Proof of Riemann hypothesis

By Toshihiko Ishiwata

June 29, 2021

Abstract. This paper is a trial to prove Riemann hypothesis according to the following process. 1. We create the infinite number of infinite series from one equation that gives $\zeta(s)$ analytic continuation to $Re(s) > 0$ and 2 formulas $(1/2 + a + bi,\ 1/2 - a - bi)$ which show zero point of $\zeta(s)$. 2. We find that a cannot have any value but zero from the above infinite number of infinite series. Therefore zero point of $\zeta(s)$ must be $1/2 \pm bi$.

1. Introduction

The following (1) gives Riemann zeta function $\zeta(s)$ analytic continuation to $Re(s) > 0$. "$+ - --$" means infinite series in all equations in this paper.

$$1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + - - - = (1 - 2^{1-s})\zeta(s) \tag{1}$$

The following (2) shows non-trivial zero point of $\zeta(s)$. S_0 is the zero point of the left side of (1) and also zero point of $\zeta(s)$.

$$S_0 = 1/2 + a + bi \tag{2}$$

The range of a is $0 \leq a < 1/2$ by the critical strip of $\zeta(s)$. The range of b is $b > 0$ due to the following reasons. And i is $\sqrt{-1}$.

1.1 There is no zero point on the real axis of the critical strip.

1.2 [Conjugate complex number of S_0] $= 1/2 + a - bi$ is also zero point of $\zeta(s)$. Therefore $b > 0$ is necessary and sufficient range for investigation.

The following (3) also shows zero point of $\zeta(s)$ by the functional equation of $\zeta(s)$.

$$S_1 = 1 - S_0 = 1/2 - a - bi \tag{3}$$

We have the following (4) and (5) by substituting S_0 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b \log 2)}{2^{1/2+a}} - \frac{\cos(b \log 3)}{3^{1/2+a}} + \frac{\cos(b \log 4)}{4^{1/2+a}} - \frac{\cos(b \log 5)}{5^{1/2+a}} + - - - \tag{4}$$

$$0 = \frac{\sin(b \log 2)}{2^{1/2+a}} - \frac{\sin(b \log 3)}{3^{1/2+a}} + \frac{\sin(b \log 4)}{4^{1/2+a}} - \frac{\sin(b \log 5)}{5^{1/2+a}} + - - - \tag{5}$$

2020 *Mathematics Subject Classification.* Primary 11M26.
Key Words and Phrases. Riemann hypothesis.

We also have the following (6) and (7) by substituting S_1 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + - - - \tag{6}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + - - - \tag{7}$$

2. Infinite number of infinite series

We define $f(n)$ as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5,\ - - -) \tag{8}$$

We have the following (9) from (4) and (6) with the method shown in item 1 of [Appendix 1: Equation construction].

$$0 = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + - - - \tag{9}$$

We also have the following (10) from (5) and (7) with the method shown in item 2 of [Appendix 1: Equation construction].

$$0 = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + - - - \tag{10}$$

We can have the following (11) (which is the function of real number x) from the above (9) and (10) with the method shown in item 3 of [Appendix 1: Equation construction]. And the value of (11) is always zero at any value of x.

$$\begin{aligned} 0 &\equiv \cos x\{\text{right side of (9)}\} + \sin x\{\text{right side of (10)}\} \\ &= \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + - - -\} \\ &\quad + \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + - - -\} \\ &= f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x) \\ &\quad - f(5)\cos(b\log 5 - x) + f(6)\cos(b\log 6 - x) - \ - - - \end{aligned} \tag{11}$$

We have (12-1) by substituting $b\log 1$ for x in (11).

$$\begin{aligned} 0 &= f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1) + f(4)\cos(b\log 4 - b\log 1) \\ &\quad - f(5)\cos(b\log 5 - b\log 1) + f(6)\cos(b\log 6 - b\log 1) - \ - - - \end{aligned} \tag{12-1}$$

We have (12-2) by substituting $b\log 2$ for x in (11).

$$\begin{aligned} 0 &= f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2) + f(4)\cos(b\log 4 - b\log 2) \\ &\quad - f(5)\cos(b\log 5 - b\log 2) + f(6)\cos(b\log 6 - b\log 2) - \ - - - \end{aligned} \tag{12-2}$$

We have (12-3) by substituting $b\log 3$ for x in (11).

$$\begin{aligned}0 &= f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3) + f(4)\cos(b\log 4 - b\log 3)\\ &\quad - f(5)\cos(b\log 5 - b\log 3) + f(6)\cos(b\log 6 - b\log 3) - \ --- \end{aligned} \tag{12-3}$$

