Mathematical Survey of the Action Principle

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Abstract. Defining the principle of least action in concise mathematical terms, it is shown that continuous paths exist, for which the action integrals are well-defined and extremal, but that do not not allow the Lagrange equations to be derived from. Instead, Lagrange functions need to be locally integrable (in an open region of space), in order that the Lagrange equations hold. The principle of extremal action therefore reduces to the condition of local integrability of the Lagrange function to a (locally defined) Hamilton-Jacobi function.

1. (Euclidean) Lagrange Function

In current mathematics, the Lagrangrian is defined on a symplectic 2n-dimensional manifold of 2-forms. That conceils from some basic problems by the additional layer of a manifold structure; it seems worth to restrict the problem domain to the simplest possible model, which is the global (2n+1)-dimensional Euclidean space of time t, the n generalized momenta $p := (p_1, \ldots, p_n)$, and the n generalized location coordinates $q := (q_1, \ldots, q_n)$. (All results can later be expressed in straightfoward manner to tangent spaces on these manifolds.)
With this, the classical action integral for a path $\omega : [t_1, t_2] \ni t \mapsto q(t) \in \mathbb{R}^n$ is physically defined as

$$S(\omega) := -\int_{t_0}^{t_1} (H(t, p(t), q(t)) - p(t) \cdot \dot{q(t)})dt, \tag{1.1}$$

and the principle of extremal action is stated in the form that all dynamically possible paths from $(t_0, q(t_0))$ to $(t1, q(t_1))$ are to be extremals of the action integral.

In order to get this physical definition also mathematically defined, we need a proper topology on the space of time curves $\omega : [t_0, t_1] \to \mathbb{R}^n$ that allows for a definition of a derivative of $S : \omega \mapsto S(\omega)$. The simplest one would be vector space of all continuously differentiable curves $\omega : [0, 1] \ni$

$t \mapsto q(t) \in \mathbb{R}^n$, which is a Banachspace (i.e. complete normed space) with its natural norm $\|\omega\|_1 := sup_{0\leq t\leq 1}\big(|q(t)| + \left|\frac{d}{dt}q(t)\right|\big)$.
Sadly, the topology is a bit too strong to meet the physical requirements of $q(0)$ and $q(1)$ being fixed endpoints, only. So things become more complicated: The space of all continuously differentiable paths has two closed subspaces: one is the space of all closed continuously differentiable paths, and the other one its subspace of all continuosly differentiable closed paths that start and end in the origin $0 \in \mathbb{R}^n$. We denote this 2^{nd} subspace as $\mathcal{C}_0^1(\mathbb{R}^n)$. Then the set of all continuous differentiable paths from $[0,1]$ in $\mathbb{R}^n$ with fixed endpoints $q(0)$ and $q(1)$ is an affine Banachspace $\omega_0 + \mathcal{C}_0^1(\mathbb{R}^n)$, where ω_0 is an arbitrary continuously differentiable path from $q(0)$ to $q(1)$. We can then define the differentiability of $S : \omega \mapsto S(\omega)$ to be:

Definition 1.1 (Differentiability). Let U be an open subset of $\omega_0+\mathcal{C}_0^1(\mathbb{R}^n)$ and $S : U \to \mathbb{R}$ be defined and continuous on U. Then S is defined to be differentiable at $\omega \in U$ if and only if there exists a continuous linear operator $DS : \mathcal{C}_0^1(\mathbb{R}^n) \to \mathcal{C}_0^1(\mathbb{R}^n)$ such that for all $\eta \in \mathcal{C}_0^1(\mathbb{R}^n)$ with $\omega + \eta \in U$ and $\|\eta\|_1 \to 0$: $S(\omega + \eta) = S(\omega) + DS(\eta) + o(\|\eta\|_1)$, where $o(h)$ denotes the rest term with the asymptotic condition $\frac{o(h)}{h} \to 0$ as $h \to 0$.
DS is called derivative of S at ω. S is said to be extremal in ω, if S is differentiable at ω and if its derivative is the zero operator.

