On identical periodic solutions between differential equations

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Abstract

We show the existence for the first time of a singular quadratic Lienard type equation having the linear harmonic oscillator solution. We show secondly the existence of a singular quadratic Lienard type equation having also the Ermakov-Pinney equation solution.

Keywords: Linear harmonic oscillator, singular quadratic Lienard type equation, exact periodic solution.

Introduction

In theory of second-order autonomous differential equations, the linear harmonic oscillator

$$\ddot{x} + \omega^2 x = 0 \tag{1}$$

where the overdot denotes differentiation with respect to time, and ω is a constant, is taken as the prototype. It is known that the general solution of the equation (1) is

$$x(t) = A_0 \sin(\omega t + \alpha) \tag{2}$$

where A_0 and α are arbitrary parameters.

As the analytical properties of the sine function are well known and very convenient for the engineering practice, the problem of finding sinusoidal solution to differential equations has becomes an attractive research field for the physics and engineering. However, to day, there are no known nonlinear differential equations having the formula (2) as exact and explicit general solution. Another celebrated differential equation like the linear harmonic oscillator is the Ermakov-Pinney equation [1]

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$$\ddot{x} + a x + \frac{b}{x^3} = 0 \tag{3}$$

where a and b are arbitrary parameters. Recently in [2] it has been shown the existence of a new periodic solution to this equation in the form

$$x(t) = \pm\sqrt{\sin\left[2\sqrt{a}\,(t+K)\right]} \tag{4}$$

where $a = b$, and K is an arbitrary parameter. The solution (4) is such that

$$x^2(t) = \sin\left(2\sqrt{a}\,(t+K)\right) \tag{5}$$

which is identical to (2) when $A_0 = 1$, $\omega = 2\sqrt{a}$ and $\alpha = 2\sqrt{a}K$. In this situation the objective in this paper is to show that the formula (5) is the solution of a singular quadratic Lienard type equation and that the formula (4) is also the solution of a singular quadratic Lienard type equation. In section (2) we show the first prediction, and the second prediction is shown in the section (3). Finally a conclusion is carried out for the work.

2. Equation having (5) as solution

Let us consider the point transformation

$$u = x^p \tag{6}$$

Using the equation (6) one may get the first derivative

$$\frac{dx}{dt} = \frac{1}{p} u^{\frac{1-p}{p}} \frac{du}{dt} \tag{7}$$

and the second derivative

$$\frac{d^2x}{dt^2} = \frac{1}{p}\left\{\ddot{u}\, u^{\frac{1-p}{p}} + \frac{1-p}{p}\dot{u}^2 u^{\frac{1-2p}{p}}\right\} \tag{8}$$

Substituting (6) and (8) into (3) yields

$$\ddot{u}\, u^{\frac{1-p}{p}} + \frac{1-p}{p}\dot{u}^2 u^{\frac{1-2p}{p}} + ap\, u^{\frac{1}{p}} + bp\, u^{\frac{-3}{p}} = 0 \tag{9}$$

from which one may secure the quadratic Lienard type equation

$$\ddot{u} + \frac{1-p}{p}\frac{\dot{u}^2}{u} + apu + bpu^{\frac{p-4}{p}} = 0 \tag{10}$$

Putting $p = 2$, reduces (10) to

$$\ddot{u} - \frac{1}{2}\frac{\dot{u}^2}{u} + 2au + \frac{2b}{u} = 0 \tag{11}$$

which admits the solution (5), that is

$$u(t) = \sin\left(2\sqrt{a}\,(t+K)\right) \tag{12}$$

where $a = b$.

References

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