In the same way as above we can have (12-N) by substituting $b\log N$ for x in (11). $(N = 4, 5, 6, 7, 8, ---)$

$$\begin{aligned}0 &= f(2)\cos(b\log 2 - b\log N) - f(3)\cos(b\log 3 - b\log N) + f(4)\cos(b\log 4 - b\log N)\\ &\quad - f(5)\cos(b\log 5 - b\log N) + f(6)\cos(b\log 6 - b\log N) - \ --- \end{aligned} \tag{12-N}$$

3. Proof

We define $g(k)$ as follows. $(k = 2, 3, 4, 5, ---)$

$$\begin{aligned}g(k) &= \cos(b\log k - b\log 1) + \cos(b\log k - b\log 2) + \cos(b\log k - b\log 3) + ---\\ &= \cos(b\log 1 - b\log k) + \cos(b\log 2 - b\log k) + \cos(b\log 3 - b\log k) + ---\\ &= \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cos(b\log 4/k) + --- \end{aligned} \tag{13}$$

From [Appendix 2: Investigation of $g(k)$] we can have the following (23-4).

$$g(k) = \frac{\lim_{N\to\infty} N\sin(b\log N/k + \tan^{-1} 1/b)}{\sqrt{1+b^2}} \tag{23-4}$$

We can have the following (14) from infinite equations of (12-1), (12-2), (12-3), ———, (12-N), ———— with the method shown in item 4 of [Appendix 1: Equation construction].

$$\begin{aligned}0 &= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3) + ---\}\\ &\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3) + ---\}\\ &\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3) + ---\}\\ &\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3) + ---\}\\ &\quad + f(6)\{\cos(b\log 6 - b\log 1) + \cos(b\log 6 - b\log 2) + \cos(b\log 6 - b\log 3) + ---\}\\ &\quad - \ -----------\\ &= f(2)g(2) - f(3)g(3) + f(4)g(4) - f(5)g(5) + f(6)g(6) - f(7)g(7) + ---\\ &= f(2)\frac{\lim_{N\to\infty} N\sin(b\log N/2 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - f(3)\frac{\lim_{N\to\infty} N\sin(b\log N/3 + \tan^{-1} 1/b)}{\sqrt{1+b^2}}\\ &\quad + f(4)\frac{\lim_{N\to\infty} N\sin(b\log N/4 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - f(5)\frac{\lim_{N\to\infty} N\sin(b\log N/5 + \tan^{-1} 1/b)}{\sqrt{1+b^2}}\\ &\quad + \ ---\\ &= (1/\sqrt{1+b^2})\lim_{N\to\infty} N\{f(2)\sin(b\log N/2 + \tan^{-1} 1/b) - f(3)\sin(b\log N/3 + \tan^{-1} 1/b)\\ &\quad + f(4)\sin(b\log N/4 + \tan^{-1} 1/b) - f(5)\sin(b\log N/5 + \tan^{-1} 1/b) + \ ---\} \end{aligned} \tag{14}$$

As shown in [Appendix 3: Sum of infinite series of sine waves] sum of infinite series of sine waves in the above (14) converges as follows.

$$\begin{aligned}
&f(2)\sin(b\log N/2+\tan^{-1}1/b)-f(3)\sin(b\log N/3+\tan^{-1}1/b)\\
&+f(4)\sin(b\log N/4+\tan^{-1}1/b)-f(5)\sin(b\log N/5+\tan^{-1}1/b)+\ ---\\
&=f(2)\sin(b\log N-b\log 2+\tan^{-1}1/b)-f(3)\sin(b\log N-b\log 3+\tan^{-1}1/b)\\
&+f(4)\sin(b\log N-b\log 4+\tan^{-1}1/b)-f(5)\sin(b\log N-b\log 5+\tan^{-1}1/b)+\ ---\\
&=A(a,b)\sin\{b\log N-B(a,b)+\tan^{-1}1/b\} \qquad (15)
\end{aligned}$$

In the above (15) $A(a,b)$ and $B(a,b)$ are constant which depends on a and b. If $A(a,b)=0$ is true, $a=0$ holds true and if $A(a,b)\neq 0$ is true, $0<a<1/2$ holds true as shown in [Appendix 3: Sum of infinite series of sine waves]. From (14) and (15) we have the following (16).

$$0=\lim_{N\to\infty}N[A(a,b)\sin\{b\log N-B(a,b)+\tan^{-1}(1/b)\}] \qquad (16)$$

If $A(a,b)\neq 0$ is true, the right side of (16) diverges to $\pm\infty$. Therefore $A(a,b)=0$ must be true for (16) to hold. Due to $A(a,b)=0$ a cannot have any value but zero. From (2) and (3) non-trivial zero point of Riemann zeta function $\zeta(s)$ must be $1/2\pm bi$ and other zero point does not exist. Riemann hypothesis which says "All non-trivial zero points of Riemann zeta function $\zeta(s)$ exist on the line of $Re(s)=1/2$." is true.