2. Time Reversal

Definition 2.1 (Time Inversion). Given a time curve $\omega : [0,1] \to \mathbb{R}^n$, the time inverse $\mathcal{T}\omega$ is defined as: $\mathcal{T}\omega : [0,1] \ni t \mapsto \omega(1-t) \in \mathbb{R}^n$.

Now, $S : U \to \mathbb{R}$ is not an arbitrary mapping from U to $\mathbb{R}$, but a path integral of the Lagrange function along the path ω. Therefore, the following holds:

Proposition 2.2. *$S(\mathcal{T}\omega) = -S(\omega)$ and $S(\mathcal{T}\omega|_{[s_1,s_2]}) = -S(\omega|_{[s_1,s_2]})$, where $\omega|_{[s_1,s_2]}$ denote the restriction of ω to the open interval $[s_1, s_2]$ for $0 < s_1 < s_2 < 1$ and $S(\mathcal{T}\omega|_{[s_1,s_2]})$ its time inverse.*

Corollary 2.3. *Let $S : U \to \mathbb{R}$ be differentiable at $\omega \in U$. Then the derivative is zero, that is: ω is extremal.*

Proof. Because the derivative DS at ω is a linear, continuous mapping, we can split a small path $\eta \in \mathcal{C}_0^1(\mathbb{R}^n)$ into the sum $\eta = \eta_1+\cdots+\eta_n$ of its n component projections, and $DS(\eta) = \sum_k DS(\eta_k)$ is the commuting sum of the path integrals each of the components. Each curve η_k is also closed, starting and ending in the origin. Now, $DS(\eta_k)$ might not be zero. But because of the above corollary, the differentiability of S at ω implies the differentiability of S at $\mathcal{T}\omega$ with inverted derivative $-DS$, which demands $\int_0^1 DS(\eta_k) = o(\|\eta_k\|_1)$. $\square$

3. The Lagrange Equations

Let S be differentiable at ω. Then we know, it is extremal at ω, which is usually written as $\delta S(\omega) = 0$, and δ is called "virtual" displacement. Can we derive the Langrange equations from that?
The answer is no:

To get in line with the normal representation within physics, let's replace the momenta p by the (generalized) velocities $\dot{q}$ and write $L(q(t), \dot{q}(t), t) := -H(t, p(t), q(t)) + p(t) \cdot \dot{q}(t)$. Then the condition of extremality at a path ω is usually written as: $\delta \int_\omega L(q(t), \dot{q}(t), t) d\omega = \delta \int_0^1 L(q(t), \dot{q}(t), t) dt = 0$.
In the first step the differential δ is commuted with the time integral:

$$\delta \int_0^1 L dt = \int_0^1 \delta L dt.$$

Next, δL is written out as a sum of partial differentials: $\delta L = \frac{\partial L}{\partial q} \cdot \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial t} \delta t$, and then it is assumed that - given explicit time independence of L:

$$\frac{\partial L}{\partial \dot{q}} \cdot \delta \dot{q} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \cdot \delta q \right) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \cdot \delta q.$$

And, indeed, if L and therefore the Hamiltonian function is explicit time independant, δt and δq commute, that is: a differential variation of δt followed by a differential δq equals δq followed by δt. (Note, however, that under non-conserved conditions δt and δq will generally not commute.)

