1) g_2 = Q - 2 g_1' - (P - g_1) g_1 \\ \hline g_3 = P - g_1 - g_2 \\ \hline \end{array} \right.$$

*Proof:*

$$\begin{array}{|l|} \hline P \equiv (g_1 + g_2 + g_3) \\ \hline Q \equiv (g_1' + g_2 g_1) + (g_1' + g_3 g_1) + (g_2' + g_3 g_2) \\ \hline R \equiv g_1'' + (g_2 + g_3) g_1' + (g_2' + g_3 g_2) g_1 \\ \hline \end{array}$$
$$\begin{aligned} \Rightarrow R - g_1'' - (P - g_1) g_1' &= [Q - (g_1' + g_2 g_1) - (g_1' + g_3 g_1)] g_1 \\ &= [Q - 2 g_1' - (g_2 + g_3) g_1] g_1 \\ &= [Q - 2 g_1' - (P - g_1) g_1] g_1 \\ &= Q g_1 - 2 g_1 g_1' - P g_1^2 + g_1^3 \end{aligned}$$
$$\Rightarrow R - g_1'' - P g_1' + g_1 g_1' = Q g_1 - 2 g_1 g_1' - P g_1^2 + g_1^3$$
$$\Rightarrow R = g_1'' - 3 g_1 g_1' + P g_1' + Q g_1 - P g_1^2 + g_1^3$$
$$\Rightarrow \left\{ \begin{array}{|l|} \hline g_1'' - \frac{3}{2} (g_1^2)' + P g_1' + g_1^3 - P g_1^2 + Q g_1 = R \\ \hline g_1'' - 3 g_1 g_1' + P g_1' + g_1^3 - P g_1^2 + Q g_1 = R \\ \hline \end{array} \right.$$
$$\begin{aligned} \Rightarrow (g_2' + g_3 g_2) &= Q - (g_1' + g_2 g_1) - (g_1' + g_3 g_1) \\ &= Q - (g_1' + g_2 g_1) - (g_1' + g_3 g_1) \\ &= Q - 2 g_1' - (g_2 + g_3) g_1 \\ &= Q - 2 g_1' - (P - g_1) g_1 \end{aligned}$$
$$\Rightarrow (g_2' + (P - g_1 - g_2) g_2) = Q - 2 g_1' - (P - g_1) g_1$$
$$\Rightarrow g_2' - g_2^2 + (P - g_1) g_2 = Q - 2 g_1' - (P - g_1) g_1$$
$$\Rightarrow \left\{ \boxed{g_2' - g_2^2 + (P - g_1) g_2 = Q - 2 g_1' - (P - g_1) g_1} \right.$$
$$\Rightarrow \left\{ \boxed{g_3 = P - g_1 - g_2} \right.$$

□

Example 1:

$P, Q, R$ constants (Constant Coefficients)

Let: $g_1 . g_2 . g_3$ be constants:

$$\Rightarrow \left\{ \begin{array}{|l|} \hline 0 - 0 + 0 + g_1^3 - P g_1^2 + Q g_1 = R \\ \hline 0 - g_2^2 + (P - g_1) g_2 = Q - (P - g_1) g_1 \\ \hline g_3 = P - g_1 - g_2 \\ \hline \end{array} \right.$$

$$\Rightarrow \begin{cases} g_1^3 - Pg_1^2 + Qg_1 - R = 0 \\ g_2^2 - (P - g_1)g_2 + [Q - 2g_1' - (P - g_1)g_1] = 0 \\ g_3 = P - g_1 - g_2 \end{cases}$$

$$\Rightarrow \left\{ \begin{array}{l} g_1 = \sqrt[3]{s + \sqrt{s^2 + (t - r^2)^3}} + \sqrt[3]{s - \sqrt{s^2 + (t - r^2)^3}} - r \\ g_2 = \frac{1}{2}\left[ (P - g_1) \pm \sqrt{(P - g_1)^2 - 4[Q - (P - g_1)g_1]} \right] \\ g_3 = P - g_1 - g_2 \end{array} \right\} \left\{ \begin{array}{l} r = \frac{1}{3}P \\ s = r^3 + \frac{1}{6}(3R - PQ) \\ t = -\frac{1}{3}R \end{array} \right\}$$

Example 2:

$P = \frac{A}{x}, Q = \frac{B}{x^2}, R = \frac{C}{x^3}$ ; $A, B, C$ ; constants (Cauchy-Euler)