Appendix 1. Equation construction

We can construct (9),(10),(11) and (14) by applying the following Theorem 1[1].

Theorem 1

On condition that the following (Series 1) and (Series 2) converge, the following (Series 3) and (Series 4) are true.

$$\begin{aligned}
&(\text{Series } 1)=a_1+a_2+a_3+a_4+a_5+----=A\\
&(\text{Series } 2)=b_1+b_2+b_3+b_4+b_5+----=B\\
&(\text{Series } 3)=(a_1+b_1)+(a_2+b_2)+(a_3+b_3)+(a_4+b_4)+----=A+B\\
&(\text{Series } 4)=(a_1-b_1)+(a_2-b_2)+(a_3-b_3)+(a_4-b_4)+----=A-B
\end{aligned}$$

1. Construction of (9)

We can have the following (9) as (Series 4) by regarding (6) and (4) as (Series 1) and (Series 2) respectively.

$$(\text{Series } 1)=\frac{\cos(b\log 2)}{2^{1/2-a}}-\frac{\cos(b\log 3)}{3^{1/2-a}}+\frac{\cos(b\log 4)}{4^{1/2-a}}-\frac{\cos(b\log 5)}{5^{1/2-a}}+---=1 \qquad (6)$$

$$(\text{Series } 2)=\frac{\cos(b\log 2)}{2^{1/2+a}}-\frac{\cos(b\log 3)}{3^{1/2+a}}+\frac{\cos(b\log 4)}{4^{1/2+a}}-\frac{\cos(b\log 5)}{5^{1/2+a}}+---=1 \qquad (4)$$

$$(\text{Series } 4)=f(2)\cos(b\log 2)-f(3)\cos(b\log 3)+f(4)\cos(b\log 4)-f(5)\cos(b\log 5)$$

$$+ - - - - = 1 - 1 = 0 \tag{9}$$

Here

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5,\ - - -) \tag{8}$$

2. Construction of (10)

We can have the following (10) as (Series 4) by regarding (7) and (5) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + - - - - = 0 \tag{7}$$

$$\text{(Series 2)} = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + - - - - = 0 \tag{5}$$

$$\begin{aligned}\text{(Series 4)} &= f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5)\\ &\quad + - - - - = 0 - 0\end{aligned} \tag{10}$$

3. Construction of (11)

We can have the following (11) as (Series 3) by regarding the following equations as (Series 1) and (Series 2).

$$\begin{aligned}\text{(Series 1)} &= \cos x\{\text{right side of (9)}\}\\ &= \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5)\\ &\quad + - - -\} = 0\\ \text{(Series 2)} &= \sin x\{\text{right side of (10)}\}\\ &= \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5)\\ &\quad + - - -\} = 0\\ \text{(Series 3)} &= f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x)\\ &\quad - f(5)\cos(b\log 5 - x) + \ - - - - = 0 + 0\end{aligned} \tag{11}$$

4. Construction of (14)

4.1 We can have the following (12-1*2) as (Series 3) by regarding (12-1) and (12-2) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}\text{(Series 1)} &= f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1)\\ &\quad + f(4)\cos(b\log 4 - b\log 1) - f(5)\cos(b\log 5 - b\log 1)\\ &\quad + f(6)\cos(b\log 6 - b\log 1) - \ - - - - = 0\end{aligned} \tag{12-1}$$

$$\begin{aligned}\text{(Series 2)} &= f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2)\\ &\quad + f(4)\cos(b\log 4 - b\log 2) - f(5)\cos(b\log 5 - b\log 2)\\ &\quad + f(6)\cos(b\log 6 - b\log 2) - \ - - - - = 0\end{aligned} \tag{12-2}$$

$$\begin{aligned}\text{(Series 3)} &= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2)\}\\ &\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2)\}\\ &\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2)\}\\ &\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2)\}\end{aligned}$$

$$+ \ - - - = 0 + 0 \tag{12-1*2}$$

4.2 We can have the following (12-1*3) as (Series 3) by regarding (12-1*2) and (12-3) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}
\text{(Series 2)} = {} & f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3) \\
& + f(4)\cos(b\log 4 - b\log 3) - f(5)\cos(b\log 5 - b\log 3) \\
& + f(6)\cos(b\log 6 - b\log 3) - \ - - - = 0 \qquad (12\text{-}3) \\
\text{(Series 3)} = {} & f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3)\} \\
& - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3)\} \\
& + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3)\} \\
& - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3)\} \\
& + \ - - - = 0 + 0 \qquad (12\text{-}1{*}3)
\end{aligned}$$

