MY COLLECTION OF PAPERS WRITTEN TO TRY TO RESOLVE THE *abc* CONJECTURE

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Abstract. — In this book, I present my collection of 23 papers written, with different approaches to try to resolve the *abc* conjecture and others conjectures related to it like $c < rad^2(abc)$.

This monograph can give an idea about the advancement of the comprehension of the conjectures related to the problem cited above.

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Résumé **(Collection d'Articles Essayant de Résoudre la Conjecture** *abc* **(Novembre 2018 - Novembre 2020))**
Dans ce monograph, nous présentons 23 différents articles écrits pour essayer de résoudre la conjecture *abc* et d'autres en relation comme celle de $c < rad^2(abc)$.

Ce recueil peut donner une idée sur l'avancement de la compréhension des conjectures objet du problème.

CONTENTS

CHAPTER 1

FERMAT'S LAST THEOREM $\Longrightarrow$ PROOF OF THE *ABC* CONJECTURE

1.1. *Introduction and notations*

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a). \prod_i a_i^{\alpha_i - 1} \tag{1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a . rad(a) \tag{2}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph OEsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

Conjecture 1. — ***(ABC Conjecture)****: For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{3}$$

where K is a constant depending only of ϵ.

This paper about this conjecture is written after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. I try here to give a simple proof that can be understood by undergraduate students.
Our proof will use the Fermat's Last Theorem approved by Andrew John Wiles in 1993 [**3**].

We recall the Fermat's Last Theorem:

Theorem 1. — *The equation :*

$$x^n + y^n = z^n \tag{4}$$

has no solutions with x, y, z all nonzero, relatively prime integers with $n > 2$ a positive integer.

The negation of the last theorem is:
It exists A, B, C relatively prime integers and $n > 2$ a positive integer so that :

$$A^n + B^n = C^n \tag{5}$$

1.2. *Methodology of the proof*

We denote :

$$\text{A: Fermat's Last Theorem} \tag{6}$$

$$\text{B: } ABC \text{ Conjecture} \tag{7}$$

and we use the following property (the contra-positive law, [**4**]) :

$$\boxed{A(False) \Longrightarrow B(False)} \Longleftrightarrow \boxed{B(True) \Longrightarrow A(True)} \tag{8}$$

From the right equivalent expression in the box above, as A (FLT) is true, then B (ABC Conjecture) is true.

1.3. *Proof of the conjecture*

(33) We suppose that FLT is false, then it exists A, B, C positive coprime integers and m a positive integer > 2 such:

$$A^m + B^m = C^m \tag{9}$$

the integers A, B, C, m are supposed large integers. We consider in the following that $A > B$. Now, we use the ABC conjecture for equation (9). We choose the value of $\epsilon \approx 0.001$, then it exists the constant $K(\epsilon) > 0$, we want to find if :

$$C^m \overset{?}{<} K(\epsilon) rad(A^m.B^m.C^m)^{1+\epsilon}$$

$$C^m \overset{?}{<} K(\epsilon) \left(rad(A).rad(B).rad(C)\right)^{1+\epsilon} \tag{10}$$

But $rad(A) \leq A < C$, $rad(B) \leq B < C$ and $rad(C) \leq C$, then we write (10) as :

$$C^m \overset{?}{<} K(\epsilon) \left(rad(A).rad(B).rad(C)\right)^{1+\epsilon} \Longrightarrow C^m \overset{?}{<} K(\epsilon) C^{3.(1+\epsilon)} \tag{11}$$

1.3.1. *Case* $K(\epsilon) \leq 1$

In this case, we obtain:

$$C^m \overset{?}{<} C^{3.(1+\epsilon)} \tag{12}$$

As $\epsilon \ll 1 \Longrightarrow 3(1+\epsilon) \ll m$, then $C^m > K(\epsilon) rad(A^m.B^m.C^m)^{1+\epsilon}$ and the ABC conjecture is false. Using the right member of the property (8), we obtain:

$$\boxed{ABC \text{ Conjecture True} \Longrightarrow \text{ FLT True}} \tag{13}$$

But as FLT holds, hence ABC Conjecture is true.

1.3.2. *Case* $K(\epsilon) > 1$ and $C^m > K(\epsilon)$

In this case, Let $\epsilon \approx 0.001$ and we suppose that $K(\epsilon) > 1$. As FLT is supposed false, we consider that it exits a solution of (9) such that $C^m > K(\epsilon)$ with $C^m \gg_C K(\epsilon)$ that means $\exists \lambda$ a positive constant depending of C such $C^m = \lambda.K(\epsilon)$ and $\lambda \approx C^h$ with $(m-h) < \dfrac{m}{2}$. Then :

$$C^m \overset{?}{<} K(\epsilon) C^{3(1+\epsilon)} \tag{14}$$

The last equation can be written as :

$$\lambda \overset{?}{<} C^{3(1+\epsilon)} \tag{15}$$

The equation (15) indicates that we can write $\lambda \approx C^3 \Longrightarrow \dfrac{m}{2} < 3 \Longrightarrow m < 6$, then the contradiction with $6 \ll m$. Hence :

$$C^m > K(\epsilon) rad(A^m.B^m.C^m)^{1+\epsilon}$$

and the ABC conjecture is false. Using the right member of the property (8), we obtain:

$$\boxed{ABC \text{ Conjecture True} \Longrightarrow \text{ FLT True}} \tag{16}$$

But as FLT holds, hence ABC Conjecture is true.

1.3.3. *Case* $K(\epsilon) > 1$ and $C^m < K(\epsilon)$

We consider $\epsilon = 0.001$ and we suppose that $K(\epsilon) > 1$. As FLT is supposed false, we consider that it exits a unique solution of (9) such that $C^m < K(\epsilon)$:

$$C^m = A^m + B^m \tag{17}$$

We obtain that:

$$C^m < K(\epsilon) rad(A^m.B^m.C^m)^{1+\epsilon} \tag{18}$$

and the ABC conjecture is true for $C^m = A^m + B^m$, but there is a contradiction because the hypothesis of the beginning used for the proof is false, then this case is to reject.

The proof of the ABC conjecture is achieved.

1.4. *Conclusion*

In the mathematical literature, the ABC conjecture, assumed true, is used to approve the Fermat's Last Theorem, in our paper, we have given a proof that the ABC conjecture is true using the Fermat's Last Theorem. We can announce the important theorem:

Theorem 2. — *For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{19}$$

where K is a constant depending only of ϵ.

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CHAPTER 2

THE *ABC* CONJECTURE: A PROOF OF $C < rad^2(ABC)$

To the memory of my Father who taught me arithmetic.

2.1. *Introduction and notations*

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{20}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{21}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) ([**1**]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given above:

Conjecture 2. — *(* ***ABC Conjecture****): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then:*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{22}$$

where K is a constant depending only of ϵ. We know that numerically, $\dfrac{Logc}{log(rad(abc))} \leq 1.616751$ ([**2**]). Here we will give a proof that:

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{23}$$

This result, I think is the key to obtain a proof of the veracity of the *ABC* conjecture.

2.2. *A Proof of the condition (23)*

Let a, b, c positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$. If $c \leq rad(ab)$ then we obtain:

$$c \leq rad(ab) < rad^2(abc) \tag{24}$$

and the condition (23) is verified.

In the following, we suppose that $c > rad(ab)$.

2.2.1. Case $c = a + 1$

$$c = a + 1 = \mu_a rad(a) + 1 \overset{?}{<} rad^2(ac) \tag{25}$$

2.2.1.1. $\mu_a = 1$

In this case, $a = rad(a)$, it is immediately truth that :

$$c = a + 1 < 2a < rad(a)rad(c) < rad^2(ac) \tag{26}$$

Then (25) is verified.

2.2.1.2. $\mu_a \neq 1$, $\mu_a < rad(a)$

we obtain :

$$c = a + 1 < 2\mu_a.rad(a) \Rightarrow c < 2rad^2(a) \Rightarrow c < rad^2(ac) \tag{27}$$

Then (25) is verified.

2.2.1.3. $\mu_a \geq rad(a)$

We have $c = a + 1 = \mu_a.rad(a) + 1 \leq \mu_a^2 + 1 \overset{?}{<} rad^2(ac)$. We suppose that $\mu_a^2 + 1 \geq rad^2(ac) \Longrightarrow \mu_a^2 > rad^2(a).rad(c) \geq 3rad^2(a) \Longrightarrow \mu_a > \sqrt{3}rad(a) \geq 2rad(a)$ then $\mu_a > 2rad(a)$, that is the contradiction with $\mu_a \geq rad(a)$. We deduce that $c \leq \mu_a^2 + 1 < rad^2(ac) \Longrightarrow c < rad^2(ac)$ and the condition (25) is verified.

2.2.2. $c = a + b$

We can write that c verifies:

$$c = a + b = rad(a).\mu_a + rad(b).\mu_b = rad(a).rad(b)\left(\frac{\mu_a}{rad(b)} + \frac{\mu_b}{rad(a)}\right) \Longrightarrow$$

$$c = rad(a).rad(b).rad(c)\left(\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)k}\right) \tag{28}$$

We can write also:

$$c = rad(abc)\left(\frac{\mu_a}{rad(b).rad(c)k} + \frac{\mu_b}{rad(a).rad(c)}\right) \tag{29}$$

To obtain a proof of (25), one method is to prove that :

$$\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)} < rad(abc) \tag{30}$$

2.2.2.1. $\mu_a = \mu_b = 1$

In this case, it is immediately truth that :

$$\frac{1}{rad(a_i} + \frac{1}{rad(b_j} \leq \frac{5}{6} < rad(c).rad(abc) \tag{31}$$

Then (25) is verified.

2.2.2.2. $\mu_a = 1$ **and** $\mu_b > 1$

As $b < a \Longrightarrow \mu_b rad(b) < rad(a) \Longrightarrow \dfrac{\mu_b}{rad(a)} < \dfrac{1}{rad(b)}$, then we deduce that:

$$\frac{1}{rad(b)} + \frac{\mu_b}{rad(a)} < \frac{2}{rad(b)} < rad(c).rad(abc) \tag{32}$$

Then (25) is verified.

2.2.2.3. $\mu_b = 1$ **and** $\mu_a \leq (b = rad(b))$

In this case we obtain:

$$\frac{1}{rad(a)} + \frac{\mu_a}{rad(b)} \leq \frac{1}{rad(a)} + 1 < rad(c).rad(abc) \tag{33}$$

Then (25) is verified.

2.2.2.4. $\mu_b = 1$ **and** $\mu_a > (b = rad(b))$

As $\mu_a > rad(b)$, we can write $\mu_a = rad(b) + n$ where $n \geq 1$. We obtain:

(34)

$$c = \mu_a rad(a) + rad(b) = (rad(b) + n)rad(a) + rad(b) = rad(ab) + nrad(a) + rad(b)$$

We verify that $n < b$, then:

$$\begin{aligned} c < 2rad(ab) + rad(b) \Longrightarrow c < rad(abc) + rad(abc) < rad^2(abc) \\ \Longrightarrow c < rad^2(abc) \end{aligned} \tag{35}$$

2.2.2.5. $\mu_a.\mu_b \neq 1,\ \mu_a < rad(a)$ **and** $\mu_b < rad(b)$

we obtain :

$$c = \mu_c rad(c) = \mu_a.rad(a) + \mu_b.rad(b) < rad^2(a) + rad^2(b) < rad^2(abc) \tag{36}$$

2.2.2.6. $\mu_a.\mu_b \neq 1$ **,** $\mu_a \leq rad(a)$ **and** $\mu_b \geq rad(b)$

We have:

$$c = \mu_a.rad(a) + \mu_b.rad(b) < \mu_a\mu_b rad(a)rad(b) \leq \mu_b rad^2(a)rad(b) \tag{37}$$

Then if we give a proof that $\mu_b < rad(b)rad^2(c)$, we obtain $c < rad^2(abc)$. As $\mu_b \geq rad(b) \Longrightarrow \mu_b = rad(b) + \alpha$ with α a positive integer ≥ 0. Supposing that $\mu_b \geq rad(b)rad^2(c) \Longrightarrow \mu_b = rad(b)rad^2(c) + \beta$ with $\beta \geq 0$ a positive integer. We can write:

$$rad(b)rad^2(c) + \beta = rad(b) + \alpha \Longrightarrow \beta < \alpha$$

$$\alpha - \beta = rad(b)(rad^2(c) - 1) > 3rad(b) \Longrightarrow \mu_b = rad(b) + \alpha > 4rad(b) \tag{38}$$

Finally, we obtain:

$$\begin{cases} \mu_b \geq rad(b) \\ \mu_b > 4rad(b) \end{cases} \tag{39}$$

Then the contradiction and the hypothesis $\mu_b \geq rad(b)rad^2(c)$ is false. Hence:

$$\mu_b < rad(b)rad^2(c) \Longrightarrow c < rad^2(abc) \tag{40}$$

2.2.2.7. $\mu_a.\mu_b \neq 1$ **,** $\mu_a \geq rad(a)$ **and** $\mu_b \leq rad(b)$

The proof is identical to the case above.

2.2.2.8. $\mu_a.\mu_b \neq 1$ **,** $\mu_a \geq rad(a)$ **and** $\mu_b \geq rad(b)$

We write:

$$c = \mu_a rad(a) + \mu_b rad(b) \leq \mu_a^2 + \mu_b^2 < \mu_a^2.\mu_b^2 \overset{?}{<} rad^2(a).rad^2(b).rad^2(c) = rad^2(abc) \tag{41}$$

As $\mu_a \geq rad(a)$ and $\mu_b \geq rad(b)$, we can write that :

$$\mu_a = rad(a) + m$$
$$\mu_b = rad(b) + n$$

with $m, n \geq 0$ two positive integers. Let $F(x, y)$ the function :

$$F(x,y) = (x+rad(a))(y+rad(b)) - rad(abc),\ (x,y) \in I =]-rad(a),+\infty[\times]-rad(b),+\infty[\tag{42}$$

The set of points $M(x, y) \in I$ verifying $F(x, y) = 0$ is the hyperbola $\mathcal{C}$ given by :

$$y = \frac{-rad(b).x + rad(abc) - rad(ab)}{x + rad(a)} \tag{43}$$

The curve $\mathcal{C}$ intersects the axis $x = 0$ and $y = 0$ at the two points $M_1(0, y_1 = rad(b)(rad(c) - 1))$ and $M_2(x_2 = rad(a)(rad(c) - 1), 0)$. The region below the curve

$\mathcal{C}$ verifies $F(x,y) < 0$. $F(m,n) = \mu_a.\mu_b - rad(abc) < 0$ if we have $m < x_2 \Rightarrow m < rad(a)(rad(c)-1)$ and $n < y_1 \Rightarrow n < rad(b)(rad(c)-1)$. We suppose now that:

$$m \geq rad(a)(rad(c)-1) \Longrightarrow m > rad(a) \Longrightarrow \mu_a > 2rad(a) \Longrightarrow a > 2rad^2(a)$$
$$n \geq rad(b)(rad(c)-1) \Longrightarrow n > rad(b) \Longrightarrow \mu_b > 2rad(b) \Longrightarrow b > 2rad^2(b)$$
$$\textit{then} \quad c > 2(rad^2(a) + rad^2(b)) > 4rad(ab) \Longrightarrow c > 4rad(ab) \tag{44}$$

The last inequality $c > 4rad(ab)$ gives the contradiction with the condition $c > rad(ab)$ supposed above. Then we obtain $F(m,n) < 0 \Longrightarrow \mu_a.\mu_b - rad(abc) < 0 \Longrightarrow c < rad^2(abc)$.