Let: $g_1 = \frac{a}{x}, g_2 = \frac{b}{x}, g_3 = \frac{c}{x}$ ; $a, b, c$ ; constants

$$\Rightarrow \begin{cases} \frac{2a}{x^3} + 2\frac{a^2}{x^3} - \frac{A}{x}\frac{a}{x^2} + \frac{a^3}{x^3} - \frac{A}{x}\frac{a^2}{x^2} + \frac{B}{x^2}\frac{a}{x} = \frac{C}{x^3} \\ -\frac{b}{x^2} - \frac{b^2}{x^2} + \left(\frac{A}{x} - \frac{a}{x}\right)\frac{b}{x} = \frac{B}{x^2} - 2\frac{a}{x^2} - \left(\frac{A}{x} - \frac{a}{x}\right)\frac{a}{x} \\ g_3 = \frac{A}{x} - \frac{a}{x} - \frac{b}{x} \Rightarrow c = A - a - b \end{cases}$$

$$\Rightarrow \begin{cases} 2a + 2a^2 - Aa + a^3 - Aa^2 + Ba = C \\ -b - b^2 + (A - a)b = B - 2a - (A - a)a \\ g_3 = \frac{A}{x} - \frac{a}{x} - \frac{b}{x} \Rightarrow c = A - a - b \end{cases}$$

$$\Rightarrow \begin{cases} a^3 + (2 - A)a^2 + (2 - A + B)a - C = 0 \\ b^2 - (A - a - 1)b + [B - (A - a + 2)a] = 0 \\ g_3 = \frac{A}{x} - \frac{a}{x} - \frac{b}{x} \Rightarrow c = A - a - b \end{cases}$$

$$\Rightarrow \left\{ \begin{array}{l} g_1 = \frac{a}{x} \Leftrightarrow a = \left( \sqrt[3]{s + \sqrt{s^2 + (t - r^2)^3}} + \sqrt[3]{s - \sqrt{s^2 + (t - r^2)^3}} - r \right) \\ g_2 = \frac{1}{2x}\left[ (A - a - 1) \pm \sqrt{(A - a - 1)^2 - 4[B - (A - a + 2)a]} \right] \\ g_3 = \frac{A - a - b}{x} \end{array} \right\} \left\{ \begin{array}{l} r = -\frac{1}{3}(2 - A) \\ s = r^3 + \frac{1}{6}(3C - (2 - A)(2 - A + B)) \\ t = \frac{1}{3}C \end{array} \right\}$$

**Theorem III.4**: For thrice differentiable function $y$, twice differentiable function $g_1$ and differentiable functions $g_2, g_3$.

$$\left(\left(\left(\left(\left(ye^{\int g_1 dx}\right)' e^{-\int g_1 dx}\right) e^{\int g_2 dx}\right)' e^{-\int g_2 dx}\right) e^{\int g_3 dx}\right)' e^{-\int g_3 dx} = (D + g_3)(D + g_2)(D + g_1)y$$

$$= y''' + (g_1 + g_2 + g_3)y'' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]y' +$$
$$+ [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]y$$

*Proof:*

$$\begin{aligned} (D + g_3)(D + g_2)(D + g_1) &= (D + g_3)(D^2 + Dg_1 + g_1D + g_2D + g_2g_1) \\ &= (D^3 + D^2g_1 + Dg_1D + Dg_1D + g_1D^2 + Dg_2D + g_2D^2 + D(g_2g_1) + g_2g_1D + g_3D^2 + g_3Dg_1 + g_3g_1D + g_3 \\ &= (D^3 + g_1D^2 + g_2D^2 + g_3D^2 + Dg_1D + Dg_1D + Dg_2D + g_2g_1D + g_3g_1D + g_3g_2D + D^2g_1 + D(g_2g_1) + g_3 \\ &= (D^3 + (g_1 + g_2 + g_3)D^2 + (Dg_1 + Dg_1 + Dg_2 + g_2g_1 + g_3g_1 + g_3g_2)D + D^2g_1 + D(g_2g_1) + g_3Dg_1 + g_3g \\ &= (D^3 + (g_1 + g_2 + g_3)D^2 + (g_1' + g_1' + g_2' + g_2g_1 + g_3g_1 + g_3g_2)D + g_1'' + g_2'g_1 + g_2g_1' + g_3g_1' + g_3g_2g_1) \\ &= (D^3 + (g_1 + g_2 + g_3)D^2 + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]D + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2 \end{aligned}$$

$$\Rightarrow (D + g_3)(D + g_2)(D + g_1)y = y''' + (g_1 + g_2 + g_3)y'' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]y' +$$
$$+ [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]y$$
$$= \left(\left(\left(\left(\left(ye^{\int g_1 dx}\right)' e^{-\int g_1 dx}\right) e^{\int g_2 dx}\right)' e^{-\int g_2 dx}\right) e^{\int g_3 dx}\right)' e^{-\int g_3 dx}$$

□

**Theorem IV.1**: For four times differentiable function $y$, thrice differentiable functions $g_1$, twice differentiable function $g_2$ and differentiable functions $g_3, g_4$.