4.3 We can have the following (12-1*4) as (Series 3) by regarding (12-1*3) and (12-4) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}
\text{(Series 2)} = {} & f(2)\cos(b\log 2 - b\log 4) - f(3)\cos(b\log 3 - b\log 4) \\
& + f(4)\cos(b\log 4 - b\log 4) - f(5)\cos(b\log 5 - b\log 4) \\
& + f(6)\cos(b\log 6 - b\log 4) + \ - - - = 0 \qquad (12\text{-}4) \\
\text{(Series 3)} = {} & f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \ - - - \ + \cos(b\log 2 - b\log 4)\} \\
& - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \ - - - \ + \cos(b\log 3 - b\log 4)\} \\
& + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \ - - - \ + \cos(b\log 4 - b\log 4)\} \\
& - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 6 - b\log 2) + \ - - - \ + \cos(b\log 6 - b\log 4)\} \\
& + \ - - - = 0 + 0 \qquad (12\text{-}1{*}4)
\end{aligned}$$

4.4 In the same way as above we can have the following (12-1*N) as (Series 3) by regarding (12-1*N-1) and (12-N) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}
& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \ - - - \ + \cos(b\log 2 - b\log N)\} \\
& - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \ - - - \ + \cos(b\log 3 - b\log N)\} \\
& + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \ - - - \ + \cos(b\log 4 - b\log N)\} \\
& - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \ - - - \ + \cos(b\log 6 - b\log N)\} \\
& + \ - - - = 0 + 0 \qquad (12\text{-}1{*}N)
\end{aligned}$$

4.5 If we repeat this operation infinitely i.e. we do $N \to \infty$, from (13) and (23-4) in [Appendix 2: Investigation of $g(k)$] we can have (12-1*∞)=(14) as follows.

$$\begin{aligned}
0 = \lim_{N\to\infty} & [f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \ - - - \ + \cos(b\log 2 - b\log N)\} \\
& - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \ - - - \ + \cos(b\log 3 - b\log N)\} \\
& + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \ - - - \ + \cos(b\log 4 - b\log N)\} \\
& - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \ - - - \ + \cos(b\log 6 - b\log N)\}
\end{aligned}$$

$$+ \ ---] \tag{21-1}$$
$$= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3) + \ ---\}$$
$$- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3) + \ ---\}$$
$$+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 3 - b\log 3) + \ ---\}$$
$$- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 3 - b\log 3) + \ ---\}$$
$$+ \ --- \tag{12-1*∞}$$
$$= f(2)g(2) - f(3)g(3) + f(4)g(4) - f(5)g(5) + f(6)g(6) - f(7)g(7) + ---$$
$$= f(2)\frac{\lim_{N\to\infty} N\sin(b\log N/2 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - f(3)\frac{\lim_{N\to\infty} N\sin(b\log N/3 + \tan^{-1} 1/b)}{\sqrt{1+b^2}}$$
$$+ f(4)\frac{\lim_{N\to\infty} N\sin(b\log N/4 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - f(5)\frac{\lim_{N\to\infty} N\sin(b\log N/5 + \tan^{-1} 1/b)}{\sqrt{1+b^2}}$$
$$+ \ --- \tag{21-2}$$
$$= \lim_{A\to\infty}\{f(2)\frac{\lim_{N\to A} N\sin(b\log N/2 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - f(3)\frac{\lim_{N\to A} N\sin(b\log N/3 + \tan^{-1} 1/b)}{\sqrt{1+b^2}}$$
$$+ f(4)\frac{\lim_{N\to A} N\sin(b\log N/4 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - f(5)\frac{\lim_{N\to A} N\sin(b\log N/5 + \tan^{-1} 1/b)}{\sqrt{1+b^2}}$$
$$+ \ ---\} \tag{21-3}$$
$$= (1/\sqrt{1+b^2})\lim_{A\to\infty}\{f(2)A\sin(b\log A/2 + \tan^{-1} 1/b) - f(3)A\sin(b\log A/3 + \tan^{-1} 1/b)$$
$$+ f(4)A\sin(b\log A/4 + \tan^{-1} 1/b) - f(5)A\sin(b\log A/5 + \tan^{-1} 1/b) + \ ---\}$$
$$= (1/\sqrt{1+b^2})\lim_{A\to\infty} A\{f(2)\sin(b\log A/2 + \tan^{-1} 1/b) - f(3)\sin(b\log A/3 + \tan^{-1} 1/b)$$
$$+ f(4)\sin(b\log A/4 + \tan^{-1} 1/b) - f(5)\sin(b\log A/5 + \tan^{-1} 1/b) + \ ---\}$$
$$= (1/\sqrt{1+b^2})\lim_{N\to\infty} N\{f(2)\sin(b\log N/2 + \tan^{-1} 1/b) - f(3)\sin(b\log N/3 + \tan^{-1} 1/b)$$
$$+ f(4)\sin(b\log N/4 + \tan^{-1} 1/b) - f(5)\sin(b\log N/5 + \tan^{-1} 1/b) + \ ---\} \tag{14}$$

