On sinusoidal periodic oscillations of mixed Lienard type equations

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Abstract

We present in this paper a mixed Lienard type equation of physical importance. The equation can exhibit sinusoidal periodic solution. As a result, it can be used to model harmonic and isochronous periodic oscillations of dynamical systems.

Keywords: Periodic solution, sinusoidal oscillations, mixed Lienard type equations.

Introduction

As many dynamical systems exhibit periodic behavior, the problem of finding periodic solutions for nonlinear differential equations generated an attractive research field interesting pure and applied mathematics and physics. These dynamical systems in physics have been often described by conservative Lienard type equations

$$\ddot{x} + h(x) = 0 \tag{1}$$

where the overdot stands for derivative with respect to time and $h(x)$ is a nonlinear function of x. One of the most widely used Lienard type equation, is the so-called conservative cubic Duffing equation where $h(x) = \alpha x + \beta x^3$. Such an equation can model soft and hard nonlinear behavior of spring arising in several material systems. In this way the cubic Duffing equation has been used extensively to describe amplitude-dependent frequency feature of nonlinear dynamical systems, nonlinear resonances and other phenomena that could not be explained by the linear harmonic oscillator equation obtained for $\beta = 0$ [1]. The conservative cubic Duffing equation, as such, has been mentioned for a long time as a conservative nonlinear oscillator. In this case, such an oscillator can only have bounded periodic solutions. However, in some recently papers [2, 3] the authors have shown that the cubic Duffing equation can exhibit unbounded

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periodic solutions so that this equation is, in fact, a pseudo-oscillator [2, 3]. Another shortcoming in modeling periodic oscillations of mechanical systems is that the conservative cubic Duffing equation does not contain nonlinear dissipative terms characterizing these systems. It is known that, real world mechanical systems are characterized simultaneously by inertial, geometrical and damping nonlinearity properties [4-7]. As shown in [4-7] using the Bauer rheodynamics theory, an adequate and satisfactory second-order differential equation to describe mechanical vibrating systems is of the form of the mixed Lienard type equation

$$\ddot{x} + u(x)\dot{x}^2 + \vartheta(x)\dot{x} + h(x) = 0 \qquad (2)$$

where $u(x)$, $\vartheta(x)$, and $h(x)$ are arbitrary functions of x. As such, the mathematical problem to solve is to find the functions $u(x)$, $\vartheta(x)$, and $h(x)$ that ensure periodic solutions of the equation (2). In this perspective several authors studied the equation (2) from different approaches [8-11]. In [8] the authors used nonlocal transformation to investigate the equation (2). Using the modified Prelle-Singer approach the authors in [9] studied the equation (2) to give some integrable cases in connection with two dimensional Lotka-Volterra system. Tiwari and coworkers investigated the Lie point symmetries of the equation (2) in [10]. In [11] the authors studied the inverse problem of the mixed Lienard type equation (2). However, no sinusoidal periodic solution has been exhibited by these authors [8-11] in their studies in the context of the equation (2). As can be seen from the above, the simultaneous presence of several types of nonlinearity makes very difficult the possibility to find exact sinusoidal or periodic solutions of an equation of type (2). Despite this complexity of equations of type (2), Adjaï and coworkers [12] succeeded to identify the functions $u(x)$, $\vartheta(x)$ and $h(x)$ that secure exact and general periodic solution to the equation (2) for the first time. The authors [12] in the context of the selected functions $u(x)$, $\vartheta(x)$ and $h(x)$, obtained exact sinusoidal and isochronous periodic solutions for the equation (2). In this perspective the question to ask is whether there is another choice of functions $u(x)$, $\vartheta(x)$ and $h(x)$ ensuring the existence of sinusoidal or periodic solutions of the equation (2). In the present work, the objective is to show the existence of other expressions of functions $u(x)$, $\vartheta(x)$ and $h(x)$ that ensure exact and sinusoidal periodic solutions for the equation (2). We review in this regard the theory of integrable mixed Lienard type differential equations introduced by Monsia and coworkers [13-16] recently in the literature (section

2) and exhibit the sinusoidal periodic solution of the equation (2) according to the appropriate selected functions $u(x)$, $\vartheta(x)$ and $h(x)$ (section 3). We finally present a conclusion for the work.