$$\left(\left(\left(\left(\left(\left(\left(ye^{\int g_1 dx}\right)' e^{-\int g_1 dx}\right) e^{\int g_2 dx}\right)' e^{-\int g_2 dx}\right) e^{\int g_3 dx}\right)' e^{-\int g_3 dx}\right) e^{\int g_4 dx}\right)' e^{-\int g_4 dx} =$$

$$= y^{(iv)} + (g_1 + g_2 + g_3 + g_4)y''' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_1' + g_1g_4) + (g_2' + g_3g_2) + (g_2' + g_2g_4) + (g_3' + g_3g_4)]y'' +$$
$$+ \Big[[g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1] + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' +$$
$$+ \Big[[g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]g_4\Big]y$$

*Proof:*

$$\left(\left(\left(\left(\left(\left(\left(ye^{\int g_1 dx}\right)' e^{-\int g_1 dx}\right) e^{\int g_2 dx}\right)' e^{-\int g_2 dx}\right) e^{\int g_3 dx}\right)' e^{-\int g_3 dx}\right) e^{\int g_4 dx}\right)' e^{-\int g_4 dx} =$$

$$= \left(\left(\left(\left(\left(\left((y' + yg_1)e^{\int g_1 dx} e^{-\int g_1 dx}\right) e^{\int g_2 dx}\right)' e^{-\int g_2 dx}\right) e^{\int g_3 dx}\right)' e^{-\int g_3 dx}\right) e^{\int g_4 dx}\right)' e^{-\int g_4 dx}$$

$$= \left(\left(\left(\left(\left((y' + yg_1)e^{\int g_2 dx}\right)' e^{-\int g_2 dx}\right) e^{\int g_3 dx}\right)' e^{-\int g_3 dx}\right) e^{\int g_4 dx}\right)' e^{-\int g_4 dx}$$

$$
\begin{aligned}
&= \left(\left(\left(\left(\left((y' + yg_1)'e^{\int g_2dx} + (y' + yg_1)g_2e^{\int g_2dx}\right)e^{-\int g_2dx}\right)e^{\int g_3dx}\right)'e^{-\int g_3dx}\right)e^{\int g_4dx}\right)'e^{-\int g_4dx} \\
&= \left(\left(\left(\left((y' + yg_1)' + (y' + yg_1)g_2\right)e^{\int g_3dx}\right)'e^{-\int g_3dx}\right)e^{\int g_4dx}\right)'e^{-\int g_4dx} \\
&= \left(\left(\left(\left(\left((y' + yg_1)' + (y' + yg_1)g_2\right)\right)'e^{\int g_3dx} + \left(\left((y' + yg_1)' + (y' + yg_1)g_2\right)\right)g_3e^{\int g_3dx}\right)e^{-\int g_3dx}\right)e^{\int g_4dx}\right)'e^{-\int g_4dx} \\
&= \left(\left(\left(\left(\left((y' + yg_1)' + (y' + yg_1)g_2\right)\right)' + \left(\left((y' + yg_1)' + (y' + yg_1)g_2\right)\right)g_3\right)\right)e^{\int g_4dx}\right)'e^{-\int g_4dx} \\
&= \left(\left(y''' + (g_1 + g_2 + g_3)y'' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]y' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]y\right)e^{\int g_4dx}\right)'e^{-\int g_4dx} \\
&= \left(\left(y''' + (g_1 + g_2 + g_3)y'' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]y' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]y\right)'e^{\int g_4dx}\right)e^{-\int g_4dx} \\
&+ \left(\left(y''' + (g_1 + g_2 + g_3)y'' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]y' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]y\right)g_4e^{\int g_4dx}\right)e \\
&= \Big(\Big(y^{(iv)} + (g_1 + g_2 + g_3)y''' + (g_1 + g_2 + g_3)'y'' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]y'' + \\
&\quad + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]'y' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]y' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g \\
&+ \left(\left(y''' + (g_1 + g_2 + g_3)y'' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]y' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]y\right)g_4e^{\int g_4dx}\right)e \\
&= y^{(iv)} + (g_1 + g_2 + g_3)y''' + (g_1 + g_2 + g_3)'y'' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]y'' + \\
&\quad + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]'y' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]y' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g \\
&+ (y''' + (g_1 + g_2 + g_3)y'' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]y' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]y)g_4 \\
&= y^{(iv)} + (g_1 + g_2 + g_3)y''' + (g_1 + g_2 + g_3)'y'' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]y'' + \\
&\quad + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]'y' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]y' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g \\
&+ g_4y''' + (g_1 + g_2 + g_3)g_4y'' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]g_4y' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]g_4y \\
&= y^{(iv)} + (g_1 + g_2 + g_3 + g_4)y''' + (g_1 + g_2 + g_3)'y'' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]y'' + (g_1 + g_2 + g_3)g_4y'' + \\
&\quad + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]'y' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]y' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' \\
&\quad + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]'y + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]g_4y \\
&= y^{(iv)} + (g_1 + g_2 + g_3 + g_4)y''' + \Big[(g_1 + g_2 + g_3)' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)] + (g_1 + g_2 + g_3)g_4\Big]y'' + \\
&\quad + \Big[[(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1] + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + \\
&\quad + \Big[[g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]g_4\Big]y \\
&= y^{(iv)} + (g_1 + g_2 + g_3 + g_4)y''' + [g_1' + g_2' + g_3' + g_1' + g_2g_1 + g_1' + g_3g_1 + g_2' + g_3g_2 + g_1g_4 + g_2g_4 + g_3g_4]y'' + \\
&\quad + \Big[[(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1] + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + \\
&\quad + \Big[[g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]g_4\Big]y \\
&= y^{(iv)} + (g_1 + g_2 + g_3 + g_4)y''' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_1' + g_1g_4) + (g_2' + g_3g_2) + (g_2' + g_2g_4) + (g_3' + g_3g_4)]y'' + \\
&\quad + \Big[[g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1] + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + \\
&\quad + \Big[[g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]g_4\Big]y
\end{aligned}
$$