We announce the theorem:

Theorem 3. — *Let a,b,c positive integers relatively prime with $c = a+b$ and $b < a$, then $c < rad^2(abc)$.*

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CHAPTER 3

THE *ABC* CONJECTURE: THE END OF THE MYSTERY

3.1. *Introduction and notations*

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{45}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{46}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

Conjecture 3. — *(* ***ABC Conjecture****): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then:*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{47}$$

where K is a constant K depending only of ϵ. This paper about this conjecture is written after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. I try here to give a simple proof that can be understood by undergraduate students.

3.2. *Proof of the conjecture (3)*

Let a, b, c positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$, we can write that a verifies:

$$c = a + b \Rightarrow c(a - b) = a^2 - b^2 < e^{4a^2} \Longrightarrow c < \frac{e^{4a^2}}{a - b} \tag{48}$$

We can write also:

$$c < \frac{e^{4a^2}}{a-b}.\frac{K(\epsilon).rad(abc)^{1+\epsilon}}{K(\epsilon).rad(abc)^{1+\epsilon}} \tag{49}$$

We propose the constant $K(\epsilon)$ depending of ϵ as :

$$K(\epsilon) = \frac{2}{\epsilon^2} \tag{50}$$

it is a decreasing function so that $lim_{\epsilon\longrightarrow 0}K(\epsilon) = +\infty$ and $lim_{\epsilon\longrightarrow +\infty}K(\epsilon) = 0$. We write (49) as:

$$c < \frac{e^{4a^2}\epsilon^2}{a-b}.\frac{1}{2rad(abc)^{1+\epsilon}}.K(\epsilon).rad(abc)^{1+\epsilon} \tag{51}$$

It is known that $2 \leq rad(q)$ for $\forall\, q$ a positive integer ≥ 2, then $2^3 \leq rad(abc) \Longrightarrow \frac{1}{rad(abc)} \leq \frac{1}{2^3}$. As $1+\epsilon < 1 + \frac{4a^2\epsilon}{3Log2}$, we obtain:

$$c < \frac{e^{4a^2}\epsilon^2}{a-b}.\frac{1}{2^{4+\frac{4a^2\epsilon}{Log2}}}.K(\epsilon).rad(abc)^{1+\epsilon} \tag{52}$$

Let:

$$G(\epsilon,a,b) = \frac{e^{4a^2}\epsilon^2}{a-b}.\frac{1}{2^{4+\frac{4a^2\epsilon}{Log2}}} = \frac{e^{4a^2}\epsilon^2}{a-b}.\frac{1}{16e^{4a^2\epsilon}} \tag{53}$$

Then, equation (52) is written as:

$$c < G(\epsilon,a,b).K(\epsilon).rad(abc)^{1+\epsilon} \tag{54}$$

If we can give a proof that $G(\epsilon,a,b) < 1$ independently of a,b,ϵ, we will obtain:

$$c < G(\epsilon,a,b).K(\epsilon).rad(abc)^{1+\epsilon} < K(\epsilon).rad(abc)^{1+\epsilon} \tag{55}$$

then the *ABC* conjecture holds with proposing the expression of the constant $K(\epsilon) = \frac{2}{\epsilon^2}$.

3.2.1. The Proof

We write:

$$G(\epsilon,a,b) = \frac{e^{4a^2}\epsilon^2}{a-b}.\frac{1}{16e^{4a^2\epsilon}} \overbrace{<}^{?} 1 \Rightarrow (a-b)16e^{4a^2\epsilon} - e^{4a^2}\epsilon^2 \overbrace{>}^{?} 0$$

As $a > b$, the minimum value of $a-b$ is equal to 1, then we must verify if :

$$16e^{4a^2\epsilon} - e^{4a^2}\epsilon^2 \overbrace{>}^{?} 0 \tag{56}$$

We call:

$$\phi(\epsilon) = 16e^{4a^2\epsilon} - e^{4a^2}\epsilon^2 \Rightarrow \phi'(\epsilon) = 2(32a^2e^{4a^2\epsilon} - e^{4a^2}\epsilon) \tag{57}$$

$$\phi"(\epsilon) = 2(128a^4e^{4a^2\epsilon} - e^{4a^2}) > 0 \quad \forall\epsilon > 0 \quad and\ a \geq 2 \tag{58}$$

Let us study the function $\phi"(\epsilon)$:

$$\phi"(\epsilon_1) = 0 \Longrightarrow \epsilon_1 = \frac{1}{4a^2}Log\left(\frac{e^{4a^2}}{128a^4}\right) \Longrightarrow$$

$$\epsilon_1 = 1 - \frac{7}{4a^2}Log2 - \frac{Loga}{a^2} < 1; \quad max\,\epsilon_1(a) = \epsilon_1(2) = 0.6534$$

$$\phi'(\epsilon_1) = \frac{e^{4a^2}}{2a^2}\left(1 - Log\left(\frac{e^{4a^2}}{128a^4}\right)\right) \tag{59}$$

If we write the table of variations of the function ϕ when $\epsilon \in [0, +\infty[$, we obtain successively $\phi"(\epsilon) > 0, \phi'(\epsilon) > 0$ and $\phi(\epsilon) > 0$ for $\forall a \geq 2$, we deduce that $\forall\epsilon > 0, a \geq 2$:

$$16e^{3a^2\epsilon} - e^{4a^2}\epsilon^2 > 0 \Longrightarrow (a-b)16e^{3a^2\epsilon} - e^{4a^2}\epsilon > 0 \Longrightarrow$$

$$(a-b)16e^{3a^2\epsilon} - e^{4a^2}\epsilon \Longrightarrow 1 > \frac{e^{4a^2}\epsilon^2}{(a-b)2^{4+3a^2\epsilon}} \Longrightarrow G(\epsilon, a, b) < 1 \tag{60}$$

Then we obtain the important result of the paper:

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad \forall\epsilon > 0$$

$$\text{with the constant } K(\epsilon) = \frac{2}{\epsilon^2} \tag{61}$$

Q.E.D

3.3. *Examples*

In this section, we are going to verify some numerical examples.

3.3.1. Example of Eric Reyssat

We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{62}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$. For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = \frac{2}{\epsilon^2} = \frac{2}{10^{-4}} \tag{63}$$

Let us verify (55):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \overset{?}{<} 2.10^4.(3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow$$
$$6436343 < 186142827.83 \tag{64}$$

Hence (55) is verified.

3.3.2. Example of A. Nitaj

3.3.2.1. Case 1

The example of Nitaj about the ABC conjecture [**3**] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \tag{65}$$

we take $\epsilon = 100$ we have:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 2.10^{-4}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 5.5310368633286126480363 8e+1355$$

then (55) is verified.

3.3.2.2. Case 2

We take $\epsilon = 0.000001 = 10^{-6}$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 2.10^{12}.(2.3.5.7.11.13.41.79.311.953)^{1.000001} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 57\,658\,458\,237\,370\,924\,700\,998\,757.17498$$

We obtain that (55) is verified.

3.3.2.3. Case 3

We take $\epsilon = 1$, then

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 2.(2.3.5.7.11.13.41.79.311.953)^2 \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 1\,662\,145\,872\,249\,552\,942\,316\,264\,200$$

Ouf!

3.4. *Conclusion*

This is an elementary proof of the ABC conjecture, confirmed by four numerical examples. We can announce the important theorem:

Theorem 4. — *For each* $\epsilon > 0$, *there exists* $K(\epsilon) > 0$ *such that if* a, b, c *positive integers relatively prime with* $c = a + b$*, then ::*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{66}$$

where K *is a constant depending of* ϵ *equal to* $\dfrac{2}{\epsilon^2}$.

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CHAPTER 4

A COMPLETE PROOF OF THE *ABC* CONJECTURE

4.1. *Introduction and notations*

Let a a positive integer, $a=\prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a=\prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i-1} \tag{67}$$

We note:

$$\mu_a=\prod_i a_i^{\alpha_i-1} \Longrightarrow a=\mu_a.rad(a) \tag{68}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

Conjecture 4. — *(**ABC Conjecture**): For each $\epsilon>0$, there exists $K(\epsilon)>0$ such that if a,b,c positive integers relatively prime with $c=a+b$, then : :*

$$c<K(\epsilon).rad(abc)^{1+\epsilon} \tag{69}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.616751$ [**2**]. A conjecture was proposed that $c<rad^2(abc)$ [**3**]. Here we will give a proof of it.

Conjecture 5. — *Let a,b,c positive integers relatively prime with $c=a+b$, then:*

$$c<rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))}<2 \tag{70}$$

This result, I think is the key to obtain a proof of the veracity of the *ABC* conjecture.

4.2. *A Proof of the conjecture (5)*

Let a, b, c positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$.
If $c < rad(ab)$ then we obtain:

$$c < rad(ab) < rad^2(abc) \tag{71}$$

and the condition (70) is verified.

In the following, we suppose that $c \geq rad(ab)$.

4.2.1. Case $c = a + 1$

$$c = a + 1 = \mu_a rad(a) + 1 \stackrel{?}{<} rad^2(ac) \tag{72}$$

4.2.1.1. $\mu_a = 1$

In this case, $a = rad(a)$, it is immediately truth that :

$$c = a + 1 < 2a < rad(a)rad(c) < rad^2(ac) \tag{73}$$

Then (72) is verified.

4.2.1.2. $\mu_a \neq 1$, $\mu_a < rad(a)$

we obtain :

$$c = a + 1 < 2\mu_a.rad(a) \Rightarrow c < 2rad^2(a) \Rightarrow c < rad^2(ac) \tag{74}$$

Then (72) is verified.

4.2.1.3. $\mu_a \geq rad(a)$

We have $c = a + 1 = \mu_a.rad(a) + 1 \leq \mu_a^2 + 1 \stackrel{?}{<} rad^2(ac)$. We suppose that $\mu_a^2 + 1 \geq rad^2(ac) \Longrightarrow \mu_a^2 > rad^2(a).rad(c) > rad^2(a)$ as $rad(c) > 1$, then $\mu_a > rad(a)$, that is the contradiction with $\mu_a \geq rad(a)$. We deduce that $c < \mu_a^2 + 1 < rad^2(ac)$ and the condition (72) is verified.

4.2.2. $c = a + b$

We can write that c verifies:

$$c = a + b = rad(a).\mu_a + rad(b).\mu_b = rad(a).rad(b)\left(\frac{\mu_a}{rad(b)} + \frac{\mu_b}{rad(a)}\right)$$
$$\Longrightarrow c = rad(a).rad(b).rad(c)\left(\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)}\right) \tag{75}$$

We can write also:

$$c = rad(abc)\left(\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)}\right) \tag{76}$$

To obtain a proof of (70), one method is to prove that :

$$\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)} < rad(abc) \tag{77}$$

4.2.2.1. $\mu_a = \mu_b = 1$

In this case, it is immediately truth that :

$$\frac{1}{rad(a)} + \frac{1}{rad(b)} \leq \frac{5}{6} < rad(c).rad(abc) \tag{78}$$

Then (70) is verified.

4.2.2.2. $\mu_a = 1$ **and** $\mu_b > 1$

As $b < a \Longrightarrow \mu_b rad(b) < rad(a) \Longrightarrow \dfrac{\mu_b}{rad(a)} < \dfrac{1}{rad(b)}$, then we deduce that:

$$\frac{1}{rad(b)} + \frac{\mu_b}{rad(a)} < \frac{2}{rad(b)} < rad(c).rad(abc) \tag{79}$$

Then (70) is verified.

4.2.2.3. $\mu_b = 1$ **and** $\mu_a \leq (b = rad(b))$

In this case we obtain:

$$\frac{1}{rad(a)} + \frac{\mu_a}{rad(b)} \leq \frac{1}{rad(a)} + 1 < rad(c).rad(abc) \tag{80}$$

Then (70) is verified.

4.2.2.4. $\mu_b = 1$ **and** $\mu_a > (b = rad(b))$

As $\mu_a > rad(b)$, we can write $\mu_a = rad(b) + n$ where $n \geq 1$. We obtain:

(81)

$$c = \mu_a rad(a) + rad(b) = (rad(b) + n)rad(a) + rad(b) = rad(ab) + nrad(a) + rad(b)$$

We have $n < b$, if not $n \geq b \Longrightarrow \mu_a \geq 2b \Longrightarrow a \geq 2brad(a) \Longrightarrow a \geq 3b \Longrightarrow c > 3b$, then the contradiction with $c > 2b$. We can write:

(82)

$$c < 2rad(ab) + rad(b) \Longrightarrow c < rad(abc) + rad(abc) < rad^2(abc) \Longrightarrow c < rad^2(abc)$$

4.2.2.5. $\mu_a.\mu_b \neq 1$ **,** $\mu_a < rad(a)$ **and** $\mu_b < rad(b)$

we obtain :

$$c = \mu_c rad(c) = \mu_a.rad(a) + \mu_b.rad(b) < rad^2(a) + rad^2(b) < rad^2(abc) \tag{83}$$

4.2.2.6. $\mu_a.\mu_b \neq 1$, $\mu_a \leq rad(a)$ and $\mu_b \geq rad(b)$

We have:

$$c = \mu_a.rad(a) + \mu_b.rad(b) < \mu_a\mu_b rad(a)rad(b) \leq \mu_b rad^2(a)rad(b) \tag{84}$$

Then if we give a proof that $\mu_b < rad(b)rad^2(c)$, we obtain $c < rad^2(abc)$. As $\mu_b \geq rad(b) \Longrightarrow \mu_b = rad(b) + \alpha$ with α a positive integer ≥ 0. Supposing that $\mu_b \geq rad(b)rad^2(c) \Longrightarrow \mu_b = rad(b)rad^2(c) + \beta$ with $\beta \geq 0$ a positive integer. We can write:

$$rad(b)rad^2(c) + \beta = rad(b) + \alpha \Longrightarrow \beta < \alpha$$

$$\text{(85) } \alpha - \beta = rad(b)(rad^2(c) - 1) > 3rad(b) \Longrightarrow \mu_b = rad(b) + \alpha > 4rad(b)$$

Finally, we obtain:

$$\begin{cases} \mu_b \geq rad(b) \\ \mu_b > 4rad(b) \end{cases} \tag{86}$$

Then the contradiction and the hypothesis $\mu_b \geq rad(b)rad^2(c)$ is false. Hence:

$$\mu_b < rad(b)rad^2(c) \Longrightarrow c < rad^2(abc) \tag{87}$$

4.2.2.7. $\mu_a.\mu_b \neq 1$, $\mu_a \geq rad(a)$ and $\mu_b \leq rad(b)$

The proof is identical to the case above.

4.2.2.8. $\mu_a.\mu_b \neq 1$, $\mu_a \geq rad(a)$ and $\mu_b \geq rad(b)$

We write:
(88)

$$c = \mu_a rad(a) + \mu_b rad(b) \leq \mu_a^2 + \mu_b^2 < \mu_a^2.\mu_b^2 \overset{?}{<} rad^2(a).rad^2(b).rad^2(c) = rad^2(abc)$$

Supposing that $\mu_a.\mu_b \geq rad(abc)$, we obtain:

$$\begin{aligned} \mu_a.\mu_b \geq rad(abc) \Rightarrow rad(a).rad(b).\mu_a.\mu_b \geq rad^2(ab)rad(c) \Longrightarrow \\ ab \geq rad^2(ab).rad(c) \Rightarrow a^2 > ab \geq rad^2(ab).rad(c) \\ \Rightarrow a > rad(ab)\sqrt{rad(c)} \geq rad(ab)\sqrt{7} \Rightarrow \end{aligned}$$

$$\begin{cases} c > \sqrt{7}rad(ab) \geq 3rad(ab) \\ \\ c \geq rad(ab) \end{cases} \tag{89}$$

The inequality $c \geq 3rad(ab)$ gives the contradiction with the condition $c \geq rad(ab)$ supposed at the beginning of this section. Then we obtain $\mu_a.\mu_b - rad(abc) < 0 \Longrightarrow c < rad^2(abc)$.