In (21-1) all N become $N \to \infty$ simultaneously and synchronously because all N are operated by only one $\lim_{N\to\infty}$. In (21-2) all $\lim_{N\to\infty}$ work simultaneously and synchronously because (21-2) is equal to (21-1) and all N in (21-2) also must become $N \to \infty$ simultaneously and synchronously. (21-3) shows the situation where in (21-2) all $\lim_{N\to\infty}$ work simultaneously and synchronously. A is natural number. Therefore we can combine all $\lim_{N\to\infty}$ in (21-2) into one $\lim_{N\to\infty}$ and make (14).

Appendix 2. Investigation of $g(k)$

We define $g(k, N)$ as follows.

$g(k, N)$: the partial sum from the first term of $g(k)$ to the N-th term of $g(k)$.
$(k = 2, 3, 4, 5, ---)$

From (13) $g(k,N)$ is as follows.

$$
\begin{aligned}
&g(k,N) = \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \ --- \ + \cos(b\log N/k)\\
&= N\frac{1}{N}\{\cos(b\log\frac{1}{N}\frac{N}{k}) + \cos(b\log\frac{2}{N}\frac{N}{k}) + \cos(b\log\frac{3}{N}\frac{N}{k}) + \ --- \cos(b\log\frac{N}{N}\frac{N}{k})\}\\
&= N\frac{1}{N}\{\cos(b\log\frac{1}{N} + b\log\frac{N}{k}) + \cos(b\log\frac{2}{N} + b\log\frac{N}{k}) + \cos(b\log\frac{3}{N} + b\log\frac{N}{k})\\
&+ \ --- \ + \cos(b\log\frac{N}{N} + b\log\frac{N}{k})\}\\
&= N\frac{1}{N}\{\cos(b\log\frac{N}{k})\}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \ --- \ + \cos(b\log\frac{N}{N})\}\\
&- N\frac{1}{N}\{\sin(b\log\frac{N}{k})\}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \ --- \ + \sin(b\log\frac{N}{N})\}
\end{aligned}
$$

Here we do $N \to \infty$ as follows. $\lim_{N\to\infty} g(k,N)$ means $g(k)$.

$$
\begin{aligned}
&\lim_{N\to\infty} g(k,N) = g(k)\\
&= \lim_{N\to\infty}\{\cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \ --- \ + \cos(b\log N/k)\}\\
&= \lim_{N\to\infty}[N\frac{1}{N}\{\cos(b\log\frac{N}{k})\}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \ --- \ + \cos(b\log\frac{N}{N})\}\\
&- N\frac{1}{N}\{\sin(b\log\frac{N}{k})\}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \ --- \ + \sin(b\log\frac{N}{N})\}] \qquad \text{(22-1)}\\
&= \lim_{N\to\infty}\{N\cos(b\log\frac{N}{k})\}\lim_{N\to\infty}\frac{1}{N}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \ --- \ + \cos(b\log\frac{N}{N})\}\\
&- \lim_{N\to\infty}\{N\sin(b\log\frac{N}{k})\}\lim_{N\to\infty}\frac{1}{N}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \ --- \ + \sin(b\log\frac{N}{N})\}\\
&= \lim_{N\to\infty}\{N\cos(b\log\frac{N}{k})\}\int_0^1 \cos(b\log x)dx - \lim_{N\to\infty}\{N\sin(b\log\frac{N}{k})\}\int_0^1 \sin(b\log x)dx \qquad \text{(22-2)}
\end{aligned}
$$

We define A and B as follows.

$$A = \int_0^1 \cos(b\log x)dx \qquad B = \int_0^1 \sin(b\log x)dx$$

We calculate A and B.

$$
\begin{aligned}
A &= [x\cos(b\log x)]_0^1 + bB = 1 + bB\\
B &= [x\sin(b\log x)]_0^1 - bA = -bA
\end{aligned}
$$

Then we can have the values of A and B from the above equations as follows.