□

**Theorem IV.2**: For four times differentiable function $y$, thrice differentiable functions $g_1$, twice differentiable function $g_2$ and differentiable functions $g_3, g_4$.

$$
\begin{aligned}
&\left(\left(\left(\left(\left(\left(\left(ye^{\int g_1dx}\right)'e^{-\int g_1dx}\right)e^{\int g_2dx}\right)'e^{-\int g_2dx}\right)e^{\int g_3dx}\right)'e^{-\int g_3dx}\right)e^{\int g_4dx}\right)'e^{-\int g_4dx} = \\
&= (D + g_4)(D + g_3)(D + g_2)(D + g_1)y \\
&= y^{(iv)} + (g_1 + g_2 + g_3 + g_4)y''' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_1' + g_1g_4) + (g_2' + g_3g_2) + (g_2' + g_2g_4) + (g_3' + g_3g_4)]y'' + \\
&\quad + \Big[[g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1] + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + g_3g_2)]' + [(g_1' + g_2g_1) + (g_1' + g_3g_1) + (g_2' + \\
&\quad + \Big[[g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]' + [g_1'' + (g_2 + g_3)g_1' + (g_2' + g_3g_2)g_1]g_4\Big]y
\end{aligned}
$$

*Proof:*

Similar to theorem III.4.

□

Obviously, theorrems II.4 & IV.2 may be generalized for any integral order LODE.

The *g*'s are determined just as with Lemma III.1 , and it's following Examples 1 & 2.

Just as 2nd order LODEs may be factored for solution, LPDEs may also be factored.

**Theorem V.1**: For differentiable functions $u_1, u_2, g, v_1, v_2, h$ :

$$
\begin{aligned}
&\left(u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + g\right)\left(u_2\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} + h\right) = \\
&= u_1u_2\frac{\partial^2}{\partial x^2} + (u_1v_2 + v_1u_2)\frac{\partial^2}{\partial y\partial x} + v_1v_2\frac{\partial^2}{\partial y^2} + \\
&+ \left(\left[u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + g\right]u_2 + hu_1\right)\frac{\partial}{\partial x} + \left(\left[u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + g\right]v_2 + hv_1\right)\frac{\partial}{\partial y} + \\
&+ \left[\left(u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + g\right)h\right]
\end{aligned}
$$

*Proof:*

$$
\begin{aligned}
&\left(u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + g\right)\left(u_2\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} + h\right) = \\
&= u_1\frac{\partial u_2}{\partial x}\frac{\partial}{\partial x} + u_1u_2\frac{\partial^2}{\partial x^2} + u_1\frac{\partial v_2}{\partial x}\frac{\partial}{\partial y} + u_1v_2\frac{\partial^2}{\partial x\partial y} + u_1\frac{\partial h}{\partial x} + u_1h\frac{\partial}{\partial x} +
\end{aligned}
$$