We announce the theorem:

Theorem 5. — *Let a, b, c positive integers relatively prime with $c = a + b$ and $1 \leq b < a$, then $c < rad^2(abc)$.*

4.3. *The Proof of the* abc *conjecture (4)*

We denote $R = rad(abc)$.

4.3.1. Case: $\epsilon \geq 1$

Using the result of the theorem above, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon} e^{\left(\frac{1}{\epsilon^2} - \epsilon\right)}, \ \epsilon \geq 1 \tag{90}$$

4.3.2. Case: $\epsilon < 1$

4.3.2.1. Case: $c \leq R$

In this case, we can write :

$$c \leq R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon} e^{\left(\frac{1}{\epsilon^2} - \epsilon\right)}, \ \epsilon < 1 \tag{91}$$

and the ABC conjecture is true.

4.3.2.2. Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon} e^{\left(\frac{1}{\epsilon^2} - \epsilon\right)}, 0 < \epsilon < 1 \tag{92}$$

If not, then $\exists \epsilon_0 \in]0, 1[$, so that the triplets (a, b, c) checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{93}$$

are in finite number. We have:

$$\begin{aligned} c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow \\ R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > c.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > K(\epsilon_0) \end{aligned} \tag{94}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow c > K(\epsilon_0)^{\left(\frac{1}{1-\epsilon_0}\right)} \tag{95}$$

We deduce that it exists an infinity of triples (a, b, c) verifying (93), hence the contradiction. Then the proof of the ABC conjecture is finished. We obtain that $\forall \epsilon > 0$,

$c = a + b$ with a, b, c relatively coprime:

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad with \quad K(\epsilon) = 6^{1+\epsilon} e^{\left(\frac{1}{\epsilon^2} - \epsilon\right)} \tag{96}$$

Q.E.D

4.4. *Examples*

In this section, we are going to verify some numerical examples.

4.4.1. Example of Eric Reyssat

We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{97}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$. For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = 6^{1.01} e^{9999.99} = 1.8884880155640644914779227374022e + 4343 \tag{98}$$

Let us verify (96):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \overset{?}{<} K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow$$
$$6436343 \ll K(0.01) \times 15042 \tag{99}$$

Hence (96) is verified.

4.4.2. Example of A. Nitaj

4.4.2.1. Case 1

The example of Nitaj about the ABC conjecture [**1**] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \tag{100}$$

$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \tag{101}$$

$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \tag{102}$$

$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \tag{103}$$

we take $\epsilon = 100$ we have:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 6^{101} e^{-99.9999}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 8.2558649305610435609546415285004e + 48$$

then (96) is verified.

4.4.2.2. Case 2

We take $\epsilon = 0.5$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \tag{104}$$

$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 6^{1.5}.e^{3.5}.(2.3.5.7.11.13.41.79.311.953)^{1.5} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 < 75\,333\,109\,597\,556\,257\,182\,261.66 \tag{105}$$

We obtain that (96) is verified.

4.4.2.3. Case 3

We take $\epsilon = 1$, then

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 6^{2}.(2.3.5.7.11.13.41.79.311.953)^{2} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 < 29\,918\,625\,700\,491\,952\,961\,692\,755\,600 \tag{106}$$

We obtain that (96) is verified.

4.4.3. Example of Ralf Bonse

The example of Ralf Bonse about the ABC conjecture [**2**] is:

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \tag{107}$$

$$a = 2543^4.182587.2802983.85813163$$

$$b = 2^{15}.3^{77}.11.173$$

$$c = 5^{56}.245983$$

$$rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$

$$rad(abc) = 1,5683959920004546031461002610848e+33 \tag{108}$$

For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = 6^{1.01}.e^{9999.99} = 5.2903884296336672264108948608106e+4343$$

Let us verify (96):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<}$$

$$6^{1.01}.e^{9999.99}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.01}$$

$$\Longrightarrow 3.4136998783296235160378273576498e+44 <$$

$$1.7819595478010681971905561514574e+4377 \tag{109}$$

The equation (96) is verified.

Ouf, end of the mystery!

4.5. *Conclusion*

This is an elementary proof of the ABC conjecture, confirmed by four numerical examples. We can announce the important theorem:

Theorem 6. — *For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{110}$$

where K is a constant depending of ϵ equal to $6^{1+\epsilon}.e^{\left(\dfrac{1}{\epsilon^2} - \epsilon\right)}$.

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CHAPTER 5

THE END OF THE MYSTERY OF THE *ABC* CONJECTURE

5.1. *Introduction and notations*

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a). \prod_i a_i^{\alpha_i - 1} \tag{111}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a . rad(a) \tag{112}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

Conjecture 6. — *(**ABC Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then : :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{113}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.616751$ [**2**]. A conjecture was proposed that $c < rad^2(abc)$ [**3**]. Here we will give the proof of it.

Conjecture 7. — *Let a, b, c positive integers relatively prime with $c = a + b$, then:*

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{114}$$

This result, I think is the key to obtain a proof of the veracity of the *ABC* conjecture.

5.2. *A Proof of the conjecture (7)*

Let a, b, c positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$.
If $c < rad(ab)$ then we obtain:

$$c < rad(ab) < rad^2(abc) \tag{115}$$

and the condition (114) is verified.

If $c = rad(ab)$, then a, b, c are not relatively coprime. In the following, we suppose that $c > rad(ab)$. We can write :

$$c \overset{?}{<} rad^2(abc) \Longrightarrow \mu_c.rad(c) \overset{?}{<} rad^2(abc) \Longrightarrow$$
$$\mu_c \overset{?}{<} rad^2(ab)rad(c) \Longrightarrow \frac{\mu_c}{rad(c)} \overset{?}{<} rad^2(ab) < c^2 \Longrightarrow$$
$$\frac{\mu_c}{rad(c)} \overset{?}{<} c^2 \Longrightarrow \mu_c \overset{?}{<} c^2 rad(c) \Longrightarrow \mu_c rad(c) \overset{?}{<} c^2 rad^2(c) \Longrightarrow$$
$$c \overset{?}{<} c^2 rad^2(c) \Longrightarrow 1 < c.rad^2(c) \tag{116}$$

Then $c < rad^2(abc)$. We announce the theorem:

Theorem 7. — *Let a, b, c positive integers relatively prime with $c = a + b$ and $1 \leq b < a$, then $c < rad^2(abc)$.*

5.3. *The Proof of the* abc *conjecture (6)*

We denote $R = rad(abc)$.

5.3.1. Case: $\epsilon \geq 1$

Using the result of the theorem above, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon} e^{\left(\frac{1}{\epsilon^2} - \epsilon\right)}, \ \epsilon \geq 1 \tag{117}$$

We verify easily that $K(\epsilon) > 1$ for $\epsilon \geq 1$.

5.3.2. Case: $\epsilon < 1$

5.3.2.1. Case: $c < R$

In this case, we can write :

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon} e^{\left(\frac{1}{\epsilon^2} - \epsilon\right)}, \ \epsilon < 1 \tag{118}$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the ABC conjecture is true.

5.3.2.2. Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon} e^{\left(\dfrac{1}{\epsilon^2} - \epsilon\right)}, 0 < \epsilon < 1 \tag{119}$$

If not, then $\exists \epsilon_0 \in]0,1[$, so that the triplets (a,b,c) checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{120}$$

are in finite number. We have:

$$\begin{gathered} c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow \\ R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > c.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > K(\epsilon_0) \end{gathered} \tag{121}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow$$

$$c^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow c > K(\epsilon_0)^{\left(\dfrac{1}{1-\epsilon_0}\right)} \tag{122}$$

We deduce that it exists an infinity of triples (a,b,c) verifying (120), hence the contradiction. Then the proof of the ABC conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with a, b, c relatively coprime:

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad with \quad K(\epsilon) = 6^{1+\epsilon} e^{\left(\dfrac{1}{\epsilon^2} - \epsilon\right)} \tag{123}$$

Q.E.D

5.4. *Examples*

In this section, we are going to verify some numerical examples.

5.4.1. Example of Eric Reyssat

We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{124}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$. For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = 6^{1.01} e^{9999.99} = 1.8884880155640644914779227374022e + 4343 \tag{125}$$

Let us verify (123):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \overset{?}{<} K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow$$
$$6436343 \ll K(0.01) \times 15042 \tag{126}$$

Hence (123) is verified.

5.4.2. Example of A. Nitaj

5.4.2.1. Case 1

The example of Nitaj about the ABC conjecture [**1**] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \tag{127}$$

we take $\epsilon = 100$ we have:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 6^{101}e^{-99.9999}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 8.2558649305610435609546415285004e+48$$

then (123) is verified.

5.4.2.2. Case 2

We take $\epsilon = 0.5$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \tag{128}$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 6^{1.5}.e^{3.5}.(2.3.5.7.11.13.41.79.311.953)^{1.5} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 75\,333\,109\,597\,556\,257\,182\,261.66 \tag{129}$$

We obtain that (123) is verified.

5.4.2.3. Case 3

We take $\epsilon = 1$, then

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 6^2.(2.3.5.7.11.13.41.79.311.953)^2 \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 29\,918\,625\,700\,491\,952\,961\,692\,755\,600 \tag{130}$$

We obtain that (123) is verified.

5.4.3. Example of Ralf Bonse

The example of Ralf Bonse about the ABC conjecture [**2**] is:

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \tag{131}$$

$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983$$

$$rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$rad(abc) = 1.5683959920004546031461002610848e+33 \tag{132}$$

For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = 6^{1.01}.e^{9999.99} = 5.2903884296336672264108948608106e+4343$$

Let us verify (123):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<}$$
$$6^{1.01}.e^{9999.99}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.01}$$
$$\Longrightarrow 3.4136998783296235160378273576498e+44 <$$
$$1.7819595478010681971905561514574e+4377 \tag{133}$$

The equation (123) is verified.

Ouf, end of the mystery!

5.5. *Conclusion*

This is an elementary proof of the ABC conjecture, confirmed by five numerical examples. We can announce the important theorem:

Theorem 8. — *For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{134}$$

where K is a constant depending of ϵ equal to $6^{1+\epsilon}.e^{\left(\frac{1}{\epsilon^2}-\epsilon\right)}$.

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CHAPTER 6

A PROOF OF THE *ABC* CONJECTURE: THE END OF MYSTERY

6.1. *Introduction and notations*

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{135}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{136}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

Conjecture 8. — (***ABC Conjecture***): *Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{137}$$

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.616751$ [**2**]. Here we will give a proof that:

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{138}$$

This result, I think is the key to obtain a proof of the veracity of the *ABC* conjecture.

6.2. *A Proof of the condition (138)*

Let a, b, c positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$.

If $c < rad(ab)$ then we obtain:

$$c < rad(ab) < rad^2(abc) \tag{139}$$

and the condition (138) is verified.

In the following, we suppose that $c \geq rad(ab)$.

6.2.1. Case $c = a + 1$

$$c = a + 1 = \mu_a rad(a) + 1 \stackrel{?}{<} rad^2(ac) \tag{140}$$

6.2.1.1. $\mu_a = 1$

In this case, $a = rad(a)$, it is immediately truth that :

$$c = a + 1 < 2a < rad(a) rad(c) < rad^2(ac) \tag{141}$$

Then (140) is verified.

6.2.1.2. $\mu_a \neq 1$, $\mu_a < rad(a)$

we obtain :

$$c = a + 1 < 2\mu_a.rad(a) \Rightarrow c < 2rad^2(a) \Rightarrow c < rad^2(ac) \tag{142}$$

Then (140) is verified.

6.2.1.3. $\mu_a \geq rad(a)$

We have $c = a + 1 = \mu_a.rad(a) + 1 \leq \mu_a^2 + 1 \stackrel{?}{<} rad^2(ac)$. We suppose that $\mu_a^2 + 1 \geq rad^2(ac) \Longrightarrow \mu_a^2 > rad^2(a).rad(c) > rad^2(a)$ as $rad(c) > 1$, then $\mu_a > rad(a)$, that is the contradiction with $\mu_a \geq rad(a)$. We deduce that $c < \mu_a^2 + 1 < rad^2(ac)$ and the condition (140) is verified.

6.2.2. $c = a + b$

We can write that c verifies:

$$c = a + b = rad(a).\mu_a + rad(b).\mu_b = rad(a).rad(b)\left(\frac{\mu_a}{rad(b)} + \frac{\mu_b}{rad(a)}\right)$$

$$\Longrightarrow c = rad(a).rad(b).rad(c)\left(\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)}\right) \tag{143}$$

We can write also:

$$c = rad(abc)\left(\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)}\right) \tag{144}$$

To obtain a proof of (138), one method is to prove that :

$$\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)} < rad(abc) \tag{145}$$

6.2.2.1. $\mu_a = \mu_b = 1$

In this case, it is immediately truth that :

$$\frac{1}{rad(a_i} + \frac{1}{rad(b_j} \leq \frac{5}{6} < rad(c).rad(abc) \tag{146}$$

Then (138) is verified.

6.2.2.2. $\mu_a = 1$ and $\mu_b > 1$

As $b < a \Longrightarrow \mu_b rad(b) < rad(a) \Longrightarrow \frac{\mu_b}{rad(a)} < \frac{1}{rad(b)}$, then we deduce that:

$$\frac{1}{rad(b)} + \frac{\mu_b}{rad(a)} < \frac{2}{rad(b)} < rad(c).rad(abc) \tag{147}$$

Then (138) is verified.

6.2.2.3. $\mu_b = 1$ and $\mu_a \leq (b = rad(b))$

In this case we obtain:

$$\frac{1}{rad(a)} + \frac{\mu_a}{rad(b)} \leq \frac{1}{rad(a)} + 1 < rad(c).rad(abc) \tag{148}$$

Then (138) is verified.

