# IS THE $abc$ CONJECTURE TRUE?

ABDELMAJID BEN HADJ SALEM

ABSTRACT. In this paper, we consider the *abc* conjecture. In the first part, we give the proof of the conjecture $c < rad^{1.63}(abc)$ that constitutes the key to resolve the *abc* conjecture. The proof of the *abc* conjecture is given in the second part of the paper, supposing that the *abc* conjecture is false, we arrive in a contradiction.

## 1. INTRODUCTION AND NOTATIONS

Let $a$ be a positive integer, $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of $a$ the integer $\prod_i a_i$ noted by $rad(a)$. Then $a$ is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [8]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

**Conjecture 1.1.** *(**abc Conjecture**): For each $\varepsilon > 0$, there exists $K(\varepsilon)$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then :*

$$c < K(\varepsilon).rad^{1+\varepsilon}(abc) \tag{3}$$

*where $K$ is a constant depending only of $\varepsilon$.*

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [5]. It concerned the best example given by E. Reyssat [5]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{4}$$

A conjecture was proposed that $c < rad^2(abc)$ [3]. In 2012, A. Nitaj [4] proposed the following conjecture:

**Conjecture 1.2.** *Let $a, b, c$ be positive integers relatively prime with $c = a + b$, then:*

$$c < rad^{1.63}(abc) \tag{5}$$

$$abc < rad^{4.42}(abc) \tag{6}$$

Firstly, we will give the proof of the conjecture given by (5) that constitutes the key to obtain the proof of the *abc* conjecture. Secondly, we present in section three of the paper the proof that the *abc* conjecture is true.

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## 2. A Proof of the conjecture (1.2) case $c = a + b$

Let $a,b,c$ be positive integers, relatively prime, with $c = a+b, b < a$ and $R = rad(abc)$, $c = \prod_{j'\in J'} c_{j'}^{\beta_{j'}}, \beta_{j'} \geq 1$.

In a previous paper [1], we has given, for the case $c = a+1$, the proof that $c < rad^{1.63}(ac)$. In the following, we will give the proof for the case $c = a+b$.

*Proof.* If $c < rad(abc)$, then we obtain:

$$c < rad(abc) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

and the condition (5) is satisfied.

If $c = rad(abc)$, then $a,b,c$ are not coprime, case to reject. In the following, we suppose that $c > rad(abc)$ and $a,b$ and $c$ are not prime numbers.

$$c = a+b = \mu_a rad(a) + \mu_b rad(b) \overset{?}{<} rad^{1.63}(abc) \tag{7}$$

2.1. $\mu_a \neq 1, \mu_a \leq rad^{0.63}(a)$**.** We obtain :

$$c = a+b < 2a \leq 2rad^{1.63}(a) < rad^{1.63}(abc) \Longrightarrow c < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

Then (7) is satisfied.

2.2. $\mu_c \neq 1$, $\mu_c \leq rad^{0.63}(c)$**.** We obtain :

$$c = \mu_c rad(c) \leq rad^{1.63}(c) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

and the condition (7) is satisfied.

2.3. $\mu_a > rad^{0.63}(a)$ **and** $\mu_c > rad^{0.63}(c)$**.**

2.3.1. ***Case:*** $rad^{0.63}(c) < \mu_c \leq rad^{1.63}(c)$ ***and*** $rad^{0.63}(a) < \mu_a \leq rad^{1.63}(a)$: We can write:

$$\left.\begin{array}{l} \mu_c \leq rad^{1.63}(c) \Longrightarrow c \leq rad^{2.63}(c) \\ \\ \mu_a \leq rad^{1.63}(a) \Longrightarrow a \leq rad^{2.63}(a) \end{array}\right\} \Longrightarrow ac \leq rad^{2.63}(ac) \Longrightarrow a^2 < ac \leq rad^{2.63}(ac)$$

$$\Longrightarrow a < rad^{1.315}(ac) \Longrightarrow c < 2a < 2rad^{1.315}(ac) < rad^{1.63}(abc)$$

$$\Longrightarrow \boxed{c = a+b < R^{1.63}}$$

2.3.2. ***Case:*** $\mu_c > rad^{1.63}(c)$ ***or*** $\mu_a > rad^{1.63}(a)$. I- We suppose that $\mu_c > rad^{1.63}(c)$ and $\mu_a \leq rad^2(a)$:

I-1- Case $rad(a) < rad(c)$: In this case $a = \mu_a.rad(a) \leq rad^3(a) \leq rad^{1.63}(a)rad^{1.37}(a) < rad^{1.63}(a).rad^{1.37}(c) \Longrightarrow c < 2a < 2rad^{1.63}(a).rad^{1.37}(c) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

I-2- Case $rad(c) < rad(a) < rad^{\frac{1.63}{1.37}}(c)$: As $a \leq rad^{1.63}(a).rad^{1.37}(a) < rad^{1.63}(a).rad^{1.63}(c) \Longrightarrow c < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

I-3- Case $rad^{\frac{1.63}{1.37}}(c) < rad(a)$:

I-3-1- We suppose $c \leq rad^{3.26}(c)$, we obtain:

$$c \leq rad^{3.26}(c) \Longrightarrow c \leq rad^{1.63}(c).rad^{1.63}(c) \Longrightarrow$$
$$c < rad^{1.63}(c).rad(a)^{1.37} < rad^{1.63}(c).rad(a)^{1.63}.rad^{1.63}(b) = R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$$

I-3-2- We suppose $c > rad^{3.26}(c) \Longrightarrow \mu_c > rad^{2.26}(c)$. We consider the case $\mu_a = rad^2(a) \Longrightarrow a = rad^3(a)$. Then, we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$X^3 + 1 = c - b + 1 = c' \tag{8}$$

But it is the case $c' = 1 + a$. If $c' = rad^n(c')$ with $n \geq 4$, we obtain the equation:

$$rad^n(c') - rad^3(a) = 1 \tag{9}$$

But the solutions of the equation (9) are [2] :$(rad(c') = 3, n = 2, rad(a) = +2)$, it follows the contradiction with $n \geq 4$ and the case $c' = rad^n(c'), n \geq 4$ is to reject.

In the following, we will study the cases $\mu'_c = A.rad^n(c')$ with $rad(c') \nmid A, n \geq 0$. The above equation (8) can be written as :

$$(X+1)(X^2 - X + 1) = c' \tag{10}$$

Let $\delta$ any divisor of $c'$, then:

$$X + 1 = \delta \tag{11}$$

$$X^2 - X + 1 = \frac{c'}{\delta} = c" = \delta^2 - 3X \tag{12}$$

We recall that $rad(a) > rad^{\frac{1.63}{1.37}}(c)$.

I-3-2-1- We suppose $\delta = l.rad(c')$. We have $\delta = l.rad(c') < c' = \mu'_c.rad(c') \Longrightarrow l < \mu'_c$. As $\delta$ is a divisor of $c'$, then $l$ is a divisor of $\mu'_c$, we write $\mu'_c = l.m$. From $\mu'_c = l(\delta^2 - 3X)$, we obtain:

$$m = l^2 rad^2(c') - 3rad(a) \Longrightarrow 3rad(a) = l^2 rad^2(c') - m$$

A- Case $3|m \Longrightarrow m = 3m', m' > 1$: As $\mu'_c = ml = 3m'l \Longrightarrow 3|rad(c')$ and $(rad(c'), m')$ not coprime. We obtain:

$$rad(a) = l^2 rad(c').\frac{rad(c')}{3} - m'$$

It follows that a,c' are not coprime, then the contradiction.

B - Case $m = 3 \Longrightarrow \mu'_c = 3l \Longrightarrow c' = 3lrad(c') = 3\delta = \delta(\delta^2 - 3X) \Longrightarrow \delta^2 = 3(1+X) = 3\delta \Longrightarrow \delta = lrad(c') = 3$, then the contradiction.

I-3-2-2- We suppose $\delta = l.rad^2(c'), l \geq 2$. If $lrad(c') \nmid \mu'_c$ then the case is to reject. We suppose $lrad(c')|\mu'_c \Longrightarrow \mu'_c = m.lrad(c')$, then $\dfrac{c'}{\delta} = m = \delta^2 - 3rad(a)$.