$$
\begin{aligned}
&+ v_1 \frac{\partial u_2}{\partial y}\frac{\partial}{\partial x} + v_1 u_2 \frac{\partial^2}{\partial y \partial x} + v_1 \frac{\partial v_2}{\partial y}\frac{\partial}{\partial y} + v_1 v_2 \frac{\partial^2}{\partial y^2} + v_1 \frac{\partial h}{\partial y} + v_1 h \frac{\partial}{\partial y} + \\
&+ g u_2 \frac{\partial}{\partial x} + g v_2 \frac{\partial}{\partial y} + gh \\
&= u_1 u_2 \frac{\partial^2}{\partial x^2} + u_1 v_2 \frac{\partial^2}{\partial x \partial y} + v_1 u_2 \frac{\partial^2}{\partial y \partial x} + v_1 v_2 \frac{\partial^2}{\partial y^2} + \\
&+ u_1 \frac{\partial u_2}{\partial x}\frac{\partial}{\partial x} + v_1 \frac{\partial u_2}{\partial y}\frac{\partial}{\partial x} + g u_2 \frac{\partial}{\partial x} + u_1 h \frac{\partial}{\partial x} + u_1 \frac{\partial v_2}{\partial x}\frac{\partial}{\partial y} + v_1 \frac{\partial v_2}{\partial y}\frac{\partial}{\partial y} + g v_2 \frac{\partial}{\partial y} + v_1 h \frac{\partial}{\partial y} + \\
&+ u_1 \frac{\partial h}{\partial x} + v_1 \frac{\partial h}{\partial y} + gh \\
&= u_1 u_2 \frac{\partial^2}{\partial x^2} + (u_1 v_2 + v_1 u_2)\frac{\partial^2}{\partial y \partial x} + v_1 v_2 \frac{\partial^2}{\partial y^2} + \\
&+ \left( u_1 \frac{\partial u_2}{\partial x} + v_1 \frac{\partial u_2}{\partial y} + u_1 h + g u_2 \right)\frac{\partial}{\partial x} + \left( u_1 \frac{\partial v_2}{\partial x} + v_1 \frac{\partial v_2}{\partial y} + v_1 h + g v_2 \right)\frac{\partial}{\partial y} + \\
&+ \left( u_1 \frac{\partial h}{\partial x} + v_1 \frac{\partial h}{\partial y} + gh \right) \\
&= u_1 u_2 \frac{\partial^2}{\partial x^2} + (u_1 v_2 + v_1 u_2)\frac{\partial^2}{\partial y \partial x} + v_1 v_2 \frac{\partial^2}{\partial y^2} + \\
&+ \left[ \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right) u_2 + u_1 h \right]\frac{\partial}{\partial x} + \left[ \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right) v_2 + v_1 h \right]\frac{\partial}{\partial y} + \\
&+ \left[ \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right) h \right]
\end{aligned}
$$

□

**Theorem V.1**: For differentiable functions $u_1, u_2, g, v_1, v_2, h$ :

If:

$$
\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right)\left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right) = \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right)\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right)
$$

then

$$
\begin{cases}
\boxed{\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} \right) u_2 = \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} \right) u_1} \\
\boxed{\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} \right) v_2 = \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} \right) v_1} \\
\boxed{\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} \right) h = \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} \right) g}
\end{cases}
$$

*Proof:*

$$
\begin{aligned}
&u_1 u_2 \frac{\partial^2}{\partial x^2} + (u_1 v_2 + v_1 u_2)\frac{\partial^2}{\partial y \partial x} + v_1 v_2 \frac{\partial^2}{\partial y^2} + \\
&+ \left[ \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right) u_2 + u_1 h \right]\frac{\partial}{\partial x} + \left[ \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right) v_2 + v_1 h \right]\frac{\partial}{\partial y} + \\
&+ \left[ \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right) h \right] = \\
&\quad = u_2 u_1 \frac{\partial^2}{\partial x^2} + (u_2 v_1 + v_2 u_1)\frac{\partial^2}{\partial x \partial y} + v_2 v_1 \frac{\partial^2}{\partial y^2} + \\
&\quad + \left[ \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right) u_1 + u_2 g \right]\frac{\partial}{\partial x} + \left[ \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right) v_1 + v_2 g \right]\frac{\partial}{\partial y} + \\
&\quad + \left[ \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right) g \right] \\
&\Rightarrow \left[ \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right) u_2 + u_1 h \right]\frac{\partial}{\partial x} + \left[ \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right) v_2 + v_1 h \right]\frac{\partial}{\partial y} + \left[ \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right) h \right] = \\
&\quad = \left[ \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right) u_1 + u_2 g \right]\frac{\partial}{\partial x} + \left[ \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right) v_1 + v_2 g \right]\frac{\partial}{\partial y} + \left[ \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right) g \right]
\end{aligned}
$$

$$
\Rightarrow \begin{cases}
\boxed{\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right) u_2 + u_1 h = \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right) u_1 + u_2 g} \\
\boxed{\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right) v_2 + v_1 h = \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right) v_1 + v_2 g} \\
\boxed{\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right) h = \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right) g}
\end{cases}
$$

$$
\Rightarrow \begin{cases}
\boxed{\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} \right) u_2 = \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} \right) u_1} \\
\boxed{\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} \right) v_2 = \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} \right) v_1} \\
\boxed{\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} \right) h = \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} \right) g}
\end{cases}
$$