6.2.2.4. $\mu_b = 1$ and $\mu_a > (b = rad(b))$

As $\mu_a > rad(b)$, we can write $\mu_a = rad(b) + n$ where $n \geq 1$. We obtain:

(149)

$$c = \mu_a rad(a) + rad(b) = (rad(b) + n)rad(a) + rad(b) = rad(ab) + nrad(a) + rad(b)$$

We have $n < b$, if not $n \geq b \Longrightarrow \mu_a \geq 2b \Longrightarrow a \geq 2brad(a) \Longrightarrow a \geq 3b \Longrightarrow c > 3b$, then the contradiction with $c > 2b$. We can write:

(150)

$$c < 2rad(ab) + rad(b) \Longrightarrow c < rad(abc) + rad(abc) < rad^2(abc) \Longrightarrow c < rad^2(abc)$$

6.2.2.5. $\mu_a.\mu_b \neq 1\,, \mu_a < rad(a)$ and $\mu_b < rad(b)$

we obtain :

$$c = \mu_c rad(c) = \mu_a.rad(a) + \mu_b.rad(b) < rad^2(a) + rad^2(b) < rad^2(abc) \tag{151}$$

6.2.2.6. $\mu_a.\mu_b \neq 1\,, \mu_a \leq rad(a)$ and $\mu_b \geq rad(b)$

We have:

$$c = \mu_a.rad(a) + \mu_b.rad(b) < \mu_a\mu_b rad(a)rad(b) \leq \mu_b rad^2(a)rad(b) \tag{152}$$

Then if we give a proof that $\mu_b < rad(b)rad^2(c)$, we obtain $c < rad^2(abc)$. As $\mu_b \geq rad(b) \Longrightarrow \mu_b = rad(b) + \alpha$ with α a positive integer ≥ 0. Supposing that $\mu_b \geq rad(b)rad^2(c) \Longrightarrow \mu_b = rad(b)rad^2(c) + \beta$ with $\beta \geq 0$ a positive integer. We can write:

$$rad(b)rad^2(c) + \beta = rad(b) + \alpha \Longrightarrow \beta < \alpha$$

$$\alpha - \beta = rad(b)(rad^2(c) - 1) > 3rad(b) \Longrightarrow \mu_b = rad(b) + \alpha > 4rad(b) \tag{153}$$

Finally, we obtain:

$$\begin{cases} \mu_b \geq rad(b) \\ \mu_b > 4rad(b) \end{cases} \tag{154}$$

Then the contradiction and the hypothesis $\mu_b \geq rad(b)rad^2(c)$ is false. Hence:

$$\mu_b < rad(b)rad^2(c) \Longrightarrow c < rad^2(abc) \tag{155}$$

6.2.2.7. $\mu_a.\mu_b \neq 1\,, \mu_a \geq rad(a)$ and $\mu_b \leq rad(b)$

The proof is identical to the case above.

6.2.2.8. $\mu_a.\mu_b \neq 1\,, \mu_a \geq rad(a)$ and $\mu_b \geq rad(b)$

We write:

(156)

$$c = \mu_a rad(a) + \mu_b rad(b) \leq \mu_a^2 + \mu_b^2 < \mu_a^2.\mu_b^2 \stackrel{?}{<} rad^2(a).rad^2(b).rad^2(c) = rad^2(abc)$$

Supposing that $\mu_a.\mu_b \geq rad(abc)$, we obtain:

$$\mu_a.\mu_b \geq rad(abc) \Rightarrow rad(a).rad(b).\mu_a.\mu_b \geq rad^2(ab)rad(c) \Longrightarrow$$
$$ab \geq rad^2(ab).rad(c) \Rightarrow a^2 > ab \geq rad^2(ab).rad(c)$$
$$\Rightarrow a > rad(ab)\sqrt{rad(c)} \geq rad(ab)\sqrt{7} \Rightarrow$$
$$\begin{cases} c > \sqrt{7}rad(ab) \geq 3rad(ab) \\ \\ c \geq rad(ab) \end{cases} \tag{157}$$

The inequality $c \geq 3rad(ab)$ gives the contradiction with the condition $c \geq rad(ab)$ supposed at the beginning of this section. Then we obtain $\mu_a.\mu_b - rad(abc) < 0 \Longrightarrow c < rad^2(abc)$.

We announce the theorem:

Theorem 9. — (Abdelmajid Ben Hadj Salem, 2019) *Let a, b, c positive integers relatively prime with $c = a + b$ and $b < a$, then $c < rad^2(abc)$.*

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CHAPTER 7

A PROOF OF THE BEAL'S CONJECTURE

7.1. *Introduction and notations*

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{158}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{159}$$

A paper about the proof of the ABC conjecture, is accepted [**1**]. We have obtained the following theorem:

Theorem 10. — ***(ABC Theorem)****: For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{160}$$

where K is a constant depending only of ϵ equal to $6^{1+\epsilon} e^{\left(\dfrac{1}{\epsilon^2} - \epsilon\right)}$.

In 1997, Andrew Beal [**2**] announced the following conjecture:
Conjecture 1: (Beal Conjecture)
Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:

$$A^m + B^n = C^l \tag{161}$$

then $A, B,$ and C have a common factor.

7.2. *Methodology of the proof*

We note :

$$\text{A: Beal Conjecture} \tag{162}$$

$$\text{B: } ABC \text{ Theorem} \tag{163}$$

and we use the following property (the contrapositive law, [**3**]:

$$\boxed{A(False) \Longrightarrow B(False)} \Longleftrightarrow \boxed{B(True) \Longrightarrow A(True)} \tag{164}$$

From the right equivalent expression in the box above, we obtain that B (ABC Theorem) which is true implies A (Beal Conjecture) is true.

7.3. *Proof of the conjecture* (7.1)

We suppose that Beal conjecture is false, then it exists A, B, C positive **coprime** integers and m, n, l positive integers > 2 such:

$$A^m + B^n = C^l \tag{165}$$

the integers A, B, C, m, n, l are supposed large integers. We consider in the following that $A^m > B^n$. Now, we use the ABC theorem for equation (165) because A^m, B^n, C^m are relatively coprime. We obtain :

$$C^l < K(\epsilon) rad(A^m B^n C^l)^{1+\epsilon} \Longrightarrow C^l < K(\epsilon) \left(rad(A).rad(B).rad(C)\right)^{1+\epsilon} \tag{166}$$

As $rad(A) \leq A, rad(B) \leq B$ and $rad(C) \leq C$, the last equation becomes:

$$C^l < \frac{2}{\epsilon^2}(A.B.C)^{1+\epsilon} \tag{167}$$

But $rad(A) \leq A < C^{\frac{l}{m}}$, $rad(B) \leq B < C^{\frac{l}{n}}$, then we write (167) as :

$$\frac{\epsilon^2}{2} < C^{\left(1 + \frac{l}{m} + \frac{l}{n}\right).(1+\epsilon) - l} \tag{168}$$

7.3.1. *Case* $m > l$ *and* $n > l$

In this case, $\left(1 + \dfrac{l}{m} + \dfrac{l}{n}\right).(1+\epsilon) - l \approx 3 - l + 3\epsilon$. We take $\epsilon = 1$. As $6 \ll l \Longrightarrow \dfrac{1}{C^{l-6}} \ll 0.5$, then the contradiction.

7.3.2. *Case* $m < l$ *and* $n < l$

In this case, if $C > A \Rightarrow C^m > A^m > B^n \Rightarrow C^m > B^n \Rightarrow C^m > C^l - A^m \Rightarrow A^m > C^l - C^m \Rightarrow A^m > C^m(C^{l-m} - 1)$. As $l > m \Rightarrow C^{l-m} - 1 > 1$, then

$A^m > C^m \Longrightarrow A > C$ that is a contradiction with $C > A$. Hence $C < A$. We rewrite equation (167):

$$C^l < K(\epsilon)rad(A^mB^nC^l)^{1+\epsilon} \Longrightarrow C^l < K(\epsilon)(A.B.C)^{1+\epsilon}$$
$$\Rightarrow A^m < C^l < K(\epsilon)\left(A.A^{\frac{m}{n}}.A\right)^{1+\epsilon} \tag{169}$$

Then:

$$A^m < \frac{2}{\epsilon^2}.A^{\left(2+\frac{m}{n}\right)(1+\epsilon)} \tag{170}$$

7.3.2.1. Case $n > m$

If $n > m$, we have $\left(2+\dfrac{m}{n}\right)(1+\epsilon) \approx 3+3\epsilon$. We take $\epsilon = 1$, as $6 \ll m \Longrightarrow \dfrac{1}{A^{m-6}} \ll 0.5$, then the contradiction.

7.3.2.2. Case $n < m$

We have:

$$C^l < K(\epsilon)(A.B.C)^{1+\epsilon} \tag{171}$$

As $A^m < C^l, C < A$ and $B^n < A^m \Longrightarrow B < A^{m/n}$, the last equation becomes:

$$\frac{\epsilon^2}{2} < A^{(2+m/n)(1+\epsilon)-m} \tag{172}$$

We choose $\epsilon = \dfrac{1}{m}$, we obtain :

$$\frac{1}{2m^2} < A^{2-m+\frac{2}{m}+\frac{1}{n}} \Longrightarrow \frac{1}{2m^2} < A^{3-m} \tag{173}$$

But $3 \ll m$ and $1 \ll A \Longrightarrow \dfrac{1}{2m^2} > A^{3-m}$, then the contradiction.

7.3.3. Case $m < l$ and $n > l$

If $C < A$, as $l < n \Rightarrow C^l < A^n \Rightarrow 0 < A^m < A^n - B^n$ then $A > B$. As $C^n > C^l > B^n \Rightarrow C^n > B^n \Rightarrow C > B$. So we obtain :

$$\boxed{B < C < A} \tag{174}$$

Then the equation (167) becomes:

$$C^l\frac{\epsilon^2}{2} < (A.B.C)^{1+\epsilon} \Longrightarrow C^l\frac{\epsilon^2}{2} < (A.A^{l/n}.A)^{1+\epsilon} \Rightarrow C^l\frac{\epsilon^2}{2} < A^{(2+l/n)(1+\epsilon)} \tag{175}$$

As $A^m < C^l$, we arrive to:

$$\frac{\epsilon^2}{2} < A^{3-3m+3\epsilon} \tag{176}$$

We take $\epsilon = 1/3 \Longrightarrow A^{3m-4} < 18$, then the contradiction because $18 \ll A^{3m-4}$.

If $A < C \Rightarrow A^l < C^l$ but $B^n < A^m \Rightarrow A^l < 2A^m \Rightarrow A^l < A^{m+1} \Rightarrow l < m+1$, as $m < l \Rightarrow m+1 \leq l < m+1$ that is a contradiction, then $C < A$ and this case is studied above.

7.3.4. Case $m > l$ and $n < l$

We have $n < l < m$. As $A^m < C^l \Rightarrow A < C^{\frac{l}{m}} < C \Rightarrow A < C$. The equation (167) becomes:

$$C^l < \frac{2}{\epsilon^2}\left(C^{l/m}.C^{l/n}.C\right)^{1+\epsilon} \tag{177}$$

We take $\epsilon = 0.1$, we obtain:

$$0.005 < C^{2.2+1.1\frac{l}{n}-l} \approx C^{3+\frac{l}{n}-l} \tag{178}$$

But as $3 \ll l \Longrightarrow l > 3 + \frac{l}{n}$, then the contradiction.

All the cases give contradiction, then ABC theorem is false. We deduce from :

$$\boxed{\text{Beal Conjecture (False)} \Rightarrow \text{ABC Theorem (False)}} \Leftrightarrow$$

$$\boxed{\text{ABC Theorem (True)} \Rightarrow \text{Beal Conjecture (True)}}$$

that Beal Conjecture is true.

The proof of the Beal conjecture is achieved[(1)].

7.4. *Conclusion*

From the ABC theorem, we have given a proof that the ABC conjecture is true. We can announce the theorem:

Theorem 11. — (Abdelmajid Ben Hadj Salem, Andrew Beal, 2019): *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{179}$$

then $A, B,$ and C have a common factor.

[(1)] A paper giving another proof of Beal conjecture is under reviewing [**4**]

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CHAPTER 8

THE *ABC* CONJECTURE: THE END OF THE MYSTERY

8.1. *Introduction and notations*

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{180}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{181}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

Conjecture 9. — (***ABC Conjecture***)*: Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{182}$$

This paper about this conjecture is written after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. I try here to give a simple proof that can be understood by undergraduate students.

8.2. *Proof of the conjecture (9)*

Let a, b, c positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$. We propose for the constant $K(\epsilon)$ the formula:

$$K(\epsilon) = \frac{2}{\epsilon^2} \tag{183}$$

Function of ϵ, $K(\epsilon)$ is a decreasing function so that $lim_{\epsilon \longrightarrow 0} K(\epsilon) = +\infty$ and $lim_{\epsilon \longrightarrow +\infty} K(\epsilon) = 0$. It permits an equilibrium when $rad(abc)^{1+\epsilon} \nearrow$ with $\epsilon > 1$ or $rad(abc)^{1+\epsilon} \searrow$ with $\epsilon \searrow 0$. We want to see if :

$$c \overbrace{<}^{?} K(\epsilon).rad(abc)^{1+\epsilon} = \frac{2}{\epsilon^2}.e^{(1+\epsilon)Log(rad(abc))} \tag{184}$$

We denote:

$$R = Log(rad(abc)) > 0 \tag{185}$$

$$F(\epsilon) = 2e^{(1+\epsilon)R} - c.\epsilon^2 \tag{186}$$

If we arrive to give a proof of $F(\epsilon) > 0$ for $\forall \epsilon > 0$, then we deduce that the ABC conjecture is true and it is the end of the mystery of the ABC conjecture!

Let us study the function $F(\epsilon)$ with $\epsilon \in [0, +\infty[$. The function $F(\epsilon)$ is of class C^∞. We have $F(0) = 2e^R = 2rad(abc) \geq 12$, and $lim_{\epsilon \longrightarrow +\infty} F(\epsilon) = +\infty$ because $min(rad(abc)) > e \Rightarrow R > 1$. We calculate $F'(\epsilon)$:

(187)

$$F'(\epsilon) = 2.Log(rad(abc))e^{(1+\epsilon)R} - 2c\epsilon = 2.Re^{(1+\epsilon)R} - 2c \longrightarrow +\infty, if\ \epsilon \longrightarrow +\infty$$

$F"(\epsilon) = 2.Log^2(rad(abc))e^{(1+\epsilon)R} - 2 = 2R^2e^{(1+\epsilon)R} - 2$, $F'(\epsilon) = 0$ gives:

$$2.Log(rad(abc))e^{(1+\epsilon)R} = 2c\epsilon \tag{188}$$

Let :

$$\alpha = \frac{c}{R^2e^R} \tag{189}$$

$$X = R\epsilon \tag{190}$$

The equation (188) becomes:

$$e^X = \alpha X \tag{191}$$

The equation above represents the determination of the coordinates (X, Y) of the points of the intersection of the two curves: $Y = e^X$ and $Y = \alpha X$. The tangent line to the curve $Y = e^X$ passing at the origin $(0, 0)$ is represented by $Y = eX$. Then three cases are to study:

- Case 1: $\alpha < e \Longrightarrow c < e.rad(abc).Log^2rad(abc)$, and $e^X \neq \alpha X$ and the exponential curve is above the line $Y = \alpha X \Longrightarrow F'(\epsilon) > 0\ \forall \epsilon > 0$, then $F(\epsilon) > 0$ and the ABC conjecture holds.

- Case 2 : $\alpha = e$, the line $Y = \alpha X = eX$ is tangent to the curve $Y = e^X$ at the point $(1, e)$. $\alpha = e \Longrightarrow c = e.rad(abc).Log^2rad(abc)$. Supposing that $eLog^2rad(abc)$ is an integer, we obtain that a, b, c are not relatively prime. Then the contradiction.

- Case 3: $\alpha > e \Longrightarrow c > e.rad(abc).Log^2 rad(abc)$, and $e^X = \alpha X$ has one solution $X_1 = R\epsilon_1 < 1$ with $\epsilon_1 < 1$ and for $\epsilon > \epsilon_1$, we obtain $F'(\epsilon) < 0 \Longrightarrow F(\epsilon)$ is decreasing for $\epsilon \in [\epsilon_1, +\infty[$, but $lim_{\epsilon \longrightarrow +\infty} F(\epsilon) = +\infty$ then the contradiction.

Finally, only the case 1 is correct and we obtain that the ABC conjecture is true. Then we obtain the important result of the paper:

$$\begin{gathered} c < K(\epsilon).rad(abc)^{1+\epsilon} \quad \forall \epsilon > 0 \\ \text{with the constant } K(\epsilon) = \frac{2}{\epsilon^2} \end{gathered} \tag{192}$$

Q.E.D

8.3. *Examples*

In this section, we are going to verify some numerical examples.