$$A = \frac{1}{1+b^2} \qquad B = \frac{-b}{1+b^2}$$

We have the following (23-1) by substituting the above values of A and B for

$\int_0^1 \cos(b\log x)dx$ and $\int_0^1 \sin(b\log x)dx$ in (22-2).

$$g(k) = \lim_{N\to\infty}\{N\cos(b\log\frac{N}{k})\}\frac{1}{1+b^2} - \lim_{N\to\infty}\{N\sin(b\log\frac{N}{k})\}\frac{-b}{1+b^2} \tag{23-1}$$

$$= \lim_{A\to\infty}[\lim_{N\to A}\{N\cos(b\log\frac{N}{k})\}\frac{1}{1+b^2} - \lim_{N\to A}\{N\sin(b\log\frac{N}{k})\}\frac{-b}{1+b^2}] \tag{23-2}$$

$$= \frac{\lim_{A\to\infty} A\{\cos(b\log\frac{A}{k}) + b\sin(b\log\frac{A}{k})\}}{1+b^2}$$

$$= \frac{\lim_{N\to\infty} N\{\cos(b\log\frac{N}{k}) + b\sin(b\log\frac{N}{k})\}}{1+b^2} = \frac{\lim_{N\to\infty} N\sin(b\log\frac{N}{k} + \tan^{-1}\frac{1}{b})}{\sqrt{1+b^2}} \tag{23-4}$$

In (22-1) all N become $N \to \infty$ simultaneously and synchronously because all N are operated by only one $\lim\limits_{N\to\infty}$. In (23-1) 2 $\lim\limits_{N\to\infty}$ work simultaneously and synchronously because (22-1) is equal to (23-1) and all N in (23-1) also must become $N \to \infty$ simultaneously and synchronously. (23-2) shows the situation where in (23-1) 2 $\lim\limits_{N\to\infty}$ work simultaneously and synchronously. A is natural number. Therefore we can combine 2 $\lim\limits_{N\to\infty}$ in (23-1) into one $\lim\limits_{N\to\infty}$ and make (23-4).

Appendix 3. Sum of infinite series of sine waves

Sum of infinite series of sine waves in the following (15) converges to one sine wave like the rightmost side of (15) due to the following reasons.

$$\begin{aligned}
&f(2)\sin(b\log N/2 + \tan^{-1}1/b) - f(3)\sin(b\log N/3 + \tan^{-1}1/b)\\
&+ f(4)\sin(b\log N/4 + \tan^{-1}1/b) - f(5)\sin b\log N/5 + \tan^{-1}1/b) + \ ---\\
&= f(2)\sin(b\log N - b\log 2 + \tan^{-1}1/b) - f(3)\sin(b\log N - b\log 3 + \tan^{-1}1/b)\\
&+ f(4)\sin(b\log N - b\log 4 + \tan^{-1}1/b) - f(5)\sin(b\log N - b\log 5 + \tan^{-1}1/b) + \ ---\\
&= A(a,b)\sin\{b\log N - B(a,b) + \tan^{-1}1/b\}
\end{aligned} \tag{15}$$

1 The general term of the infinite series in (15) is

$$(-1)^n f(n)\sin\{b\log N - b\log n + \tan^{-1}1/b\} \qquad (n = 2,3,4,5,---).$$

If n is large natural number, the value of $b\log n$ increases very slowly with increase of n and the sign of $\sin\{b\log N - b\log n + \tan^{-1}1/b\}$ does not change often. Therefore +term and −term appear alternately and 2 +terms or 2 −terms appear in succession only when the sign of $\sin\{b\log N - b\log n + \tan^{-1}1/b\}$ changes. +term and −term in the above explanation are defined as follows.
$(b\log N - b\log n + \tan^{-1}1/b = \alpha\ )$

+term : $+f(n)\sin\alpha$ when the sign of $\sin\alpha$ is " + ".
$-f(n)\sin\alpha$ when the sign of $\sin\alpha$ is " − ".

−term : $+f(n)\sin\alpha$ when the sign of $\sin\alpha$ is " − ".
$-f(n)\sin\alpha$ when the sign of $\sin\alpha$ is " + ".