2- Theory of integrable mixed Lienard type equations

Let us briefly review the theory of integrable mixed Lienard type equations introduced recently in the literature by Monsia and his group [13-16]. Under differentiation with respect to time, the first integral

$$g(x)\dot{x} + a\, f(x)x^{\ell} = b \tag{3}$$

where ℓ, b and a are constants, $g(x)$ and $f(x)$ are functions of x, and the dot over a symbol stands for derivative with respect to time, allows one to secure the class of Lienard type equations

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a\ell x^{\ell-1}\frac{f(x)}{g(x)}\dot{x} + a\,x^{\ell}\frac{f'(x)}{g(x)}\dot{x} = 0 \tag{4}$$

where prime means differentiation with respect to the variable x. Substituting the equation (3) into the fourth term of the equation (4) yields the class of mixed Lienard type equations

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a\,\ell x^{\ell-1}\frac{f(x)}{g(x)}\dot{x} - a^2 x^{2\ell}\frac{f'(x)f(x)}{g^2(x)} + ab\,x^{\ell}\frac{f'(x)}{g^2(x)} = 0 \tag{5}$$

Using the equation (5), the substitution of $g(x) = x^{\ell}$, secures the class of mixed Lienard type equations

$$\ddot{x} + \ell\frac{\dot{x}^2}{x} + a\,\ell\, f(x)\frac{\dot{x}}{x} - a^2 f'(x)f(x) + ab\, f'(x)x^{-\ell} = 0 \tag{6}$$

Now, making $b = 0$, yields the mixed Lienard type equation

$$\ddot{x} + \ell\frac{\dot{x}^2}{x} + a\,\ell\, f(x)\frac{\dot{x}}{x} - a^2 f'(x)f(x) = 0 \tag{7}$$

with the corresponding first integral

$$g(x)\dot{x} + a\, f(x)x^{\ell} = 0 \tag{8}$$

that is

$$\dot{x} + a\, f(x) = 0 \tag{9}$$

where $g(x) = x^{\ell} \neq 0$. From the equation (9) one can ensure the general solution of the mixed Lienard type equation (7) by the quadrature

$$-a(t+K) = \int \frac{dx}{f(x)} \tag{10}$$

where K is a constant of integration. As can be seen there are several expressions of $f(x)$ to ensure explicit and general periodic solutions of the equation (7). However, there is only an expression of $f(x) = \sqrt{\beta^2 - x^2}$, to obtain immediately the desired sinusoidal periodic solution of the equation (7). Thus, using $f(x) = \sqrt{\beta^2 - x^2}$, the equation (7) becomes

$$\ddot{x} + \ell \frac{\dot{x}^2}{x} + a\,\ell \sqrt{\beta^2 - x^2}\,\frac{\dot{x}}{x} + a^2 x = 0 \tag{11}$$

where β is an arbitrary parameter, such that $u(x) = \frac{\ell}{x}$, $\vartheta(x) = a\ell \frac{\sqrt{\beta^2 - x^2}}{x}$ and $h(x) = a^2 x$. In this context, after integrating the integral in the equation (10) one can obtain

$$\sin^{-1}\left(\frac{x}{\beta}\right) = -a(t+K) \tag{12}$$

from which the exact and sinusoidal periodic solutions of (11) can be expressed as

$$x(t) = -\beta \sin[a(t+K)] \tag{13}$$

where $\beta < 0$. The solution (13) is identical to the solution of the linear harmonic oscillator equation

$$\ddot{x} + a^2 x = 0 \tag{14}$$

where the amplitude of oscillations is taken equal to $-\beta > 0$. As such, the equation (11) has the same exact sinusoidal periodic solution with the mixed Lienard type equation solved in [12] while the two equations are quite different.

Conclusion

We present a remarkable mixed Lienard type equation in this paper. We have successfully shown that this equation can exhibit sinusoidal periodic solution. In this way, it can be used to model harmonic and isochronous oscillations of nonlinear oscillators.

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