There is however one major problem in the last step: $\delta \dot{q}(t)$ is parallel to the tangent of ω at time t, so $\delta_{\|} := \dot{q}(t) \delta t$ is on the tangent of ω, while δq is in any of the directions spanned by the n location coordinates $q = (q_1, \ldots, q_n)$. So, in the time-independant situation, one ends up with

$$\delta \int_0^1 L dt = \int_0^1 \left(\frac{\partial L}{\partial q} \cdot \delta q - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \cdot \delta q_{\|} \right) dt = 0,$$

which is not sufficient (unless one includes an arbitrary rest term): instead, one needs

$$\int_0^1 \left(\frac{\partial L}{\partial q} \cdot \delta q - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \cdot \delta q \right) dt = 0$$

to derive the Lagrange equations, which mandates that for each small $s > 0$ there must exist an $\epsilon > 0$, such that $\int_s^{1-s} L dt$ is to be path independant for all paths $\omega|_{[s,1-s]} + \eta$ with $\eta \in \mathcal{C}_0^1(\mathbb{R}^n)$ and $\|\eta\|_1 < \epsilon$. The union of ranges of all the paths $\omega|_{[s,1-s]} + \eta$ contains the ϵ-balls in $\mathbb{R}^n$ around each $\omega(t) \in \mathbb{R}^n$, ($t \in [s, 1-s]$). And, since these ϵ-balls are convex, the conclusion is that within these ϵ-balls L is integrable to a unique function, i.e.: it is locally integrable. Whether L also also is globally integrable to a unique function in the whole environment of the range of the paths $\omega|_{[s,1-s]} + \eta$, that depends on whether or not this environment is globally convex or at least simply connected (as is well-known: see [1]).

The following example shows the necessity of local integrability of the Lagrange function:

4. Example of a non-integrable Lagrangian with paths of extremal action integrals

The Hamilton function of a free 2-dimensional mechanical system is given by $H = \frac{1}{2m}(p_1^2 + p_2^2)$, so $L = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2)$, which - as any free n-dimensioinal mechanical sysytem - is known to be globally integrable, as $S : \mathbb{R}^3 \ni (t, q_1, q_2) \mapsto -Et + p_1 q_1 - p_2 q_2 \in \mathbb{R}$ is its (Hamilton-Jacobi) action function, where E and $p = (p_1, p_2)$ are its constant energy and momentum. Let's write that Lagrange function in polar coordinates (ϕ, r), where $\phi in[0, 2\pi]$ is the angle and $r \in [0, \infty)$ the radius, which then is: $L = \frac{m}{2}\left(\dot{r}^2 + r\dot{\phi}^2\right)$. Now, let's add to it the potential $V = -r^2\phi$, which introduces a curl around the origin, and consider the curve ω that goes in a straight line from $q_1 = -1/2$ to $q_1 = 1/2$ along the q_1 axis. In polar coordinates, ω is the step function $\omega : [0, 1/2) \ni t \mapsto (\pi, 1/2 - t)$ and $\omega : [1/2, 1] \ni t \mapsto (0, 1/2 - t)$. Integration of V along an arc η_1, say, from ϕ to $\phi + \Delta\phi$ at a fixes radius $r_1 > 0$ gives $(1/2)r_1^2(\Delta\phi)^2)$, therefore integration in the opposite direction η_2 fom $\phi + \Delta\phi$ to ϕ at some smaller radius $r_0 < r_1$ adds $-(1/2)r_0^2(\Delta\phi)^2)$, while integration along the closing pathes η_3 and η_4 from r_0 to r_1 at ϕ and from r_1 to r_0 at $\phi + \Delta\phi$ cancel eachother. Therefore, the path integral of V or L along any piecewise continuously differentiable closed path in $\mathbb{R}^2$ is unequal zero. Yet, for $r_1 \to r_0$, the value is $o(r_1 - r_o)$, and integration along a circle at a distance r_1 around the origin, likewise gives $2\pi^2 r_1^2$, which again is $o(r_1)$. So, the path integrals of L and V are differentiable at the straight line curve ω: although the path integration of V along ω and $\omega + \eta$ with $\eta \in \mathcal{C}_0^1(\mathbb{R}^2)$ might differ, the difference is $o(\|\eta\|_1)$ as $\eta \to 0$ in $\mathcal{C}_0^1(\mathbb{R}^n)$. Hence, according to infinitesimal calculus, $\delta \int_0^1 L dt$ is zero at ω.