C - Case $m = 1 = c'/\delta \Longrightarrow \delta^2 - 3rad(a) = 1 \Longrightarrow (\delta - 1)(\delta + 1) = 3rad(a) = rad(a)(\delta + 1) \Longrightarrow \delta = 2 = l.rad^2(c')$, then the contradiction.

D - Case $m = 3$, we obtain $3(1 + rad(a)) = \delta^2 = 3\delta \Longrightarrow \delta = 3 = lrad^2(c')$. Then the contradiction.

E - Case $m \neq 1,3$, we obtain: $3rad(a) = l^2 rad^4(c') - m \Longrightarrow rad(a)$ and $rad(c')$ are not coprime. Then the contradiction.

I-3-2-3- We suppose $\delta = l.rad^n(c'), l \geq 2$ with $n \geq 3$. From $c' = \mu'_c.rad(c') = lrad^n(c')(\delta^2 - 3rad(a))$, we denote $m = \delta^2 - 3rad(a) = \delta^2 - 3X$.

F - As seen above (paragraphs C,D), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G - Case $m \neq 1,3$. Let $q$ be a prime that divides $m$, it follows $q|\mu'_c \Longrightarrow q = c'_{j'_0} \Longrightarrow c'_{j'_0}|\delta^2 \Longrightarrow c'_{j'_0}|3rad(a)$. Then $rad(a)$ and $rad(c')$ are not coprime. It follows the contradiction.

I-3-2-4- We suppose $\delta = \prod_{j \in J_1} c'^{\beta_j}_j$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with $\beta_{j_0} \geq 2$, $rad(c') \nmid \delta$. We can write:

$$\delta = \mu_\delta.rad(\delta), \quad rad(c') = m.rad(\delta), \quad m > 1, \quad (m, \mu_\delta) = 1 \tag{13}$$

Then, we obtain:

$$c' = \mu'_c.rad(c') = \mu'_c.m.rad(\delta) = \delta(\delta^2 - 3X) = \mu_\delta.rad(\delta)(\delta^2 - 3X) \Longrightarrow$$
$$m.\mu'_c = \mu_\delta(\delta^2 - 3X) \tag{14}$$

- If $\mu'_c = \mu_\delta \Longrightarrow m = \delta^2 - 3X = (\mu'_c.rad(\delta))^2 - 3X$. As $\delta < \delta^2 - 3X \Longrightarrow m > \delta \Longrightarrow rad(c') > m > \mu'_c.rad(\delta) > rad^3(c')$ because $\mu'_c > rad^{2.26}(c')$, it follows $rad(c') > rad^2(c')$. Then the contradiction.

- We suppose $\mu'_c < \mu_\delta$. As $rad(a) = \mu_\delta rad(\delta) - 1$, we obtain:

$$rad(a) > \mu'_c.rad(\delta) - 1 > 0 \Longrightarrow rad(ac') > c'.rad(\delta) - rad(c') > 0 \Longrightarrow$$
$$c' > rad(ac') > c'.rad(\delta) - rad(c') > 0 \Longrightarrow 1 > rad(\delta) - \frac{rad(c')}{c'} > 0, \quad rad(\delta) \geq 2$$
$$\Longrightarrow \text{The contradiction} \tag{15}$$

- We suppose $\mu_\delta < \mu'_c$. In this case, from the equation (14) and as $(m, \mu_\delta) = 1$, it follows we can write:

$$\mu'_c = \mu_1.\mu_2, \quad \mu_1, \mu_2 > 1 \tag{16}$$
$$c' = \mu'_c rad(c') = \mu_1.\mu_2.rad(\delta).m = \delta.(\delta^2 - 3X) \tag{17}$$
$$\text{so that} \quad m.\mu_1 = \delta^2 - 3X, \quad \mu_2 = \mu_\delta \Longrightarrow \delta = \mu_2.rad(\delta) \tag{18}$$

** We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists c'_{j_0}$ so that $c'_{j_0}|\mu_1$ and $c'_{j_0}|\mu_2$. But $\mu_\delta = \mu_2 \Rightarrow c'^2_{j_0}|\delta$. From $3X = \delta^2 - m\mu_1 \Longrightarrow c'_{j_0}|3X \Longrightarrow c'_{j_0}|X$ or $c'_{j_0} = 3$.

- If $c'_{j_0}|X$, it follows the contradiction with $(c', a) = 1$.