□

**Theorem V.2**: For differentiable functions $u_1, v_1, g, \Psi$ :

If: $g = 0$ or $u_1$ & $v_1$ are constants, then:

$$
\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right)\Psi = 0
$$

$$
\Rightarrow \Psi(r,s) = \psi\left( x - \frac{u_1}{v_1} y \right) e^{-\int g(x,y)\partial r}
$$

*Proof:*

Let:

$r = r(x,y)$ , $s = s(x,y)$ , $\Phi(x,y) = \Phi(r)$ , $\Psi(x,y) = \Psi(s)$

$u_1 = u_1(x,y) = u_1(r,s)$ , $v_1 = v_1(x,y) = v_1(r,s)$ , $g(x,y) = g(r,s)$

$$
\Rightarrow \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right)\Psi = \left( u_1 \left( \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial s}{\partial x}\frac{\partial}{\partial s} \right) + v_1 \left( \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial s}{\partial y}\frac{\partial}{\partial s} \right) + g \right)\Psi(r,s)
$$

$$= \left[ \left( u_1 \frac{\partial r}{\partial x} + v_1 \frac{\partial r}{\partial y} \right) \frac{\partial}{\partial r} + \left( u_1 \frac{\partial s}{\partial x} + v_1 \frac{\partial s}{\partial y} \right) \frac{\partial}{\partial s} \right] \Psi(r,s) + g\Psi(r,s)$$

choose $p,q,w,z$ such that: $r = px + qy$ & $s = wx + zy$ & $pz - qw \neq 0$ AND::

$$0 = \frac{\partial s}{\partial x} + \frac{v_1}{u_1}\frac{\partial s}{\partial y} = w + \frac{v_1}{u_1} z \Rightarrow w = -\frac{v_1}{u_1} z \Rightarrow p \neq q\frac{w}{z} = -q\frac{v_1}{u_1} \quad , \quad (u_1 \neq 0)$$

$$\Rightarrow u_1 \frac{\partial r}{\partial x} + v_1 \frac{\partial r}{\partial y} = u_1 p + v_1 q = u_1\left(p + q\frac{v_1}{u_1}\right)$$

$$\Rightarrow 0 = \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right)\Psi = u_1\left(p + q\frac{v_1}{u_1}\right)\frac{\partial}{\partial r}\Psi(r,s) + g\Psi(r,s)$$

So, let: $u_1\left(p + q\frac{v_1}{u_1}\right) = 1 \Rightarrow u_1 p + q v_1 \Rightarrow p = \frac{1}{u_1} - q\frac{v_1}{u_1}$

$$\Rightarrow r = \left(\frac{1}{u_1} - q\frac{v_1}{u_1}\right)x + qy \quad \& \quad s = -\frac{v_1}{u_1} zx + zy$$

So, choose: $q = \frac{1}{v_1}$ & $z = -\frac{u_1}{v_1}$

$$\Rightarrow \begin{cases} r = \frac{1}{v_1} y & x = s - u_1 r \\ s = x - \frac{u_1}{v_1} y & y = v_1 r \end{cases} \quad , \quad \left(u_1 \neq 0 \ \& \ v_1 \neq 0\right)$$

$$\Rightarrow 0 = \frac{\partial}{\partial r}\Psi(r,s) + g\Psi(r,s) = \frac{\partial}{\partial r}\left(\Psi(r,s) e^{\int g \partial r}\right) \Rightarrow \Psi(r,s) = \psi(s) e^{-\int g \partial r}$$

$$\Rightarrow \Psi(r,s) = \psi\left(x - \frac{u_1}{v_1} y\right) e^{-\int g(x,y) \partial r}$$

□

With these tansformations, if $g(x,y) \neq 0$ then $g$ cannot be explicitly written as $g = g(r,s)$ unless: $u_1$ & $v_1$ are constants.