On the condition of

$$\sin\{b\log N - b\log X_0 + \tan^{-1} 1/b\} = 0 \qquad (n_0 + 1 > X_0 > n_0)$$

the followings are true.

$$\log(n_0+1) - \log n_0 > \log X_0 - \log n_0 > 0$$
$$\log(n_0+1) - \log n_0 > \log(n_0+1) - \log X_0 > 0$$

Due to $\lim_{n_0\to\infty}\{\log(n_0+1) - \log n_0\} = 0$ the following equations hold.

$$\lim_{n_0\to\infty}\log(n_0+1) = \log X_0 = \lim_{n_0\to\infty}\log n_0$$
$$\lim_{n_0\to\infty}\sin\{b\log N - b\log(n_0+1) + \tan^{-1} 1/b\} = \sin\{b\log N - b\log X_0 + \tan^{-1} 1/b\}$$
$$= \lim_{n_0\to\infty}\sin\{b\log N - b\log n_0 + \tan^{-1} 1/b\} = 0$$

Therefore 2 +terms or 2 −terms ($\pm f(n_0)\sin\{b\log N - b\log n_0 + \tan^{-1} 1/b\}$ and $\mp f(n_0+1)\sin\{b\log N - b\log(n_0+1) + \tan^{-1} 1/b\}$) which appear in succession only when the sign of $\sin\{b\log N - b\log n + \tan^{-1} 1/b\}$ changes have almost the values of zero, if n_0 is large natural number. If we regard the sum of these 2 +terms or 2 −terms that exist in succession as one +terms or one −terms, we can consider this infinite series as alternating series and this alternating series converges due to $\lim_{n\to\infty} f(n) = 0$.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \ ---) \tag{8}$$

2 In $f(n)\sin\{b\log N - b\log n + \tan^{-1} 1/b\}$ even if N is multiplied by $e^{2\pi/b}$, the value of the equation does not change as follows. Therefore $f(n)\sin\{b\log N - b\log n + \tan^{-1} 1/b\}$ has a period of $e^{2\pi/b}$ times. $1 < e^{2\pi/b} < \infty$ is true due to $0 < b < \infty$.

$$f(n)\sin\{b\log(e^{2\pi/b}N) - b\log n + \tan^{-1} 1/b\}$$
$$= f(n)\sin\{b\log N + 2\pi - b\log n + \tan^{-1} 1/b\} = f(n)\sin\{b\log N - b\log n + \tan^{-1} 1/b\}$$

If we calculate sum or difference of 2 sine waves which have the same period, the result becomes another sine wave which has the same period as shown in (Figure 1) and (Figure 2).

Figure 1: Sum of 2 sine waves

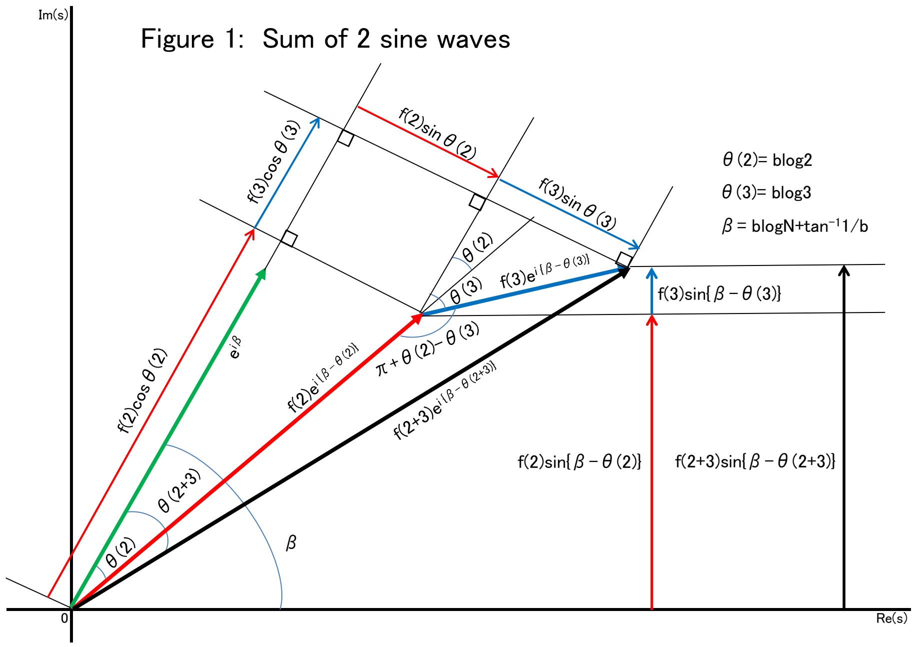
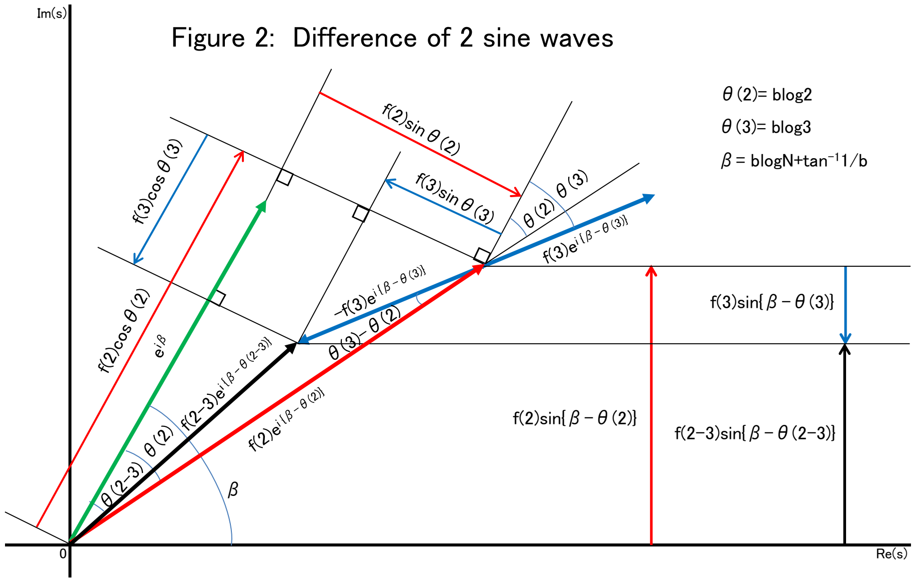