- If $c'_{j_0} = 3$. We have $m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1) \Longrightarrow \delta^2 - 3\delta + 3 - m.\mu_1 = 0$. As $3|\mu_1 \Longrightarrow \mu_1 = 3^k\mu'_1, 3 \nmid \mu'_1, k \geq 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - 3^{k-1}m\mu'_1) = 0 \tag{19}$$

- We consider the case $k > 1 \Longrightarrow 3 \nmid (1 - 3^{k-1}m\mu'_1)$. Let us recall the Eisenstein criterion [7]:

**Theorem 2.1.** ***(Eisenstein Criterion)*** *Let $f = a_0 + \cdots + a_nX^n$ be a polynomial $\in \mathbb{Z}[X]$. We suppose that $\exists p$ a prime number so that $p \nmid a_n$, $p|a_i$, $(0 \leq i \leq n-1)$, and $p^2 \nmid a_0$, then $f$ is irreducible in $\mathbb{Q}$.*

We apply Eisenstein criterion to the polynomial $R(Z)$ given by:

$$R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1}m\mu'_1) \tag{20}$$

then:
- $3 \nmid 1$, - $3|(-3)$,- $3|3(1-3^{k-1}m\mu_1')$, and - $3^2 \nmid 3(1-3^{k-1}m\mu_1')$.
It follows that the polynomial $R(Z)$ is irreducible in $\mathbb{Q}$, then, the contradiction with $R(\delta)=0$.

- We consider the case $k=1$, then $\mu_1 = 3\mu_1'$ and $(\mu_1',3)=1$, we obtain:

$$\delta^2 - 3\delta + 3(1-m\mu_1') = 0 \tag{21}$$

* If $3 \nmid (1-m.\mu_1')$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z) = Z^2 - 3Z + 3(1-m\mu_1')$$

and we find a contradiction with $R'(\delta)=0$.

* We consider that $3|(1-m.\mu_1') \Longrightarrow m\mu_1' - 1 = 3^i.h, i \geq 1, 3 \nmid h, h \in \mathbb{N}^*$. $\delta$ is an integer root of the polynomial $R'(Z)$:

$$R'(Z) = Z^2 - 3Z + 3(1-m\mu_1') = 0 \Rightarrow \text{the discriminant of } R'(Z) \text{ is :} \Delta = 3^2 + 3^{i+1} \times 4.h \tag{22}$$

As the root $\delta$ is an integer, it follows that $\Delta = l^2 > 0$ with $l$ a positive integer. We obtain:

$$\Delta = 3^2(1+3^{i-1} \times 4h) = l^2 \tag{23}$$

$$\Longrightarrow 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^* \tag{24}$$

We can write the equation (21) as :

$$\delta(\delta-3) = 3^{i+1}.h \Longrightarrow 3^3\mu_1' \frac{rad(\delta)}{3}.\left(\mu_1' rad(\delta) - 1\right) = 3^{i+1}.h \Longrightarrow \tag{25}$$

$$\mu_1' \frac{rad(\delta)}{3}.\left(\mu_1' rad(\delta) - 1\right) = h \tag{26}$$

We obtain $i=2$ and $q^2 = 1+12h = 1+4\mu_1' rad(\delta)(\mu_1' rad(\delta)-1)$. Then, $q$ satisfies :

$$q^2 - 1 = 12h \Rightarrow \tfrac{(q-1)}{2}.\tfrac{(q+1)}{2} = 3h = (\mu_1' rad(\delta)-1).\mu_1' rad(\delta) \Rightarrow \tag{27}$$

$$q - 1 = 2\mu_1' rad(\delta) - 2 \tag{28}$$

$$q + 1 = 2\mu_1' rad(\delta) \tag{29}$$

It follows that $(q=x, 1=y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{30}$$

with $N = 12h > 0$. Let $Q(N)$ be the number of the solutions of (30) and $\tau(N)$ is the number of suitable factorization of $N$, then we announce the following result concerning the solutions of the Diophantine equation (30) (see theorem 27.3 in [6]):
- If $N \equiv 2 (\bmod 4)$, then $Q(N)=0$.
- If $N \equiv 1$ or $N \equiv 3 (\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0 (\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.
$[x]$ is the integral part of $x$ for which $[x] \leq x < [x]+1$.