More generally:

**Theorem V.3**: For differentiable functions $u_1, u_2, g, v_1, v_2, h$ :

If:

$$\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right)\left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right) = \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right)\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right)$$

*Proof:*

$$\Psi = \left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right)\Phi$$

, whenever: $u_1 = u_1(x)$ & $v_1 = v_1(x)$ & $g = g(x)$ :

$$\frac{d}{dx}\Psi(x,y(x)) = \left( \frac{\partial}{\partial x}\Psi(x,y(x)) + \frac{dy}{dx}\frac{\partial}{\partial y}\Psi(x,y(x)) \right) = \left( \frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y} \right)\Psi(x,y(x))$$

With: $\frac{dy}{dx} = \frac{v_1}{u_1} \Rightarrow y = \int \frac{v_1}{u_1} dx + y(0)$ :

$$0 = \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right)\Psi = \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} \right)\Psi + g\Psi$$

$$= \left( \frac{\partial}{\partial x} + \frac{v_1}{u_1}\frac{\partial}{\partial y} \right)\Psi(x,y(x)) + \frac{g}{u_1}\Psi(x,y(x)) = \frac{d}{dx}\Psi(x,y(x)) + \frac{g}{u_1}\Psi(x,y(x))$$

$$= e^{-\int \frac{g}{u_1} dx}\frac{d}{dx}\left[ \Psi(x,y(x)) e^{\int \frac{g(x)}{u_1(x)} dx} \right]$$

$$\Rightarrow 0 = \frac{d}{dx}\left[ \Psi(x,y(x)) e^{\int \frac{g(x)}{u_1(x)} dx} \right]$$

$$\Rightarrow \psi(0,y(0)) = \Psi(x,y(x)) e^{\int \frac{g(x)}{u_1(x)} dx}$$

$$\Rightarrow \Psi(x,y) = \psi\left(y - \int \frac{v_1}{u_1} dx\right) e^{-\int \frac{g(x)}{u_1(x)} dx}$$

Examples:

The free-space wave equation:

$$\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right)\left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right) =$$

$$= u_1 u_2 \frac{\partial^2}{\partial x^2} + (u_1 v_2 + v_1 u_2)\frac{\partial^2}{\partial y \partial x} + v_1 v_2 \frac{\partial^2}{\partial y^2} +$$

$$+ \left( \left[ u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right] u_2 + h u_1 \right)\frac{\partial}{\partial x} + \left( \left[ u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right] v_2 + h v_1 \right)\frac{\partial}{\partial y} +$$

$$+ \left[ \left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right) h \right]$$

$u_1 = u_2 = 1$ , $v_1 = \frac{1}{c}$ , $v_2 = -\frac{1}{c}$ , $u_1 v_2 + v_1 u_2 = 0$ , $g = h = 0$ , $y = t$

$$\Rightarrow 0 = \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right)\phi = \left( \frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t} \right)\left( \frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t} \right)\phi = \left( \frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t} \right)\left( \frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t} \right)\phi$$

$$= \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c}\frac{\partial^2}{\partial t^2} \right) & 0 \\ 0 & \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$= \begin{pmatrix} \left( \frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t} \right) & 0 \\ 0 & \left( \frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t} \right) \end{pmatrix}\begin{pmatrix} \left( \frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t} \right) & 0 \\ 0 & \left( \frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t} \right) \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\Rightarrow 0 = \phi_1 = \psi\left(x - \frac{1}{c}t\right) \ , \ \phi_2 = \psi\left(x + \frac{1}{c}t\right)$$

Instead of writing the wave function as a sum of arbitrary functions, this way it is written as a doublet of these arbitrary functions.

a heat equation equation is not factorable this way :

$$\left( u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + g \right)\left( u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + h \right) =$$

$$= u_1u_2\frac{\partial^2}{\partial x^2} + (u_1v_2 + v_1u_2)\frac{\partial^2}{\partial t\partial x} + v_1v_2\frac{\partial^2}{\partial t^2} +$$
$$+ \left(\left[u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + g\right]u_2 + hu_1\right)\frac{\partial}{\partial x} + \left(\left[u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + g\right]v_2 + hv_1\right)\frac{\partial}{\partial y} +$$
$$+ \left[\left(u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial t} + g\right)h\right]$$

$u_1u_2 = 1$ , $u_1v_2 + v_1u_2 = 0$ , $v_1v_2 = 0$ , $y = t$

$$v_2 = 0 \Rightarrow v_1u_2 = 0 \ \& \ \left(u_1\frac{\partial}{\partial x} + g\right)\left(u_2\frac{\partial}{\partial x} + h\right) \text{ or } \left(u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial t} + g\right)(h)$$

neither of which is of the heat equation form:

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{k}\frac{\partial\phi}{\partial t} \Rightarrow \left(\frac{\partial^2}{\partial x^2} - \frac{1}{k}\frac{\partial}{\partial t}\right)\phi = f(x,t)$$

a 'damped wave equation with source/sink:

$$\left(u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + g\right)\left(u_2\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} + h\right) =$$
$$= u_1u_2\frac{\partial^2}{\partial x^2} + (u_1v_2 + v_1u_2)\frac{\partial^2}{\partial y\partial x} + v_1v_2\frac{\partial^2}{\partial y^2} +$$
$$+ \left(\left[u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + g\right]u_2 + hu_1\right)\frac{\partial}{\partial x} + \left(\left[u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + g\right]v_2 + hv_1\right)\frac{\partial}{\partial y} +$$
$$+ \left[\left(u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + g\right)h\right]$$