8.3.1. Example of Eric Reyssat

We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{193}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$. For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = \frac{2}{\epsilon^2} = \frac{2}{10^{-4}} \tag{194}$$

Let us verify (192):

$$\begin{gathered} c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \overset{?}{<} 2.10^4.(3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow \\ 6436343 < 331213962.07 \end{gathered} \tag{195}$$

Hence (192) is verified.

8.3.2. Example of A. Nitaj

8.3.2.1. Case 1

The example of Nitaj about the ABC conjecture [**3**] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \tag{196}$$

$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \tag{197}$$

$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \tag{198}$$

$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \tag{199}$$

we take $\epsilon = 100$ we have:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 2.10^{-4}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 < 5.5310368633286126480363 8e + 1355 \tag{200}$$

then (192) is verified.

8.3.2.2. Case 2

We take $\epsilon = 0.000001 = 10^{-6}$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \tag{201}$$

$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 2.10^{12}.(2.3.5.7.11.13.41.79.311.953)^{1.000001} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 < 57\,658\,458\,237\,370\,924\,700\,998\,757\,174\,980.$$

We obtain that (192) is verified.

8.3.2.3. Case 3

We take $\epsilon = 1$, then

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 2.(2.3.5.7.11.13.41.79.311.953)^2 \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 < 1\,662\,145\,872\,249\,552\,942\,316\,264\,200. \tag{202}$$

Ouf, end of the mystery!

8.4. *Conclusion*

This is an elementary proof of the *ABC* conjecture, confirmed by four numerical examples. We can announce the important theorem:

Theorem 12. — *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2019) For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{203}$$

where K is a constant depending of ϵ equal to $\dfrac{2}{\epsilon^2}$.

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CHAPTER 9

THE *ABC* CONJECTURE: THE END OF THE MYSTERY

9.1. *Introduction and notations*

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{204}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{205}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

Conjecture 10. — *(**ABC Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then ::*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{206}$$

where K is a constant depending only of ϵ.

This paper about this conjecture is written after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. I try here to give a simple proof that can be understood by undergraduate students.

9.2. *Proof of the conjecture (10)*

Let a, b, c positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$. We propose for the constant $K(\epsilon)$ the formula:

$$K(\epsilon) = \frac{2}{\epsilon^2} \tag{207}$$

Function of ϵ, $K(\epsilon)$ is a decreasing function so that $lim_{\epsilon \longrightarrow 0} K(\epsilon) = +\infty$ and $lim_{\epsilon \longrightarrow +\infty} K(\epsilon) = 0$. It permits an equilibrium when $rad(abc)^{1+\epsilon} \nearrow$ with $\epsilon > 1$ or $rad(abc)^{1+\epsilon} \searrow$ with $\epsilon \searrow 0$. We want to see if :

$$c \overbrace{<}^{?} K(\epsilon).rad(abc)^{1+\epsilon} = \frac{2}{\epsilon^2}.e^{(1+\epsilon)Log(rad(abc))} \tag{208}$$

We denote:

$$R = Log(rad(abc)) > 0 \tag{209}$$

$$\alpha = \frac{c}{2R^2 e^R} \tag{210}$$

$$X = R\epsilon \tag{211}$$

$$F(\epsilon) = 2e^{(1+\epsilon)R} - c.\epsilon^2 = 2e^R \left(e^{\epsilon R} - \frac{c}{2R^2 e^R} R^2 \epsilon^2 \right) = 2e^R (e^X - \alpha X^2) \tag{212}$$

$$G(X) = e^X - \alpha X^2 \tag{213}$$

If we arrive to give a proof of $F(\epsilon) > 0$ for $\forall \epsilon > 0$ or $G(X) > 0$ for $\forall X > 0$, then we deduce that the *ABC* conjecture is true and it is the end of the mystery of the *ABC* conjecture!

Let us study the function $G(X)$ with $X \in]0, +\infty[$. The function $G(X)$ is of class C^∞. We have $G(0^+) = 1^+$, and $lim_{X \longrightarrow +\infty} G(X) = +\infty$ because $min(rad(abc)) > e \Rightarrow R > 1$. We calculate $G'(X)$:

$$G'(X) = e^X - 2\alpha X, \quad G'(X) \longrightarrow +\infty, if\ X \longrightarrow +\infty \tag{214}$$

$$G"(X) = e^X - 2\alpha \tag{215}$$

$G"(X) = 0$ gives:

$$e^X = 2\alpha \tag{216}$$

- Case 1: As $X > 0 \Rightarrow e^X > 1$. If $2\alpha < 1 \Rightarrow c < R^2 e^R \Rightarrow G"(X) > 0, \forall X > 0$ (Fig.1), then $G'(X) > 0, \forall X > 0$ and $G(X)$ is an increasing function from 1^+ to $+\infty$ hence, $F(\epsilon) > 0, \forall \epsilon > 0$ then the *ABC* conjecture holds.

- Case 1.1: $2\alpha = 1 \Longrightarrow c = R^2 e^R = Log^2(rad(abc)).rad(abc)$. Supposing that $Log^2 rad(abc)$ is an integer, we obtain that a, b, c are not relatively coprime. Then

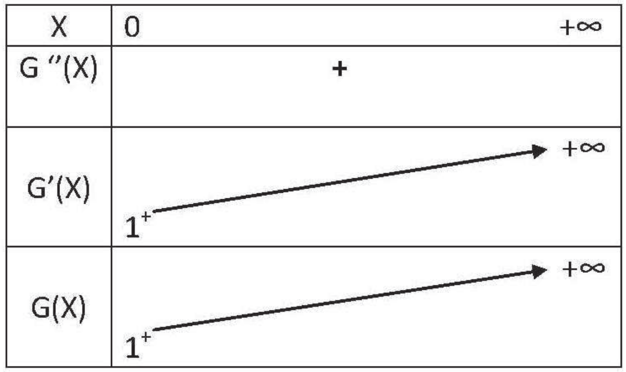

FIGURE 1. Case 1

the contradiction, and the ABC conjecture holds.

- Case 2 : If $2\alpha > 1 \Longrightarrow c > R^2 e^R$, the equation $e^X = 2\alpha$ has an unique solution $X_1 = Log 2\alpha \Longrightarrow X_1 > 0$. $G'(X_1) = e^{X_1} - 2\alpha X_1 = 2\alpha(1 - X_1)$.

- Case 2.1. If $X_1 = 1 \Longrightarrow Log 2\alpha = 1 \Longrightarrow 2\alpha = e \Longrightarrow c = eR^2 e^R$ or $c = e.rad(abc).Log^2 rad(abc)$. Supposing that $eLog^2 rad(abc)$ is an integer, we obtain that a, b, c are not relatively coprime. Then the contradiction, and the ABC conjecture holds.

- Case 2.2. If $X_1 < 1 \Longrightarrow Log 2\alpha < 1 \Longrightarrow 2\alpha < e \Longrightarrow c < eR^2 e^R$. $G"(X_1) = 0, G'(X_1) > 0$ and $G(X) > 0, \forall X > 0$ (Fig.2). Then $F(\epsilon) > 0 \, \forall \epsilon > 0$ and the ABC conjecture holds.

- Case 2.3. If $X_1 > 1 \Longrightarrow Log 2\alpha > 1 \Longrightarrow 2\alpha > e \Longrightarrow \alpha > e/2$. We obtain the condition that $c > e.rad(abc).Log^2 rad(abc)$. We have $e^{X_1} = 2\alpha$, $G'(X_1) = 2\alpha(1 - X_1) < 0$ and $G'(1) = e - 2\alpha < 0$, then $\exists X_3 < 1 < X_1$ so that $G'(X_3) = 0$ and $X_2 > X_1$ so that $G'(X_2) = 0$ (Fig.3). For the case $X_1 > 1$, we will give a proof that c does not verify $c > e.rad(abc).Log^2 rad(abc)$ and we will deduce that the case $X_1 > 1$ is to reject.

We verify easily that if $c \leq rad(abc)$, we obtain $c < e.rad(abc).Log^2 rad(abc)$, then the contradiction and the ABC conjecture holds. We suppose in the following that $c > rad(abc)$.

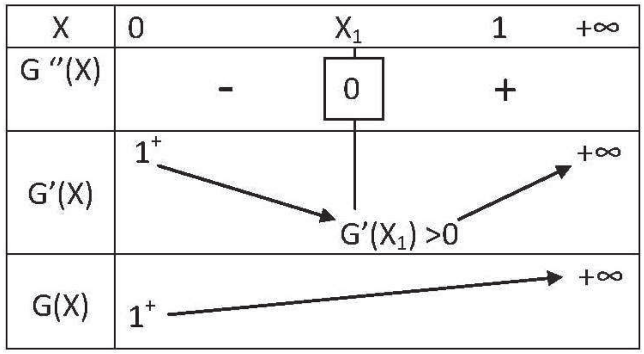

FIGURE 2. Case 2.2

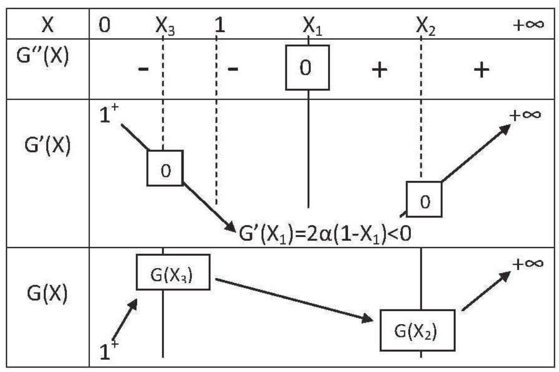

FIGURE 3. Case 3

If we consider the first example cited in section (9.3):

$$3^{10} \times 109 + 2 = 23^5 = 6436343$$

$a = 3^{10}.109 \Longrightarrow rad(a) = 3 \times 109$,
$b = 2 \Longrightarrow rad(b) = 2$,
$c = 23^5 = 6436343 \Longrightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042 < c$, $Log(rad(abc)) = Log(15042) = 9.618601 \Rightarrow 2.718\,281 \times 15042 \times 92.517\,485\,197\,201 = 245330.296812 < c$. Then we have found a, b, c relatively coprime with $c > rad(abc)$ and c does not verify $c > e.rad(abc).Log^2(rad(abc))$. We deduce that the case 2.3 is to reject.

Finally, only the cases 1, 1.1, 2.1 and 2.2 are corrects and we obtain that the ABC conjecture is true. Then we obtain the important result of the paper:

$$\begin{gathered} c < K(\epsilon).rad(abc)^{1+\epsilon} \quad \forall \epsilon > 0 \\ \text{with the constant } K(\epsilon) = \frac{2}{\epsilon^2} \end{gathered} \tag{217}$$

Q.E.D

9.3. *Examples*

In this section, we are going to verify some numerical examples.

9.3.1. Example of Eric Reyssat

We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{218}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$. For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = \frac{2}{\epsilon^2} = \frac{2}{10^{-4}} \tag{219}$$

Let us verify (217):

$$\begin{gathered} c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \overset{?}{<} 2.10^4.(3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow \\ 6436343 < 331213962.07 \end{gathered} \tag{220}$$

Hence (217) is verified.

9.3.2. Example of A. Nitaj

9.3.2.1. Case 1

The example of Nitaj about the ABC conjecture [**3**] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \tag{221}$$

$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \tag{222}$$

$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \tag{223}$$

$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \tag{224}$$

we take $\epsilon = 100$ we have:

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 \stackrel{?}{<} 2.10^{-4}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 < 5.531036863328612648036 38e + 1355 \tag{225}$$

then (217) is verified.

9.3.2.2. Case 2

We take $\epsilon = 0.000001 = 10^{-6}$, then:

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \tag{226}$$

$$613\,474\,845\,886\,230\,468\,750 \stackrel{?}{<} 2.10^{12}.(2.3.5.7.11.13.41.79.311.953)^{1.000001} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 < 57\,658\,458\,237\,370\,924\,700\,998\,757\,174\,980.$$

We obtain that (217) is verified.

9.3.2.3. Case 3

We take $\epsilon = 1$, then

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 \stackrel{?}{<} 2.(2.3.5.7.11.13.41.79.311.953)^{2} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 < 1\,662\,145\,872\,249\,552\,942\,316\,264\,200. \tag{227}$$

9.3.3. Example of Ralf Bonse

9.3.3.1. Case 1

The example of Ralf Bonse about the ABC conjecture [**4**] is:

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \tag{228}$$

$$a = 2543^4.182587.2802983.85813163$$

$$b = 2^{15}.3^{77}.11.173$$

$$c = 5^{56}.245983$$

$$rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$

For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = \frac{2}{\epsilon^2} = \frac{2}{10^{-4}} = 2.10^4$$

Let us verify (217):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$

$$c = 5^{56}.245983 \overset{?}{<} 2.10^4.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.01}$$

$$\Longrightarrow 3,4136998783296235160378273576498e+44 <$$

$$6,7365924884440287789307666776768e+37 \tag{229}$$

Ouf, end of the mystery!

9.4. *Conclusion*

This is an elementary proof of the *ABC* conjecture, confirmed by four numerical examples. We can announce the important theorem:

Theorem 13. — *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2019) For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{230}$$

where K is a constant depending of ϵ equal to $\dfrac{2}{\epsilon^2}$.

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CHAPTER 10

THE FINAL PROOF OF THE *ABC* CONJECTURE

10.1. *Introduction and notations*

Let a a positive integer, $a=\prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a=\prod_i a_i^{\alpha_i}=rad(a).\prod_i a_i^{\alpha_i-1} \tag{231}$$

We note:

$$\mu_a=\prod_i a_i^{\alpha_i-1} \Longrightarrow a=\mu_a.rad(a) \tag{232}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 11. — *(* ***abc Conjecture****): For each $\epsilon>0$, there exists $K(\epsilon)>0$ such that if a,b,c positive integers relatively prime with $c=a+b$, then : :*

$$c<K(\epsilon).rad(abc)^{1+\epsilon} \tag{233}$$

K is a constant depending only of ϵ.

The idea to try to write a paper about this conjecture was born after after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. I try here to give a simple proof that can be understood by undergraduate students.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))}\leq 1.629912$ [**1**]. A conjecture was proposed that $c<rad^2(abc)$ [**3**]. It is the key to resolve the *abc* conjecture. The paper is organized as fellow: in the second section, we give for the case $c=a+1$ the proof that $c<rad^2(ac)$. For the case $c=a+b$, the proof of $c<rad^2(abc)$ is given in

the third section. The main proof of the *abc* conjecture is presented in section four. The numerical examples are discussed in section five.

10.2. *The Proof of the conjecture* $c < rad^2(abc)$, *Case :* $c = a + 1$

Below is given the definition of the conjecture $c < rad^2(abc)$:

Conjecture 12. — *Let a, b, c positive integers relatively prime with $c = a+b$, then:*

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{234}$$

In the case $c = a + 1$, the definition of the conjecture is:

Definition 10.1. — *Let a, c positive integers, relatively prime, with $c = a + 1$, then:*

$$c < rad^2(ac) \Longrightarrow \frac{Logc}{Log(rad(ac))} < 2 \tag{235}$$

1 - If $c < rad(ac)$ then we obtain:

$$c < rad(ac) < rad^2(ac) \tag{236}$$

and the condition (235) is verified.

2 - If $c = rad(ac)$, then a, c are not relatively coprime. Case to reject.