Let $(\alpha', m')$, $\alpha', m' \in \mathbb{N}^*$ be another pair, solution of the equation (30), then $\alpha'^2 - m'^2 = x^2 - y^2 = N = 12h$, but $q=x$ and $1=y$ satisfy the equation (29) given by $x+y = 2\mu_1' rad(\delta)$, it follows $\alpha', m'$ verify also $\alpha' + m' = 2\mu_1' rad(\delta)$, that gives $\alpha' - m' = 2(\mu_1' rad(\delta) - 1)$, then $\alpha' = x = q = 2\mu_1' rad(\delta)$ and $m' = y = 1$. So, we have given the proof of the uniqueness of the solutions of the equation (30) with the condition $x+y = 2\mu_1' rad(\delta)$. As $N = 12h \equiv 0 (\bmod 4) \Longrightarrow Q(N) = [\tau(N/4)/2] = [\tau(3h)/2]$, the expression of $3h = \mu_1'.rad(\delta).\left(\mu_1' rad(\delta) - 1\right)$, then $Q(N) = [\tau(3h)/2] > 1$. But $Q(N)=1$, then the contradiction and the case $3|(1-m.\mu_1')$ is to reject.

** We suppose that $(\mu_1,\mu_2)=1$.

From the equation $m\mu_1=\delta^2-3X=\delta^2-3(\delta-1)$, we obtain that $\delta$ is a root of the following polynomial :

$$R(Z)=Z^2-3Z+3-m.\mu_1=0 \tag{31}$$

The discriminant of $R(Z)$ is:

$$\Delta=9-4(3-m.\mu_1)=4m.\mu_1-3=q^2 \quad \text{with } q\in\mathbb{N}^* \quad \text{as } \delta\in\mathbb{N}^* \tag{32}$$

- We suppose that $2|m\mu_1 \Longrightarrow c'$ is even. Then $q^2\equiv 5(\bmod 8)$, it gives a contradiction because a square is $\equiv 0,1$ or $4(\bmod 8)$.

- We suppose $c'$ an odd integer, then $a$ is even. It follows $a=rad^3(a)\equiv 0(\bmod 8)\Longrightarrow c'\equiv 1(\bmod 8)$. As $c'=\delta^2-3X.\delta$, we obtain $\delta^2-3X.\delta\equiv 1(\bmod 8)$. If $\delta^2\equiv 1(\bmod 8)\Longrightarrow -3X.\delta\equiv 0(\bmod 8)\Longrightarrow 8|X.\delta\Longrightarrow 4|\delta\Longrightarrow c'$ is even. Then, the contradiction. If $\delta^2\equiv 4(\bmod 8)\Longrightarrow\delta\equiv 2(\bmod 8)$ or $\delta\equiv 6(\bmod 8)$. In the two cases, we obtain $2|\delta$. Then, the contradiction with $c'$ an odd integer.

It follows that the case $c>rad^{3.26}(c)$ and $a=rad^3(a)$ is impossible.

I-3-3- We suppose $c>rad^{3.26}(c)$ and large, then $c=rad^3(c)+h, h>rad^3(c)$, $h$ a positive integer and $\mu_a<rad^2(a)\Longrightarrow a+l=rad^3(a), l>0$. Then we obtain :

$$rad^3(c)+h=rad^3(a)-l+b\Longrightarrow rad^3(a)-rad^3(c)=h+l-b>0 \tag{33}$$

as $rad(a)>rad^{\frac{1.63}{1.37}}(c)$. We obtain the equation:

$$rad^3(a)-rad^3(c)=h+l-b=m>0 \tag{34}$$

Let $X=rad(a)-rad(c)$, then $X$ is an integer root of the polynomial $H(X)$ defined as:

$$H(X)=X^3+3rad(ac)X-m=0 \tag{35}$$

To resolve the above equation, we denote $X=u+v$, It follows that $u^3,v^3$ are the roots of the polynomial $G(t)$ given by:

$$G(t)=t^2-mt-rad^3(ac)=0 \tag{36}$$

The discriminant of $G(t)$ is $\Delta=m^2+4rad^3(ac)=\alpha^2,\quad \alpha>0$. The two real roots of (36) are:

