On the Irrationality of Roots of 2

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It is well known that $\sqrt{2}$ is irrational. Here we prove that $\sqrt[n]{2}$ is also irrational for integers $n > 2$ using induction on n with the base case $n = 3$. First we need to prove a few preliminary lemmas.

Lemma 1: $\sqrt[3]{2}$ is irrational.

Proof: Suppose for the sake of contradiction that $\sqrt[3]{2}$ is rational. So it can be expressed as $\frac{p}{q}$ where $p \geq q \geq 1 \in \mathbb{N}$ and $(p, q) = 1$. So,

$$2 = \frac{p^3}{q^3}$$

so

$$p^3 = q^3 + q^3. \tag{1}$$

There are no solutions to the equation $a^3 = b^3 + c^3$ for positive integers a, b, and c (confirm; the proof should not take more than the margins of this paper), so there are no solutions to (1). $\square$

Lemma 2: If r satisfies $\frac{1}{r} = \sqrt[n]{2r}$ then r is irrational.

Proof: This implies that r is a root to

$$\frac{1}{x^n} = 2x \Longleftrightarrow 2x^{n+1} - 1 = 0.$$

By the Rational Root Theorem the only possible rational roots are ± 1 and $\pm\frac{1}{2}$ but none of them are roots to this particular equation (confirm). Therefore r must be irrational. $\square$

Lemma 3: If $\sqrt[n]{2}$ is irrational, so is $\sqrt[n+1]{2}$.

Proof: Suppose for the sake of contradiction that $\sqrt[n]{2}$ is irrational but $\sqrt[n+1]{2}$ is not. So $\sqrt[n+1]{2} = \frac{p}{q}$ for some positive integers $p \geq q$ and $(p, q) = 1$. So,

$$\sqrt[n]{2} = \left(\sqrt[n+1]{2}\right)^{1+\frac{1}{n}} = \frac{p}{q} \cdot \sqrt[n]{\frac{p}{q}}$$

So

$$\frac{p}{q} = \sqrt[n]{\frac{2q}{p}}$$

is rational. But by putting $r = \frac{q}{p}$, we get from Lemma 2 that there is no rational solution to that equation! There is a contradiction, so $\sqrt[n+1]{2}$ is also irrational. $\square$

Now we can complete the proof. Lemma 1 shows that the base case, $n = 3$, holds. Lemma 3 shows that the inductive step holds. So, the claim is true. $\sqrt[n]{2} \in \mathbb{P} \quad \forall n \in \mathbb{Z}_{\geq 3}$. $\blacksquare$