3 - In the following, we suppose that $c > rad(ac) \Longrightarrow c = rad(ac) + \alpha$, with $\alpha \geq 1$ an integer. We write:

$$rad^2(ac) - c = (c - \alpha)^2 - c\mu_a.rad(a) + 1 \stackrel{?}{<} rad^2(a).rad^2(c) \Longrightarrow 0 \stackrel{?}{<} rad^2(c).rad^2(a) - \mu_a.rad(a) - 1$$

$$\Longrightarrow 0 \stackrel{?}{<} rad^2(a) - \frac{\mu_a}{rad^2(c)} rad(a) - \frac{1}{rad^2(c)} \tag{237}$$

Let the function $P(X) = X^2 - \dfrac{\mu_a}{rad^2(c)} X - \dfrac{1}{rad^2(c)}$. The discriminant of $P(X)$ is:

$$\Delta = \frac{\mu_a^2}{rad^4(c)} + \frac{4}{rad^2(c)} = \frac{\mu_a^2}{rad^4(c)} \left(1 + \frac{4rad^2(c)}{\mu_a^2}\right) > 0 \tag{238}$$

The roots $X_1 < X_2$ of $P(X) = 0$ are given by:

$$X_1 = \frac{\mu_a}{2rad^2(c)} \left(1 - \sqrt{1 + \frac{4rad^2(c)}{\mu_a^2}}\right) < 0$$

$$X_2 = \frac{\mu_a}{2rad^2(c)} \left(1 + \sqrt{1 + \frac{4rad^2(c)}{\mu_a^2}}\right) > 0 \tag{239}$$

c verifies (237) $\Longrightarrow P(rad(a)) \overset{?}{>} 0 \Longrightarrow rad(a) \overset{?}{>} X_2$, we obtain:

$$rad(a) \overset{?}{>} \frac{\mu_a}{2rad^2(c)}\left(1+\sqrt{1+\frac{4rad^2(c)}{\mu_a^2}}\right) \Longrightarrow$$

$$2rad^2(a)rad^2(c) \overset{?}{>} a\left(1+\sqrt{1+\frac{4rad^2(c)}{\mu_a^2}}\right) \tag{240}$$

We denote $R = rad(ac)$. We can write:

$$2R^2 - a \overset{?}{>} a\sqrt{1+\frac{4R^2}{a^2}} \tag{241}$$

On suppose that $2R^2 - a > 0$. We obtain:

$$(2R^2-a)^2 \overset{?}{>} a^2\left(1+\frac{4R^2}{a^2}\right) \Longrightarrow$$

$$4R^4 - 4R^2a + a^2 \overset{?}{>} a^2 + 4R^2 \Longrightarrow R^2 - a \overset{?}{>} 1 \Longrightarrow \boxed{R^2 > c} \tag{242}$$

As $R^2 > c \Longrightarrow R^2 > a \Longrightarrow 2R^2 > 2a \Longrightarrow 2R^2 - a > a > 0$ and the condition above $2R^2 - a > 0$ is justified.

We announce the theorem:

Theorem 14. — *Let a, c positive integers relatively prime with $c = a+1, a \geq 2$, then $c < rad^2(ac)$.*

10.3. *The Proof of the conjecture* $c < rad^2(abc)$, *Case:* $c = a + b$

We denote $R = rad(abc)$. Let a, b, c positive integers, relatively prime, with $c = a + b \Longrightarrow rad(a) = \dfrac{c-b}{\mu_a}$. Let us verify that :

$$c \overset{?}{<} rad^2(a).rad^2(bc) \tag{243}$$

We obtain :

$$c \overset{?}{<} \left(\frac{c-b}{\mu_a}\right)^2 .rad^2(bc) \Longrightarrow$$

$$0 \overset{?}{<} c^2 - \left(2b + \frac{\mu_a^2}{rad^2(bc)}\right)c + b^2 \tag{244}$$

Let $P(X) = X^2 - \left(2b + \dfrac{\mu_a^2}{rad^2(bc)}\right)X + b^2$. The discriminant Δ of $P(X)$ is:

$$\Delta = \frac{\mu_a^4}{rad^4(bc)}\left(1+\frac{4brad^2(bc)}{\mu_a^2}\right) > 0 \tag{245}$$

The roots $X_1 < X_2$ of $P(X)$ are given by:

$$\begin{aligned} X_1 &= \frac{1}{2}\left[2b + \frac{\mu_a^2}{rad^2(bc)} - \frac{\mu_a^2}{rad^2(bc)}\sqrt{1+\frac{4b.rad^2(bc)}{\mu_a^2}}\right] \\ X_2 &= \frac{1}{2}\left[2b + \frac{\mu_a^2}{rad^2(bc)} + \frac{\mu_a^2}{rad^2(bc)}\sqrt{1+\frac{4b.rad^2(bc)}{\mu_a^2}}\right] \end{aligned} \tag{246}$$

If c will verify (244) $\Longrightarrow c > X_2$ or $c < X_1$, we obtain:

$$\begin{gathered} c \stackrel{?}{<} X_1 \Longrightarrow c \stackrel{?}{<} \frac{1}{2}\left[2b + \frac{\mu_a^2}{rad^2(bc)} - \frac{\mu_a^2}{rad^2(bc)}\sqrt{1+\frac{4b.rad^2(bc)}{\mu_a^2}}\right] \Longrightarrow \\ 2a \stackrel{?}{<} \frac{\mu_a^2}{rad^2(bc)} - \frac{\mu_a^2}{rad^2(bc)}\sqrt{1+\frac{4b.rad^2(bc)}{\mu_a^2}} \end{gathered} \tag{247}$$

As the right term of the above inequality is < 0 and $2a > 0$, then $c < X_1$ is to reject. Now, we will see if $c \stackrel{?}{>} X_2$, we obtain:

$$\begin{gathered} c \stackrel{?}{>} X_2 \Longrightarrow c \stackrel{?}{>} \frac{1}{2}\left[2b + \frac{\mu_a^2}{rad^2(bc)} + \frac{\mu_a^2}{rad^2(bc)}\sqrt{1+\frac{4b.rad^2(bc)}{\mu_a^2}}\right] \Longrightarrow \\ 2a \stackrel{?}{>} \frac{\mu_a^2}{rad^2(bc)} + \frac{\mu_a^2}{rad^2(bc)}\sqrt{1+\frac{4b.rad^2(bc)}{\mu_a^2}} \Longrightarrow \\ 2arad^2(abc) \stackrel{?}{>} a^2\left(1+\sqrt{1+4b\frac{rad^2(bc)}{\mu_a^2}}\right) \Longrightarrow \frac{2rad^2(abc)}{a} - 1 \stackrel{?}{>} \sqrt{1+4b\frac{rad^2(bc)}{\mu_a^2}} \end{gathered}$$

We suppose that $\dfrac{2rad^2(abc)}{a} - 1 > 0$, we obtain:

$$\begin{gathered} \left(\frac{2rad^2(abc)}{a} - 1\right)^2 \stackrel{?}{>} 1 + 4b\frac{rad^2(abc)}{a^2} \Longrightarrow \\ rad^2(abc) - a \stackrel{?}{>} b \Longrightarrow rad^2(abc) > a + b \Longrightarrow \boxed{rad^2(abc) > c} \end{gathered} \tag{248}$$

As $rad^2(abc) > c \Longrightarrow \dfrac{2rad^2(abc)}{a} - 1 > 0$, then the proof of $c < rad^2(abc)$ is justified.

We can announce the theorem:

Theorem 15. — *Let a, b, c positive integers relatively prime with $c = a + b, b \geq 2$, then $c < rad^2(abc)$.*

10.4. *The Proof of the* abc *conjecture (11)*

10.4.1. Case: $\epsilon \geq 1$

Using the result of the theorem $c < rad^2(abc)$, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \ \epsilon \geq 1 \tag{249}$$

We verify easily that $K(\epsilon) > 1$ for $\epsilon \geq 1$. Then the abc conjecture is true.

10.4.2. Case: $\epsilon < 1$

10.4.2.1. Case: $c < R$

In this case, we can write :

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \ \epsilon < 1 \tag{250}$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the ABC conjecture is true.

10.4.2.2. Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, 0 < \epsilon < 1 \tag{251}$$

If not, then $\exists \epsilon_0 \in]0, 1[$, so that the triplets (a, b, c) checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{252}$$

are in finite number. We have:

$$\begin{gathered} c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow \\ R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > c.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > K(\epsilon_0) \end{gathered} \tag{253}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow$$

$$c^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow c > K(\epsilon_0)^{\left(\frac{1}{1-\epsilon_0}\right)} \tag{254}$$

We deduce that it exists an infinity of triples (a, b, c) verifying (394), hence the contradiction. Then the proof of the abc conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with a, b, c relatively coprime, $b \geq 2$:

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad with \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \tag{255}$$

Q.E.D

10.5. *Examples*

In this section, we are going to verify some numerical examples.

10.5.1. Example 1

The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{256}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$,
$b = 1 \Rightarrow \mu_b = 1$ and $rad(b) = 1$,
$c = 19^6 = 47\,045\,880 \Rightarrow rad(c) = 19$. Then $rad(abc) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$..

We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,880$.

10.5.1.1. Case $\epsilon = 0.01$

$c < K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow 47\,045\,880 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8,7477777149120053120152473488653e + 4342 \tag{257}$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (255) is verified.

10.5.1.2. Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2,6879363309671754205917012128876e + 43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$. And the equation (255) is verified.

10.5.1.3. Case $\epsilon = 1$

$K(1) = e \Longrightarrow c = 47\,045\,880 < e.rad^2(ac) = 697\,987\,143\,184,212.$ and the equation (255) is verified.

10.5.1.4. Case $\epsilon = 100$

$$K(100) = e^{0.0001} \Longrightarrow c = 47\,045\,880 \overset{?}{<} e^{0.0001}.506\,730^{101} = \\ 1,5222350248607608781853142687284e + 576$$

and the equation (255) is verified.

10.5.2. Example 2

We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{258}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$. For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{9999.99} = 8,7477777149120053120152473488653e + 4342 \tag{259}$$

Let us verify (255):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \overset{?}{<} K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow$$
$$6436343 \ll K(0.01) \times 15042^{1.01} \tag{260}$$

Hence (255) is verified.

10.5.3. Example 3

The example of Nitaj about the ABC conjecture [**1**] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \tag{261}$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \tag{262}$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \tag{263}$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \tag{264}$$

10.5.3.1. Case 1

we take $\epsilon = 100$ we have:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^{0.0001}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 2,7657949971494838920022381186039e + 1359$$

then (255) is verified.

10.5.3.2. Case 2

We take $\epsilon = 0.5$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \tag{265}$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^4.(2.3.5.7.11.13.41.79.311.953)^{1.5} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 8\,450\,961\,319\,227\,998\,887\,403,9993 \tag{266}$$

We obtain that (255) is verified.

10.5.3.3. Case 3

We take $\epsilon = 1$, then

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} (2.3.5.7.11.13.41.79.311.953)^2 \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 < 831\,072\,936\,124\,776\,471\,158\,132\,100 \tag{267}$$

We obtain that (255) is verified.

10.5.4. Example 4

It is of Ralf Bonse about the ABC conjecture [**3**] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \tag{268}$$

$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983$$
$$rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$

$$rad(abc) = 1.5683959920004546031461002610848e + 33 \tag{269}$$

10.5.4.1. Case 1

For example, we take $\epsilon = 10$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{0.01} = 1,007815740428295674320461741677 9$$

Let us verify (255):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<}$$
$$e^{0.01}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{11}$$
$$\Longrightarrow 3.4136998783296235160378273576498e + 44 <$$

$$1,4236200596494908176008120925721e + 365 \tag{270}$$

The equation (255) is verified.

10.5.4.2. Case 2

We take $\epsilon = 0.4 \Longrightarrow K(\epsilon) = 12,182473474251512159126256696 08$, then: The

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<}$$
$$e^{6.25}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.4}$$
$$\Longrightarrow 3.4136998783296235160378273576498e + 44 <$$

$$3,6255465680011453642792720569685e + 47 \tag{271}$$

And the equation (255) is verified.

Ouf, end of the mystery!

10.6. *Conclusion*

We have given an elementary proof of the *abc* conjecture in the two cases $c = a' + 1$ and $c = a + b$, confirmed by some numerical examples. We can announce the important theorem:

Theorem 16. — *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2019) For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{272}$$

where K is a constant depending of ϵ proposed equal to $e^{\left(\frac{1}{\epsilon^2}\right)}$.

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CHAPTER 11

TENTATIVE OF THE PROOF OF *ABC* CONJ.-CASE $c = a + 1$

To the memory of my Father who taught me arithmetic
To the memory of my colleague and friend Dr.Eng. Chedly Fezzani (1943-2019) for his important work in the field of Geodesy and the promotion of the Geographic Sciences in Africa

11.1. *Introduction and notations*

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{273}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{274}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) ([**1**]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 13. — *(* ***abc Conjecture****): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{275}$$

K is a constant depending only of ϵ. We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.616751$ ([**2**]). A conjecture was proposed that $c < rad^2(abc)$ ([**3**]). Here we will give a proof of it in the case $c = a + 1$.

Conjecture 14. — *Let a, b, c positive integers relatively prime with $c = a + b$, then:*

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{276}$$

This result, I think is the key to obtain a proof of the veracity of the abc conjecture.

11.2. *A Proof of the conjecture (14) Case* $c = a + 1$

Let a, c positive integers, relatively prime, with $c = a + 1$. If $c < rad(ac)$ then we obtain:

$$c < rad(ac) < rad^2(ac) \tag{277}$$

and the condition (276) is verified.

In the following, we suppose that $c \geq rad(ac)$.

11.2.1. Notations

We note:

$$a = \prod_i a_i^{\alpha_i} \Longrightarrow rad(a) = \prod_i a_i, \mu_a = \prod_i a_i^{\alpha_i - 1},\ i = 1, N_a \tag{278}$$

$$c = \prod_j c_j^{\beta_j} \Longrightarrow rad(c) = \prod_j c_j, \mu_c = \prod_j c_j^{\beta_j - 1},\ j = 1, N_c \tag{279}$$

with a_i, c_j prime integers and $N_a, N_c, \alpha, \beta \geq 1$ positive integers. Let:

$$R = rad(a).rad(c) = rad(ac) \tag{280}$$

$$\mathcal{R}(x) = \prod_i^{N_a} (x + a_i)^2 . \prod_j^{N_c} (x + c_j) \Longrightarrow \mathcal{R}(x) > 0, \forall x \geq 0 \tag{281}$$

$$F(x) = \mathcal{R}(x) - \mu_c \tag{282}$$

From the last equations we obtain:

$$F(0) = \mathcal{R}(0) - \mu_c = rad^2(a).rad(c) - \mu_c \tag{283}$$

Then, our main task is to prove that $F(0) > 0 \Longrightarrow R^2 > c$.

11.2.1.1. The Proof of $c < rad^2(ac)$

From the definition of the polynomial $F(x)$, its degree is $2N_a + N_c$. We have :

1. $lim_{x \longrightarrow +\infty} F(x) = +\infty$,
2. $lim_{x \longrightarrow +\infty} \dfrac{F(x)}{x} = +\infty$, F is convex for x large,
3. if x_1 is the great real root of $F(x) = 0$, and from the points 1., 2. we deduce that $F"(x_1^+) > 0$,
4. if $x_1 < 0$, then $F(0) > 0$.