$u_1 = u_2 = 1$ , $v_1 = \frac{1}{c}$ , $v_2 = -\frac{1}{c}$ , $u_1v_2 + v_1u_2 = 0$ , $y = t$

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t} + g\right)\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t} + h\right) =$$
$$= \frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + (h+g)\frac{\partial}{\partial x} + \frac{1}{c}(h-g)\frac{\partial}{\partial t} + \left[\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t} + g\right)h\right]$$

And:

$$\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t} + h\right)\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t} + g\right) =$$
$$= \frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + (h+g)\frac{\partial}{\partial x} + \frac{1}{c}(h-g)\frac{\partial}{\partial t} + \left[\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t} + h\right)g\right]$$

Thus, even in one space variable, the Helmholtz/Klein-Gordon equation may be factored.
(This, however, indicates how the Maxwell-Cassano equations of an electromagnetic-nuclear field may be modified for non-constant mass and what the general high energy Lagrangian equations really are)
While we're on the subject, if the space partial corresponds to a three-space partial the only question is whether it corresponds to a gradient or divergence.
The gradient would yield:

$$\left(\nabla + \frac{1}{c}\frac{\partial}{\partial t} + g\right)\left(\nabla - \frac{1}{c}\frac{\partial}{\partial t} + h\right)\phi =$$
$$= \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + (h+g)\nabla + \frac{1}{c}(h-g)\frac{\partial}{\partial t} + \left[\left(\nabla + \frac{1}{c}\frac{\partial}{\partial t} + g\right)h\right]\right)\phi$$
$$= \left(\Box + (h+g)\nabla + \frac{1}{c}(h-g)\frac{\partial}{\partial t} + \left[\left(\nabla + \frac{1}{c}\frac{\partial}{\partial t} + g\right)h\right]\right)\phi$$

And:

$$\left(\nabla - \frac{1}{c}\frac{\partial}{\partial t} + h\right)\left(\nabla + \frac{1}{c}\frac{\partial}{\partial t} + g\right)\phi =$$
$$= \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + (h+g)\nabla + \frac{1}{c}(h-g)\frac{\partial}{\partial t} + \left[\left(\nabla - \frac{1}{c}\frac{\partial}{\partial t} + h\right)g\right]\right)\phi$$
$$= \left(\Box + (h+g)\nabla + \frac{1}{c}(h-g)\frac{\partial}{\partial t} + \left[\left(\nabla - \frac{1}{c}\frac{\partial}{\partial t} + h\right)g\right]\right)\phi$$

for scalar field $\phi$ or scalar-doublet field $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ (analogous to the free-space wave equation above)

The divergence would yield something like:

$$\left(\vec{\nabla} + \vec{t}\frac{1}{c}\frac{\partial}{\partial t} + \vec{g}\right)\left(\vec{\nabla} - \vec{t}\frac{1}{c}\frac{\partial}{\partial t} + \vec{h}\right)\vec{\phi} =$$
$$= \left(\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{\vec{I}} + \left(\vec{h} + \vec{g}\right)\vec{\nabla} + \frac{1}{c}\left(\vec{h} - \vec{g}\right)\frac{\partial}{\partial t} + \left[\left(\vec{\nabla} + \frac{1}{c}\frac{\partial}{\partial t} + \vec{g}\right)\vec{h}\right]\right)\vec{\phi}$$

And:

$$\left(\vec{\nabla} - \vec{t}\frac{1}{c}\frac{\partial}{\partial t} + \vec{h}\right)\left(\vec{\nabla} + \frac{1}{c}\frac{\partial}{\partial t} + g\right)\vec{\phi} =$$
$$= \left(\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{\vec{I}} + \left(\vec{h} + g\right)\vec{\nabla} + \frac{1}{c}\left(\vec{h} - \vec{g}\right)\frac{\partial}{\partial t} + \left[\left(\vec{\nabla} - \frac{1}{c}\frac{\partial}{\partial t} + \vec{h}\right)\vec{g}\right]\right)\vec{\phi}$$

for four-vector field $\vec{\phi}$ or four-vector-doublet field $\begin{pmatrix}\vec{\phi}_1\\ \vec{\phi}_2\end{pmatrix}$

(analogous to the free-space wave equation above)

These being written in a Cartesian coordinate system, while the what the general high energy equations is written in Lagrangian form (confusion factor one) in general coordinates (confusion factor two); some transformations are required to match them but matching these factorings to the general high energy Lagrangians goes beyond the scope of this article, left for another.