$$t_1=u^3=\frac{m+\alpha}{2},\quad t_2=v^3=\frac{m-\alpha}{2} \tag{37}$$

As $m=rad^3(a)-rad^3(c)>0$, we obtain that $\alpha=rad^3(a)+rad^3(c)>0$, then from the expression of the discriminant $\Delta$, it follows that $(\alpha=x, m=y)$ is a solution of the Diophantine equation:

$$x^2-y^2=N \tag{38}$$

with $N=4rad^3(ac)>0$. From the expression of $\Delta$ above, we remark that $\alpha$ and $m$ verify the following equations:

$$x+y=2u^3=2rad^3(a) \tag{39}$$

$$x-y=-2v^3=2rad^3(c) \tag{40}$$

$$\text{then}\quad x^2-y^2=N=4rad^3(a).rad^3(c) \tag{41}$$

Let $Q(N)$ be the number of the solutions of (38) and $\tau(N)$ is the number of suitable factorization of $N$, and using the same method as in the paragraph I-3-2-4- (case $3|(1-m.\mu'_1)$), we obtain a contradiction.

It follows that the cases $\mu_a \leq rad^2(a)$ and $c > rad^{3.26}(c)$ are impossible.

II- We suppose that $rad^{1.63}(c) < \mu_c \leq rad^2(c)$ and $\mu_a > rad^{1.63}(a)$:

II-1- Case $rad(c) < rad(a)$ : As $c \leq rad^3(c) = rad^{1.63}(c).rad^{1.37}(c) \Longrightarrow c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(ac) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

II-2- Case $rad(a) < rad(c) < rad^{\frac{1.63}{1.37}}(a)$ : As $c \leq rad^3(c) \leq rad^{1.63}(c).rad^{1.37}(c) \Longrightarrow$
$c < rad^{1.63}(c).rad^{1.63}(a) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

II-3- Case $rad^{\frac{1.63}{1.37}}(a) < rad(c)$:

II-3-1- We suppose $rad^{2.63}(a) < a \leq rad^{3.26}(a) \Longrightarrow a \leq rad^{1.63}(a).rad^{1.63}(a) \Longrightarrow a < rad^{1.63}(a).rad^{1.37}(c)$ $\Longrightarrow c = a+b < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < rad^{1.63}(abc) \Longrightarrow c < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

II-3-2- We suppose $a > rad^{3.26}(a)$ and $\mu_c \leq rad^2(c)$. Using the same method as it was explicated in the paragraphs I-3-2, I-3-3 (permuting $a,c$), we arrive at a contradiction. It follows that the case $\mu_c \leq rad^2(c)$ and $a > rad^{3.26}(a)$ is impossible.

Finally, we have finished the study of the case $rad^{1.63}(c) < \mu_c \leq rad^2(c)$ and $\mu_a > rad^{1.63}(a)$.

2.3.3. *Case $\mu_c > rad^{1.63}(c)$ and $\mu_a > rad^{1.63}(a)$*. Taking into account the cases studied above, it remains to see the following two cases:
- $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$,
- $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c)$.

III-1- We suppose $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a) \Longrightarrow c > rad^3(c)$ and $a > rad^{2.63}(a)$. We can write $c = rad^3(c) + h$ and $a = rad^3(a) + l$ with $h$ a positive integer and $l \in \mathbb{Z}$.

III-1-1- We suppose $rad(c) < rad(a)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h - l - b = m > 0 \tag{42}$$

Let $X = rad(a) - rad(c)$, from the above equation, $X$ is a real root of the polynomial:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \tag{43}$$

As above, to resolve (43), we denote $X = u + v$, It follows that $u^3, v^3$ are the roots of the polynomial $G(t)$ given by :

$$G(t) = t^2 - mt - rad^3(ac) = 0 \tag{44}$$

The discriminant of $G(t)$ is:

$$\Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > 0 \tag{45}$$

The two real roots of (44) are:

$$t_1 = u^3 = \frac{m+\alpha}{2}, \quad t_2 = v^3 = \frac{m-\alpha}{2} \tag{46}$$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the equation (45), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{47}$$

with $N = 4rad^3(ac) > 0$. From the equations (46), we remark that $\alpha$ and $m$ verify the following equations:

$$x + y = 2u^3 = 2rad^3(a) \tag{48}$$

$$x - y = -2v^3 = 2rad^3(c) \tag{49}$$

$$\text{then} \quad x^2 - y^2 = N = 4rad^3(a).rad^3(c) \tag{50}$$

Let $Q(N)$ be the number of the solutions of (47) and $\tau(N)$ is the number of suitable factorization of $N$, and using the same method as in the paragraph I-3-2-4- (case $3|(1 - m.\mu_1'))$, we obtain a contradiction.