5. Closedness of Differential Forms

Note that allowing the end points of the curves to vary either, will just simplify the proofs as we can now express the differentiation of the path integrals within the Banachspace of continuous differentiable curves $\omega : [0, 1] \to \mathbb{R}^\ltimes$, instead of dealing with affine Banachspaces, but other than that, the results will carry over. Only the symbol of variation δ is commonly replaced by the differential symbol d, and is then called "total differential".

Albeit L is just a function, it can be rewritten as a differential 1-form α, namely by the use of the Legendre "transformation". $L = p\dot{q} - H(t, p, q)$ by taking its differentials: $\alpha := pd\dot{q} - H(t, p, q)dt$. That way, if that form is well-defined on a simply connected, or perhaps even convex open set $\Omega \subset \mathbb{R}^n$ of location coordinates, and is integrable in there, which means that path integration along piecewise continuously differentiable, closed paths $\omega : [0, 1] \to \Omega$ all vanish, then α is the exterior differential dS of an action function $S : [0, 1] \times \Omega \ni (t, q) \mapsto S(t, q) \in \mathbb{R}$, and this function also is unique up to an additive constant as well as eventually time and space dilatations $t \mapsto t - t_0$, $q \mapsto q - q_0$ of the time and space coordinates (in case of conserved energy and momentum) . And, if Ω is just open and not simply connected, then at least we can find such a function within the open ϵ-balls around each $q \in \Omega$.

Now, a differential form β, say, is said to be "closed", if and only if its exterior derivative $d\beta$ exists and vanishes, i.e.: $d\beta \equiv 0$. And conversely, if β is the exterior differential of another differential form θ, say, then β is called "exact".

That conjures up a peculiarity of the definition of closed forms: The principle theorem of differential forms is Poincaré's lemma, which states that a differential form β on a simply connected region $U \subset \mathbb{R}^n$, for which the exterior derivative is well-defined and continous on U, is exact, if and only if β is closed (see: [1]).

While a vanishing differential of a function f at some x_0 is commonly defined as $(Df(x_0))h = 0$, where $DF(x_0)$ is the derivative of f at x_0, meaning that $f(x_0 + h) = f(x_0) + o(h)$ for $h \to 0$, the notion of external derivative obviously understands it in that $f(x_0 + h) = f(x_0)$ should hold for all h in some ϵ-evironment of zero! Because otherwise, the above example of section 4 would disprove Poincaré's lemma.

6. Summary

From a mathematical standpoint, the principle of extremal action should be replaced by the more pristine principle of closedness of the exterior 1-form $p \cdot \dot{d}q - Hdt$, to which the principle of extremal action runs up to, if only one wants to derive the Lagrange equations from.
It allows to express the Lagrange equation in simpler terms:

Given any open ball $B \subset \mathbb{R}^n$ on which $p \cdot dq - Hdt$ is closed (and therefore exact, i.e. integrable), then it integrates to an action function $S : B \to \mathbb{R}$, for which

(i) $p(t, q) = \frac{\partial S(t,q)}{\partial q}$ for all $t \in [0, 1]$ and $q \in B$,
(ii) $E(t, q) = -\frac{\partial S(t,q)}{\partial t}$ for all $t \in [0, 1]$ and $q \in B$, and
(iii) $\dot{p}(t, q) = -F(t, q)$, where $F := \frac{\partial E(t,q)}{\partial q}$ for all $t \in [0, 1]$ and $q \in B$,

where the last statement follows from $\frac{\partial^2 S}{\partial t \partial q} = \frac{\partial^2 S}{\partial q \partial t}$.

Above all, the closedness of $p \cdot dq - Hdt$ is equivalent to the closedness of $\pm p \cdot dq \pm Hdt$, and it's likewise irrelevant physically, because of the symmetry of time and space inversion (parity). As such, the form $Hdt + p \cdot dq$ will suffice.

References

[1] H. Cartan, *Differential Forms* Herman Kershaw, 1971.

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