Let us study $F'(x)$ and $F"(x)$. We obtain: ;

$$F'(x) = \mathcal{R}'(x)$$

$$\mathcal{R}'(x) = \left[\prod_i^{N_a}(x+a_i)^2\right]'.\prod_j^{N_c}(x+c_j) + \prod_i^{N_a}(x+a_i)^2.\left[\prod_j^{N_c}(x+c_j)\right]' \Longrightarrow$$

$$\left[\prod_i^{N_a}(x+a_i)^2\right]' = 2\prod_i^{N_a}(x+a_i)^2.\left(\sum_i \frac{1}{x+a_i}\right)$$

$$\left[\prod_j^{N_c}(x+c_j)\right]' = \prod_j^{N_c}(x+c_j)\left(\sum_{j=1}^{j=N_b}\frac{1}{x+c_j}\right) \Longrightarrow$$

$$\mathcal{R}'(x) = \mathcal{R}(x).\left(\sum_i^{N_a}\frac{2}{x+a_i} + \sum_j^{N_c}\frac{1}{x+c_j}\right) > 0, \forall x \geq 0 \tag{284}$$

$$F'(x) = \mathcal{R}' = \mathcal{R}(x)\left(\sum_i^{N_a}\frac{2}{x+a_i} + \sum_j^{N_c}\frac{1}{x+c_j}\right) > 0, \forall x > 0 \Longrightarrow$$

$$F'(0) = \mathcal{R}(0).\left(\sum_i^{N_a}\frac{2}{a_i} + \sum_j^{N_c}\frac{1}{c_j}\right) = rad^2(a).rad(c).\left(\sum_i^{N_a}\frac{2}{a_i} + \sum_j^{N_c}\frac{1}{c_j}\right) > 0$$

For $F"(x)$, we obtain:

$$F"(x) = \mathcal{R}" =$$

$$= \mathcal{R}'(x)\left(\sum_i^{N_a}\frac{2}{x+a_i} + \sum_j^{N_c}\frac{1}{x+c_j}\right) - \mathcal{R}(x)\left(\sum_i^{N_a}\frac{2}{(x+a_i)^2} + \sum_j^{N_c}\frac{1}{(x+c_j)^2}\right) \Longrightarrow$$

$$F"(x) = \mathcal{R}(x).\left[\left(\sum_i^{N_a}\frac{2}{x+a_i} + \sum_j^{N_c}\frac{1}{x+c_j}\right)^2 - \sum_i^{N_a}\frac{2}{(x+a_i)^2} - \sum_j^{N_c}\frac{1}{(x+c_j)^2}\right]$$

$$\Longrightarrow F"(x) > 0, \forall x \geq 0 \tag{285}$$

We obtain also that $F"(0) > 0$.

Before we attack the proof, we take an example as: $1+8=9 \Longrightarrow c=9, a=8, b=1$. We obtain $rad(a)=2, rad(c)=3, \mu_c=3, R=rad(ac)=2\times 3=6<(c=9)$ and $c=9$ verifies $c<(R^2=6^2=36)$. We write the polynomial $F(x)=(x+2)^2(x+3)-3=x^3+7x^2+16x+9>0, \forall x>0$. Then $F'(x)=3x^2+14x+16$, we verifies that $F'(x)=0$ has not real roots and $F'(x)>0, \forall x\in\mathbb{R}$. We have also $F"(x)=6x+14$. $F"(x)=0 \Longrightarrow x=-7/3\approx -2.33 \Longrightarrow F(-7/3)=-79/27\approx -2.92$. The point $(-7/3,-79/27)$ is an inflexion point of the curve of $y=F(x)$. We deduce that the curve is convex for $x\geq -7/3$. Let us now find the roots of $F(x)=0$. As the degree of F is three, the number of the real roots are 1 or 3. As there is one inflexion point, we will find one real root.

11.2.2. The Resolution of $F(x) = 0$

We want to resolve:

$$F(x) = x^3 + 7x^2 + 16x + 9 = 0 \tag{286}$$

Let the change of variables $x = t - 7/3$, the equation (286) becomes:

$$t^3 - \frac{t}{3} - \frac{79}{27} = 0 \tag{287}$$

For the resolution of (287), we introduce two unknowns:

$$t = u + v \Longrightarrow (u+v)(3uv - \tfrac{1}{3}) + u^3 + v^3 - \tfrac{79}{27} = 0 \Longrightarrow$$
$$\begin{cases} u^3 + v^3 = \dfrac{79}{3^3} \\ uv = \dfrac{1}{3^2} \end{cases} \tag{288}$$

Then u^3, v^3 are solutions of the equation:

$$X^2 - \frac{79}{3^3}X + \frac{1}{3^6} = 0 \tag{289}$$

and given below:

$$u^3 = \frac{1}{2}.\frac{79 + 9\sqrt{77}}{3^3} \Longrightarrow \begin{cases} u_1 = \sqrt[3]{\dfrac{1}{2}\left(\dfrac{79 + 9\sqrt{77}}{3^3}\right)} \approx 0.97515 \\ \\ u_2 = j.u_1, \quad j = \frac{-1+i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \\ u_3 = j^2 u_1 = \bar{j}.u_1 \end{cases}$$
$$v^3 = \frac{1}{2}.\frac{79 - 9\sqrt{77}}{3^3} \Longrightarrow \begin{cases} v_1 = \sqrt[3]{\dfrac{1}{2}\left(\dfrac{79 - 9\sqrt{77}}{3^3}\right)} \approx 0.00016 \\ \\ v_2 = j^2.v_1 = \bar{j}.v_1 \\ v_3 = j.v_1 \end{cases} \tag{290}$$

Finally, taking into account the second condition of (288), we obtain the real root of (287):

$$t = u_1 + v_1 = \sqrt[3]{\frac{1}{2}\left(\frac{79 + 9\sqrt{77}}{3^3}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{79 - 9\sqrt{77}}{3^3}\right)} \approx 0.97531$$
$$x_1 = t - 7/3 \approx -1.35802 \tag{291}$$

Then the first root of $F(x) = 0$ is $x_1 \approx -1.358 < 0$, the correction to the first root of $\mathcal{R}(x) = (x+2)^2(x+3) = 0$ is $dx = x_1 - (-2) = -1.358 - (-2) = +0.642$. As in our example $F'(x) > 0$, the function $F(x)$ is an increasing function having a parabolic branch as $x \longrightarrow +\infty$, the curve $y = F(x)$ intersects the line $x = 0$ in the half-plane $y \geq 0 \Longrightarrow F(0) > 0 \Longrightarrow c < rad^2(ac)$ which is verified numerically.

11.2.3. The General Case

Let us return to the general case $c = a + 1$. We denote $q = min(a_i, c_j)$. If we consider that $F(x) = \mathcal{R}(x)$, the equation $F(x) = 0 \Longrightarrow \mathcal{R}(x) = 0$ and the first real root is $x_1 = -q$, the product of all the roots is $P = \prod_i (x_i)^2 . \prod_j (x_j) = (-1)^{2N_a+N_c} \prod_i (a_i)^2 . \prod_j (c_j)$. But $F(x) = \mathcal{R}(x) - \mu_c$, the constant coefficient of $F(x)$ will be $\prod_i (a_i)^2 . \prod_j (c_j) - \mu_c$. The new product of the roots is $P' = \prod_i (x'_i)^2 . \prod_j (x'_j) = (-1)^{2N_a+N_c} (\prod_i (a_i)^2 . \prod_j (c_j) - \mu_c)$. The first root $x_1 = -q$ becomes $x'_1 = -q + dx$. To estimate dx, we can write to the order two that:

$$F(-q + dx) = \mathcal{R}(-q + dx) - \mu_c = 0 \Longrightarrow \mathcal{R}(-q + dx) = \mu_c \Longrightarrow$$

$$\mathcal{R}(-q) + dx.\mathcal{R}'(-q) + \frac{dx^2}{2}\mathcal{R}"(-q) = \mu_c \tag{292}$$

Supposing that $a_1 = q = min(a_i, c_j)$, from the equations (281-284-285), we have :

$$\mathcal{R}(-a_1) = 0$$

$$\mathcal{R}'(-a_1) = 0$$

$$\mathcal{R}"(-a_1) = 2 \prod_{i=2}^{N_a} (a_i - a_1)^2 . \prod_{j=1}^{N_c} (c_j - a_1) > 0 \Longrightarrow$$

$$dx^2 = \frac{\mu_c}{\prod_{i=2}^{N_a} (a_i - a_1)^2 . \prod_{j=1}^{N_c} (c_j - a_1)} \tag{293}$$

We take the positive value of dx following the numerical example above, then we obtain:

$$dx = \frac{\sqrt{\mu_c}}{\prod_{i=2}^{N_a} (a_i - a_1) . \sqrt{\prod_{j=1}^{N_c} (c_j - a_1)}} \tag{294}$$

As $a_1 = min(a_i, c_j)_{i=2,N_a;j=1,N_c}$, we have $a_2 - a_1 \geq 1, a_3 - a_1 \geq 2, \cdots, a_{N_a} - a_1 \geq N_a - 1, c_1 - a_1 \geq 1, c_2 - a_1 \geq 2, \cdots, c_{N_c} - a_1 \geq N_c$, then we can write:

$$dx \leq \frac{\sqrt{\mu_c}}{(N_a - 1)! . \sqrt{N_c!}} \tag{295}$$

For the expression of $N!$, we can use the Stirling's formula to two orders:

$$N! = \sqrt{2\pi N} \left(\frac{N}{e}\right)^N \left(1 + \frac{1}{12N}\right) \tag{296}$$

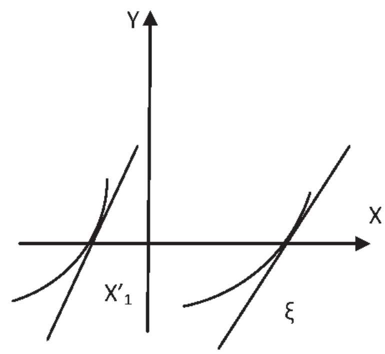

FIGURE 1

and we take $c = a + 1 < 3a/2, \mu_c = \dfrac{c}{rad(c)} < \dfrac{3a}{2a_1^{N_c}}$. We obtain :

(297)
$$dx < \left(1 - \frac{1}{12(N_a-1)}\right)\left(1 - \frac{1}{24N_c}\right)\frac{e^{-(N_a-1)Log\frac{N_a-1}{e} - (\frac{N_c}{2})Log\frac{N_c}{e}}}{\sqrt{2\pi(N_a-1)}}\sqrt{\frac{3a}{2\sqrt{2\pi N_c}a_1^{N_c}}}$$

As a, c are large positive integers, and $1 \ll N_a.N_c$, $|dx| \ll a_i, c_j$ and the sign of the first root x_1' does not change. Then the curve of $F(x) = \mathcal{R}(x) - \mu_c$ intersects the line $x = 0$ and the equation of the tangent at the point x_1' is $y = F'(x_1')(x - x_1')$. But if $x_1' = -a + dx = \xi > 0$, the equation of the tangent at the point ξ is $y = F'(\xi)(x - \xi)$, in this case $F'(\xi) < F'(x_1')$ (see Fig. 1), and for $x > -a_1, F"(x) > 0 \Longrightarrow F'$ is an increasing function. As $x_1' < \xi \Longrightarrow F'(x_1') < F'(\xi)$, then the contradiction and we obtain that $x_1' = -a_1 + dx < 0 \Longrightarrow F(0) = rad^2(a).rad(c) - \mu_c > 0 \Longrightarrow rad^2(a).rad^2(c) - c > 0 \Longrightarrow c < R^2$.

11.2.3.1. Examples

In this section, we are going to verify the above remarks with a numerical example. The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6$$
$$rad(a) = 2 \times 3 \times 5 \times 7 \times 127 = 26\,670$$
$$rad(c) = 19$$
(298)
$$c = 19^5 = 47\,045\,881, \quad \mu_c = 19^5 = 2\,476\,099$$

Using the notations of the paper, we obtain:

$$\mathcal{R}(x) = (x+2)^2(x+3)^2(x+5)^2(x+7)^2(x+127)^2(x+19)$$
$$F(x) = \mathcal{R}(x) - \mu_c$$

Let $X = x + 2$, the expression of $\mathcal{R}(x)$ becomes:

$$\overline{\mathcal{R}}(X) = X^2(X+1)^2(X+3)^2(X+5)^2(X+125)^2(X+17)$$

The calculations gives:

$$\overline{\mathcal{R}}(X) = X^{11} + 285.X^{10} + 24\,808.X^9 + 657\,728.X^8 + 7\,424\,722.X^7 + 42\,772\,898.X^6$$
$$+134\,002\,080.X^5 + 223\,508\,940.X^4 + 187\,753\,125.X^3 + 597\,656\,251.X^2$$

We want to estimate the first root of $F(x) = 0$, we write:

$$\overline{\mathcal{R}}(X) - \mu_c = 0 \Longrightarrow$$
$$X^{11} + 285.X^{10} + 24\,808.X^9 + 657\,728.X^8 + 7\,424\,722.X^7 + 42\,772\,898.X^6$$
$$+134\,002\,080.X^5 + 223\,508\,940.X^4+$$
$$187\,753\,125.X^3 + 597\,656\,251.X^2 - 2\,476\,099 = 0 \tag{299}$$

If $x = -2 \Longrightarrow X = 0 \Longrightarrow \overline{\mathcal{R}}(X) - \mu_c < 0$. If we take $x_1 = -1.936315 \Longrightarrow X_1 = 0.03685$, then we obtain that:

$$\mathcal{R}(x_1) - \mu_c = \overline{\mathcal{R}}(X_1) - \mu_c \approx 177.82 > 0 \tag{300}$$

Then, $\exists\, \xi$ with $-2 < \xi < x_1$ so that $X' = -2+\xi$ verifies $\overline{\mathcal{R}}(X') - \mu_c = 0$ and ξ is the first root of $F(x) = 0$ and $\xi < 0 \Longrightarrow F(0) > 0 \Longrightarrow rad^2(a)rad(c) - \mu_c > 0 \Longrightarrow R^2 > c$ that is true. We have also $\xi = -2 + dx = a_1 + dx$ and $0 < dx < |a_1|$.

11.3. *The Proof of The* abc *conjecture (13) Case:* $c = a + 1$

We denote $R = rad(ac)$.

11.3.1. Case: $\epsilon \geq 1$

Using the result of the theorem above, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)},\ \epsilon \geq 1 \tag{301}$$

We verify easily that $K(\epsilon) > 1$ for $\epsilon \geq 1$ and it is a decreasing function from e the base of the neperian logarithm to 1.

11.3.2. Case: $\epsilon < 1$

11.3.2.1. Case: $c \leq R$

In this case, we can write :

$$c \leq R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)},\ \epsilon < 1 \tag{302}$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the *abc* conjecture is true.

11.3.2.2. Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, 0 < \epsilon < 1 \tag{303}$$

If not, then $\exists \epsilon_0 \in]0,1[$, so that the triplets $(a,1,c)$ checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{304}$$

are in finite number. We have:

$$\begin{gathered} c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow \\ R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > c.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > K(\epsilon_0) \end{gathered} \tag{305}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow$$

$$c^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow c > K(\epsilon_0)^{\left(\frac{1}{1-\epsilon_0}\right)} \tag{306}$$

We deduce that it exists an infinity of triples $(a,1,c)$ verifying (304), hence the contradiction. Then the proof of the abc conjecture in the case $c = a+1$ is finished. We obtain that $\forall \epsilon > 0$, $c = a+1$ with a, c relatively coprime, $2 \leq a < c$:

$$c < K(\epsilon).rad(ac)^{1+\epsilon} \quad with \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \tag{307}$$

Q.E.D

11.4. *Examples*

In this section, we are going to verify some cases of one numerical example. The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{308}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$,
$b = 1 \Rightarrow \mu_b = 1$ and $rad(b) = 1$,
$c = 19^6 = 47\,045\,880 \Rightarrow rad(c) = 19$. Then $rad(abc) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$..