III-1-2- We suppose $rad(a) < rad(c)$. We obtain the equation:

$$rad^3(c) - rad^3(a) = b + l - h = m > 0 \tag{51}$$

Using the same calculations as in III-1-1-, we find a contradiction.

It follows that the case $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$ is impossible.

III-2- We suppose $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c) \Longrightarrow a > rad^3(a)$ and $c > rad^{2.63}(c)$. We can write $a = rad^3(a) + h$ and $c = rad^3(c) + l$ with $h$ a positive integer and $l \in \mathbb{Z}$.

The calculations are similar to those in case III-1-. We obtain the same results namely the cases of III-2- to be rejected.

It follows that the case $\mu_c > rad^{1.63}(c)$ and $\mu_a > rad^2(a)$ is impossible. □

We can state the following important theorem:

**Theorem 2.2.** *Let $a, b, c$ positive integers relatively prime with $c = a + b$, then $c < rad^{1.63}(abc)$.*

## 3. *The Proof of the abc conjecture*

We note $R = rad(abc)$ in the case $c = a + b$ or $R = rad(ac)$ in the case $c = a + 1$. We recall the following proposition [4]:

**Proposition 3.1.** *Let $\varepsilon \longrightarrow K(\varepsilon)$ the application verifying the abc conjecture, then:*

$$lim_{\varepsilon \to 0} K(\varepsilon) = +\infty \tag{52}$$

### 3.1. **Case : $\varepsilon \geq 0.63$.** 

As $c < R^{1.63}$ is true, we have $\forall \varepsilon \geq 0.63$:

$$c < R^{1.63} \leq R^{1+\varepsilon} < K(\varepsilon).R^{1+\varepsilon}, \quad with\ K(\varepsilon) = e^{\frac{1}{0.63^2}},\ \varepsilon \geq 0.63 \tag{53}$$

Then the *abc* conjecture is true.

### 3.2. **Case: $\varepsilon < 0.63$.**

3.2.1. *Case: $c<R$.* In this case, we can write :

$$c<R<R^{1+\varepsilon}<K(\varepsilon).R^{1+\varepsilon}, \quad with\ K(\varepsilon)=e^{\frac{1}{0.63^2}}>1,\ \varepsilon<0.63 \tag{54}$$

Then the *abc* conjecture is true.

3.2.2. *Case: $c>R$.* From the statement of the *abc* conjecture 1.1, we want to give a proof that $c<K(\varepsilon)R^{1+\varepsilon} \Longleftrightarrow Logc<LogK(\varepsilon)+(1+\varepsilon)LogR \Longleftrightarrow LogK(\varepsilon)+(1+\varepsilon)LogR-Logc>0$. For our proof, we proceed by contradiction of the abc conjecture, so we assume that the conjecture is false:

$$\exists \varepsilon_0 \in ]0,0.63[, \forall K(\varepsilon)>0, \quad \exists c_0=a_0+b_0 \quad \text{so that } c_0>K(\varepsilon_0)R_0^{1+\varepsilon_0} \Longrightarrow c_0 \text{ not a prime} \tag{55}$$

We choose the constant $K(\varepsilon)=e^{\frac{1}{\varepsilon^2}}$. Let $Y_{c_0}(\varepsilon)=\frac{1}{\varepsilon^2}+(1+\varepsilon)LogR_0-Logc_0, \varepsilon\in]0,0.63[$. From the above explications, if we will obtain $\forall\varepsilon\in]0,0.63[, Y_{c_0}(\varepsilon)>0 \Longrightarrow Y_{c_0}(\varepsilon_0)>0$, then the contradiction with (55).

