PRIMORIALS AND A FORMULA FOR ODD ABUNDANT NUMBERS

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Abstract. We conjecture that a formula that involves a difference between two primorials of different parities generates only odd abundant numbers for its arguments greater than 3. We verified this conjecture for the arguments up to $4 * 10^4$.

1. Introduction

This brief report presents some elementary formula $f(n)$ that appears to generate only odd abundant numbers for $n > 3$, where n is a natural number. The formula seems interesting in that it involves two primorials, one of odd parity, being the product of n consecutive odd primes, and the other of even parity, the ordinary primorial, i.e., the product of n consecutive primes.

Let us recall, for the sake of completeness, that an abundant number is an integer k, such that the sum of its divisors $\sigma(k)$ satisfies the condition $\sigma(k) > 2k$.

The smallest abundant number is 12, followed by 18 and 20. The smallest odd abundant number is much larger: 945. All proper multiples of 6 are abundant, but not only such multiples as 20, 40, or 70 are also abundant not being divisible by 6. All odd multiples of 945 are odd abundant numbers.

2. The formula and a conjecture

The formula we alluded to above is quite simple,

$$f(n) = oddprimorial(n) - primorial(n) = primorial(n)(prime(n+1)/2 - 1),$$

where $prime(k)$ is the k-th prime, $primorial(k)$ is the product of all consecutive primes up to the k-th prime, and $oddprimorial(k)$ is the product of the first k consecutive odd primes; as such it is linked to the latter through a simple relation: $oddprimorial(k) = primorial(k)prime(k+1)/2$.

Let us note that $f(n)$, which generates an integer sequence, can be interpreted as the difference between the smallest odd squarefree number and the smallest even such number, both with the same number of prime factors.

Using PARI/GP [1], a free software package for number theory, we can easily obtain the first terms of this sequence. Below we list the first 10 terms:

1, 9, 75, 945, 12705, 225225, 4339335, 101846745, 3011753745, 93810551835.

The 4-th term happens to be the first odd abundant number. With PARI/GP, we can verify that other terms of the sequence for $n > 3$ are also odd abundant

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numbers. We were able to do this for n up to $4 * 10^4$, and based on this we propose the following conjecture.

Conjecture. *The difference between the smallest odd squarefree number and the smallest even squarefree number containing the same number of prime factors is an odd abundant number when the number of such factors exceeds 3.*

Let us point out that the terms of the sequence studied here grow quite fast, with the 10-th term already containing 11 digits. In general, the 10^k-th term has the following number of digits for $k = 0$ to 4: 1, 11, 223, 3397, 45342.

3. The PARI/GP code

The following simple code was used to print the first ten terms listed in the previous section (one can easily modify it to print even more terms):

```
for(n=1, 10, a=prod(i=1, n, prime(i)); b=a*prime(n+1)/2-a;
print1(b, ", "))
```

The following code can be used to test which terms are not abundant:

```
for(n=1, 1000, a=prod(i=1, n, prime(i)); b=a*prime(n+1)/2-a;
sigma(b)-2*b<=0&&print1(b, ", "))
```

When used on our laptop, it revealed in about 10 minutes that among the first 10000 terms only the first three are not abundant. This is also true for the first 40000 terms, which took us about 12 hours in total to verify. The same feat for another 10000 terms would take us weeks if not months, though.

To find out the length of individual terms (the number of digits), we can employ the following code:

```
for(n=1, 1000, a=prod(i=1, n, prime(i)); b=a*prime(n+1)/2-a;
d=digits(b); print1(#d, ", "))
```

4. Conclusion

We have pointed out in this paper that the difference between the smallest odd squarefree number and the smallest even squarefree number containing the same number of prime factors may be less ordinary than it seems at first glance for being predominantly a member of a relatively rare species of odd abundant numbers.

However, it is important to keep in mind that the formula presented here may generate not only odd abundant numbers for $n > 3$. The fact that it does so for tens of thousands of cases in a row does not constitute a formal proof - such a proof is still missing, so the issue remains open. Still, the evidence we have gathered here indicates that it is likely that this is so for all $n > 3$. We hope that we or someone else will be able to prove (or disproof) it eventually. We intend to pursue this goal next and may report our progress in another version of this paper.

In connection with this, let us observe that even if our formula were to eventually fail to generate yet another odd abundant number, it seems fair to say that in relative terms it is already pretty good at generating such numbers. Let us compare it, for instance, to one of its competitors, sometimes noted in the popular (recreational) mathematics literature [2]: $945 + 630n$.

This formula generates odd abundant numbers for $n = 0$ to $n = 51$; it first fails to do so for $n = 52$ and then for other values of n. The fact it works so reasonably well has to do with some statistical property of odd abundant numbers. Namely, as evidenced by computer experiments, 630 is the most common separation between consecutive odd abundant numbers.

In contrast to this, the success of our formula is unlikely to be due to a statistical happenstance, which suggests its more fundamental nature.

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References

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[2] *Abundancy of Integers in form 945 + 630n*, https://proofwiki.org/wiki/Abundancy_of_Integers_in_form_945_%2B_630n.

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