Figure 2: Difference of 2 sine waves



$$f(2)\sin\{b\log N-\theta(2)+tan^{-1}1/b\}\pm f(3)\sin\{b\log N-\theta(3)+tan^{-1}1/b\}$$
$$=f(2\pm3)\sin\{b\log N-\theta(2\pm3)+\tan^{-1}1/b\} \qquad (24)$$

If we calculate sum or difference of 2 sine waves which have the same period and obtain the new sine wave like the right side of the above (24), the amplitude $f(2\pm3)$ and the phase $\theta(2\pm3)$ of the new sine wave become as follows. Sum (difference) of 2 sine waves takes $+(-)$ sign of $\pm$ in (24) and the following equations.

$$f(2\pm 3)=\sqrt{f(2)^2+f(3)^2\pm 2f(2)f(3)\cos\{\theta(2)-\theta(3)\}}$$
$$\theta(2\pm 3)=\tan^{-1}\frac{f(2)\sin\theta(2)\pm f(3)\sin\theta(3)}{f(2)\cos\theta(2)\pm f(3)\cos\theta(3)}$$

Therefore the partial sum of the infinite series in (15) becomes one sine wave which has the same period as that of the term of the patial sum like the following (25).

$$\begin{aligned}&f(2)\sin(b\log N-b\log 2+\tan^{-1}1/b)-f(3)\sin(b\log N-b\log 3+\tan^{-1}1/b)\\&+f(4)\sin(b\log N-b\log 4+\tan^{-1}1/b)\\&+\ ---\ +(-1)^n f(n)\sin(b\log N-b\log n+\tan^{-1}1/b)\\&=f_n(a,b)\sin\{b\log N-\theta_n(a,b)+\tan^{-1}1/b\}\end{aligned}\tag{25}$$

$f_n(a,b)$ and $\theta_n(a,b)$ converges to $A(a,b)$ and $B(a,b)$ with $n\to\infty$ respectively because as confirmed in item 1 the left side of (25) converges with $n\to\infty$.

In (15) $A(a,b)$ and $B(a,b)$ are constant which depends on a and b. If $A(a,b)=0$ is true, we have the following (15-1) which is identity regarding N.

$$\begin{aligned}&f(2)\sin(b\log N-b\log 2+\tan^{-1}1/b)-f(3)\sin(b\log N-b\log 3+\tan^{-1}1/b)\\&+f(4)\sin(b\log N-b\log 4+\tan^{-1}1/b)-f(5)\sin(b\log N-b\log 5+\tan^{-1}1/b)+\ ---\\&=A(a,b)\sin\{b\log N-B(a,b)+\tan^{-1}1/b\}\equiv 0\end{aligned}\tag{15-1}$$

For the value of the leftmost side of (15-1) to be zero at any value of N the value of $f(n)$ must be zero at any value of n like the following (8-1). In other word $a=0$ must hold.

$$f(n)=\frac{1}{n^{1/2-a}}-\frac{1}{n^{1/2+a}}\equiv 0\qquad (n=2,3,4,5,\ ---)\tag{8-1}$$

Now we can say if $A(a,b)=0$ is true, $a=0$ holds true and if $A(a,b)\neq 0$ is true, $0<a<1/2$ holds true.

References

[1] Yukio Kusunoki, Introduction to infinite series, Asakura syoten, (1972), page 22, (written in Japanese)

Toshihiko ISHIWATA
E-mail: toshihiko.ishiwata@gmail.com