About the function $Y_{c_0}$, we have $lim_{\varepsilon\longrightarrow 0.63}Y_{c_0}(\varepsilon)=1/0.63^2+Log(R_0^{1.63}/c_0)>0$ and $lim_{\varepsilon\longrightarrow 0}Y_{c_0}(\varepsilon)=+\infty$. The function $Y_{c_0}(\varepsilon)$ has a derivative for $\forall\varepsilon\in]0,0.63[$, we obtain with $R_0>2977$:

$$Y'_{c_0}(\varepsilon)=-\frac{2}{\varepsilon^3}+LogR_0=\frac{\varepsilon^3LogR_0-2}{\varepsilon^3}\Rightarrow Y'_{c_0}(\varepsilon)=0\Rightarrow\varepsilon=\varepsilon'=\sqrt[3]{\frac{2}{LogR_0}}\in]0,0.63[ \tag{56}$$

**Discussion:**

- If $Y_{c_0}(\varepsilon')\geq 0$, it follows that $\forall\varepsilon\in]0,0.63[, Y_{c_0}(\varepsilon)\geq 0$, then the contradiction with $Y_{c_0}(\varepsilon_0)<0 \Longrightarrow c_0>K(\varepsilon_0)R_0^{1+\varepsilon_0}$. Hence the *abc* conjecture is true for $\varepsilon\in]0,0.63[$.

- If $Y_{c_0}(\varepsilon')<0 \Longrightarrow \exists\ 0<\varepsilon_1<\varepsilon'<\varepsilon_2<0.63$, so that $Y_{c_0}(\varepsilon_1)=Y_{c_0}(\varepsilon_2)=0$. Then we obtain $c_0=K(\varepsilon_1)R_0^{1+\varepsilon_1}=K(\varepsilon_2)R_0^{1+\varepsilon_2}$. We recall the following definition:

**Definition 3.1.** *The number $\xi$ is called algebraic number if there is at least one polynomial:*

$$l(x)=l_0+l_1x+\cdots+a_mx^m, \quad a_m\neq 0 \tag{57}$$

*with integral coefficients such that $l(\xi)=0$, and it is called transcendental if no such polynomial exists.*

We consider the equality $c_0=K(\varepsilon_1)R_0^{1+\varepsilon_1} \Longrightarrow \dfrac{c_0}{R}=\dfrac{\mu_c}{rad(ab)}=e^{\frac{1}{\varepsilon_1^2}}R_0^{\varepsilon_1}$.

i) - We suppose that $\varepsilon_1=\beta_1$ is an algebraic number then $\beta_0=1/\varepsilon_1^2$ and $R_0=\alpha_1$ are also algebraic numbers. We obtain:

$$\frac{\mu_c}{rad(ab)}=e^{\frac{1}{\varepsilon_1^2}}R_0^{\varepsilon_1}=e^{\beta_0}.\alpha_1^{\beta_1} \tag{58}$$

From the theorem (see theorem 3, page 196 in [9]):

**Theorem 3.1.** *$e^{\beta_0}\alpha_1^{\beta_1}\ldots\alpha_n^{\beta_n}$ is transcendental for any nonzero algebraic numbers $\alpha_1,\ldots,\alpha_n,\beta_0,\ldots,\beta_n$.*

we deduce that the right member $e^{\beta_0}.\alpha_1^{\beta_1}$ of (58) is transcendental, but the term $\dfrac{\mu_c}{rad(ab)}$ is an algebraic number, then the contradiction and the *abc* conjecture is true.

ii) - We suppose that $\varepsilon_1$ is transcendental, in this case there is also a contradiction, and the *abc* conjecture is true.

Then the proof of the *abc* conjecture is finished.

## 4. *Conclusion*

We have given an elementary proof of the *abc* conjecture. We can announce the important theorem:

**Theorem 4.1.** *For each $\varepsilon > 0$, there exists $K(\varepsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then :*

$$c < K(\varepsilon).rad^{1+\varepsilon}(abc) \tag{59}$$

*where $K$ is a constant depending of $\varepsilon$.*

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RÉSIDENCE BOUSTEN 8, MOSQUÉE RAOUDHA, BLOC B, 1181 LA SOUKRA RAOUDHA,TUNISIA
*Email address*: `abenhadjsalem@gmail.com`