We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,880$.

11.4.0.1. Case $\epsilon = 0.01$

$c < K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow 47\,045\,880 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8,7477777149120053120152473488653e + 4342 \tag{309}$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (**??**) is verified.

11.4.0.2. Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2,6879363309671754205917012128876e + 43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$. And the equation (307) is verified.

11.4.0.3. Case $\epsilon = 1$

$K(1) = e \Longrightarrow c = 47\,045\,880 < e.rad^2(ac) = 697\,987\,143\,184,212.$ and the equation (307) is verified.

11.4.0.4. Case $\epsilon = 100$

$$K(100) = e^{0.0001} \Longrightarrow c = 47\,045\,880 \overset{?}{<} e^{0.0001}.506\,730^{101} = \\ 1,5222350248607608781853142687284e + 576$$

and the equation (307) is verified.

11.5. *Conclusion*

This is an elementary proof of the *abc* conjecture in the case $c = a + 1$. We can announce the important theorem:

Theorem 17. — *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2019) For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(ac)^{1+\epsilon} \tag{310}$$

where K is a constant depending of ϵ equal to $e^{\left(\frac{1}{\epsilon^2}\right)}$.

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CHAPTER 12

LAST PROOF OF *ABC* CONJECTURE - CASE $c = a + 1$

In this paper, we consider the *abc* conjecture in the case $c = a+1$. Firstly, we give the proof of the first conjecture that $c < rad^2(ac)$ using the polynomial functions. It is the key to the proof of the *abc* conjecture. Secondly, the proof of the *abc* conjecture is given for $\epsilon \geq 1$, then for $\epsilon \in]0,1[$ for the two cases: $c \leq rad(ac)$ and $c > rad(ac)$. We choose the constant $K(\epsilon)$ as $K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)}$. A numerical example is presented.

To the memory of my Father who taught me arithmetic
To the memory of my colleague and friend Dr.Eng. Chedly Fezzani (1943-2019) for his important work in the field of Geodesy and the promotion of the Geographic Sciences in Africa

12.1. *Introduction and notations*

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{311}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{312}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) ([**1**]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 15. — *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{313}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.616751$ ([**2**]). A conjecture was proposed that $c < rad^2(abc)$ ([**3**]). Here we will give the proof of it, in the case $c = a + 1$, using a polynomial function.

Conjecture 16. — *Let a, b, c positive integers relatively prime with $c = a+b$, then:*

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{314}$$

This result, I think is the key to obtain a proof of the veracity of the abc conjecture.

12.2. *A Proof of the conjecture (16) Case* $c = a + 1$

Let a, c positive integers, relatively prime, with $c = a + 1$. If $c < rad(ac)$ then we obtain:

$$c < rad(ac) < rad^2(ac) \tag{315}$$

and the condition (314) is verified.

In the following, we suppose that $c \geq rad(ac)$.

12.2.1. Notations

We note:

$$a = \prod_i a_i^{\alpha_i} \Longrightarrow rad(a) = \prod_i a_i, \mu_a = \prod_i a_i^{\alpha_i - 1},\ i = 1, N_a \tag{316}$$

$$c = \prod_j c_j^{\beta_j} \Longrightarrow rad(c) = \prod_j c_j, \mu_c = \prod_j c_j^{\beta_j - 1},\ j = 1, N_c \tag{317}$$

with a_i, c_j prime integers and $N_a, N_c, \alpha, \beta \geq 1$ positive integers. Let:

$$R = rad(a).rad(c) = rad(ac) \tag{318}$$

$$\mathcal{R}(x) = \prod_i^{N_a} (x + a_i)^2 . \prod_j^{N_c} (x + c_j) \Longrightarrow \mathcal{R}(x) > 0, \forall x \geq 0 \tag{319}$$

$$F(x) = \mathcal{R}(x) - \mu_c \tag{320}$$

From the last equations we obtain:

$$F(0) = \mathcal{R}(0) - \mu_c = rad^2(a).rad(c) - \mu_c \tag{321}$$

Then, our main task is to prove that $F(0) > 0 \Longrightarrow R^2 > c$.

12.2.1.1. The Proof of $c < rad^2(ac)$

From the definition of the polynomial $F(x)$, its degree is $2N_a + N_c$. We have :

1. $lim_{x \longrightarrow +\infty} F(x) = +\infty$,
2. $lim_{x \longrightarrow +\infty} \dfrac{F(x)}{x} = +\infty$, F is convex for x large,
3. if x_1 is the great real root of $F(x) = 0$, and from the points 1., 2. we deduce that $F"(x_1^+) > 0$,
4. if $x_1 < 0$, then $F(0) > 0$.

Let us study $F'(x)$ and $F"(x)$. We obtain:

$$F'(x) = \mathcal{R}'(x)$$

$$\mathcal{R}'(x) = \left[\prod_i^{N_a}(x+a_i)^2\right]'.\prod_j^{N_c}(x+c_j) + \prod_i^{N_a}(x+a_i)^2.\left[\prod_j^{N_c}(x+c_j)\right]' \Longrightarrow$$

$$\left[\prod_i^{N_a}(x+a_i)^2\right]' = 2\prod_i^{N_a}(x+a_i)^2.\left(\sum_i \frac{1}{x+a_i}\right)$$

$$\left[\prod_j^{N_c}(x+c_j)\right]' = \prod_j^{N_c}(x+c_j)\left(\sum_{j=1}^{j=N_b}\frac{1}{x+c_j}\right) \Longrightarrow$$

$$\mathcal{R}'(x) = \mathcal{R}(x).\left(\sum_i^{N_a}\frac{2}{x+a_i} + \sum_j^{N_c}\frac{1}{x+c_j}\right) > 0, \forall x \geq 0 \tag{322}$$

$$F'(x) = \mathcal{R}' = \mathcal{R}(x)\left(\sum_i^{N_a}\frac{2}{x+a_i} + \sum_j^{N_c}\frac{1}{x+c_j}\right) > 0, \forall x > 0 \Longrightarrow$$

$$F'(0) = \mathcal{R}(0).\left(\sum_i^{N_a}\frac{2}{a_i} + \sum_j^{N_c}\frac{1}{c_j}\right) =$$

$$rad^2(a).rad(c).\left(\sum_i^{N_a}\frac{2}{a_i} + \sum_j^{N_c}\frac{1}{c_j}\right) > 0 \tag{323}$$

For $F"(x)$, we obtain:

$$F"(x) = \mathcal{R}" = \mathcal{R}'(x)\left(\sum_i^{N_a}\frac{2}{x+a_i} + \sum_j^{N_c}\frac{1}{x+c_j}\right) - \mathcal{R}(x)\left(\sum_i^{N_a}\frac{2}{(x+a_i)^2} + \sum_j^{N_c}\frac{1}{(x+c_j)^2}\right)$$

$$\Longrightarrow F"(x) = \mathcal{R}(x).\left[\left(\sum_i^{N_a}\frac{2}{x+a_i} + \sum_j^{N_c}\frac{1}{x+c_j}\right)^2 - \sum_i^{N_a}\frac{2}{(x+a_i)^2} - \sum_j^{N_c}\frac{1}{(x+c_j)^2}\right]$$

$$\Longrightarrow F"(x) > 0, \forall x \geq 0 \tag{324}$$

We obtain also that $F"(0) > 0$.

Before we attack the proof, we take an example as: $1+8=9 \Longrightarrow c=9, a=8, b=1$. We obtain $rad(a)=2, rad(c)=3, \mu_c=3, R=rad(ac)=2\times 3=6<(c=9)$ and $c=9$ verifies $c<(R^2=6^2=36)$. We write the polynomial $F(x)=(x+2)^2(x+3)-3=x^3+7x^2+16x+9>0, \forall x>0$. Then $F'(x)=3x^2+14x+16$, we verifies that $F'(x)=0$ has not real roots and $F'(x)>0, \forall x\in\mathbb{R}$. We have also $F"(x)=6x+14$. $F"(x)=0 \Longrightarrow x=-7/3\approx -2.33 \Longrightarrow F(-7/3)=-79/27\approx -2.92$. The point $(-7/3,-79/27)$ is an inflection point of the curve of $y=F(x)$. We deduce that the curve is convex for $x\geq -7/3$. Let us now find the roots of $F(x)=0$. As the degree of F is three, the number of the real roots are 1 or 3. As there is one inflection point, we will find one real root.

12.2.2. The Resolution of $F(x)=0$

We want to resolve:

$$F(x)=x^3+7x^2+16x+9=0 \tag{325}$$

Let the change of variables $x=t-7/3$, the equation (325) becomes:

$$t^3-\frac{t}{3}-\frac{79}{27}=0 \tag{326}$$

For the resolution of (326), we introduce two unknowns:

$$t=u+v \Longrightarrow (u+v)(3uv-\tfrac{1}{3})+u^3+v^3-\tfrac{79}{27}=0 \Longrightarrow$$

$$\begin{cases} u^3+v^3=\dfrac{79}{3^3} \\ uv=\dfrac{1}{3^2} \end{cases} \tag{327}$$

Then u^3, v^3 are solutions of the equation:

$$X^2-\frac{79}{3^3}X+\frac{1}{3^6}=0 \tag{328}$$

and given below:

$$u^3=\frac{1}{2}.\frac{79+9\sqrt{77}}{3^3} \Longrightarrow \begin{cases} u_1=\sqrt[3]{\dfrac{1}{2}\left(\dfrac{79+9\sqrt{77}}{3^3}\right)}\approx 0.97515 \\ \\ u_2=j.u_1, \quad j=\frac{-1+i\sqrt{3}}{2}=e^{i\frac{2\pi}{3}} \\ u_3=j^2u_1=\bar{j}.u_1 \end{cases}$$

$$v^3=\frac{1}{2}.\frac{79-9\sqrt{77}}{3^3} \Longrightarrow \begin{cases} v_1=\sqrt[3]{\dfrac{1}{2}\left(\dfrac{79-9\sqrt{77}}{3^3}\right)}\approx 0.00016 \\ \\ v_2=j^2.v_1=\bar{j}.v_1 \\ v_3=j.v_1 \end{cases}$$

(329)

Finally, taking into account the second condition of (327), we obtain the real root of (326):

$$t = u_1 + v_1 = \sqrt[3]{\frac{1}{2}\left(\frac{79 + 9\sqrt{77}}{3^3}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{79 - 9\sqrt{77}}{3^3}\right)} \approx 0.97531$$

$$x_1 = t - 7/3 \approx -1.35802 \tag{330}$$

Then the first root of $F(x) = 0$ is $x_1 \approx -1.358 < 0$, the correction to the first root of $\mathcal{R}(x) = (x+2)^2(x+3) = 0$ is $dx = x_1 - (-2) = -1.358 - (-2) = +0.642$. As in our example $F'(x) > 0$, the function $F(x)$ is an increasing function having a parabolic branch as $x \longrightarrow +\infty$, the curve $y = F(x)$ intersects the line $x = 0$ in the half-plane $y \geq 0 \Longrightarrow F(0) > 0 \Longrightarrow c < rad^2(ac)$ which is verified numerically.

12.2.3. The General Case

Let us return to the general case $c = a + 1$. We denote $q = min(a_i, c_j)$. If we consider that $F(x) = \mathcal{R}(x)$, the equation $F(x) = 0 \Longrightarrow \mathcal{R}(x) = 0$ and the first real root is $x_1 = -q$, the product of all the roots is $P = \prod_i (x_i)^2 . \prod_j (x_j) = (-1)^{2N_a + N_c} \prod_i (a_i)^2 . \prod_j (c_j)$. But $F(x) = \mathcal{R}(x) - \mu_c$, the constant coefficient of $F(x)$ will be $\prod_i (a_i)^2 . \prod_j (c_j) - \mu_c$. The new product of the roots is $P' = \prod_i (x'_i)^2 . \prod_j (x'_j) = (-1)^{2N_a + N_c} (\prod_i (a_i)^2 . \prod_j (c_j) - \mu_c)$. The first root $x_1 = -q$ becomes $x'_1 = -q + dx$. To estimate dx, we can write to the order two that:

$$F(-q + dx) = \mathcal{R}(-q + dx) - \mu_c = 0 \Longrightarrow \mathcal{R}(-q + dx) = \mu_c \Longrightarrow$$

$$\mathcal{R}(-q) + dx.\mathcal{R}'(-q) + \frac{dx^2}{2}\mathcal{R}"(-q) = \mu_c \tag{331}$$

Supposing that $a_1 = q = min(a_i, c_j)$, from the equations (319-322-324), we have :

$$\mathcal{R}(-a_1) = 0$$

$$\mathcal{R}'(-a_1) = 0$$

$$\mathcal{R}"(-a_1) = 2\prod_{i=2}^{N_a} (a_i - a_1)^2 . \prod_{j=1}^{N_c} (c_j - a_1) > 0 \Longrightarrow$$

$$dx^2 = \frac{\mu_c}{\prod_{i=2}^{N_a} (a_i - a_1)^2 . \prod_{j=1}^{N_c} (c_j - a_1)} \tag{332}$$

We suppose that $c > rad^2(ac) \Longrightarrow \mu_c > rad^2(a).rad(c) \Longrightarrow \mu_c > \mathcal{R}(0)$. We deduce that $F(0) < 0$ and $x'_1 = -a_1 + dx > 0 \Longrightarrow dx > 0$. We take the positive value of dx, then we obtain:

$$dx = \frac{\sqrt{\mu_c}}{\prod_{i=2}^{N_a}(a_i - a_1).\sqrt{\prod_{j=1}^{N_c}(c_j - a_1)}} \tag{333}$$

But $\mu_c = \mathcal{R}(x'_1) = \prod_i^{N_a}(x'_1 + a_i)^2.\prod_j^{N_c}(x'_1 + c_j)$, we can write:

$$\mu_c = dx^2.\prod_{i=2}^{N_a}(dx + a_i - a_1)^2.\prod_j^{N_c}(dx + c_j - a_1) \Longrightarrow$$

$$\mu_c > dx^2.\prod_{i=2}^{N_a}(a_i - a_1)^2.\prod_j^{N_c}(c_j - a_1) \tag{334}$$

because all the terms $a_i - a_1$ and $c_j - a_1$ are positive numbers. Using the last inequality and the expression of dx given by the equation (333), we obtain:

$$\mu_c > \frac{\mu_c}{\prod_{i=2}^{N_a}(a_i - a_1)^2.\prod_{j=1}^{N_c}(c_j - a_1)}.\prod_{i=2}^{N_a}(a_i - a_1)^2.\prod_j^{N_c}(c_j - a_1) \Longrightarrow$$

$$1 > 1 \Longrightarrow \text{ the contradiction} \Longrightarrow \mu_c < rad^2(a)rad(c) \tag{335}$$

So, our supposition that $c > rad^2(ac)$ is false and we obtain the important result that $c < rad^2(ac)$ and the conjecture (32) is verified.

12.2.3.1. Examples

In this section, we are going to verify the above remarks with a numerical example. The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6$$
$$rad(a) = 2 \times 3 \times 5 \times 7 \times 127 = 26\,670$$
$$rad(c) = 19$$
$$c = 19^5 = 47\,045\,881, \quad \mu_c = 19^5 = 2\,476\,099 \tag{336}$$

Using the notations of the paper, we obtain:

$$\mathcal{R}(x) = (x+2)^2(x+3)^2(x+5)^2(x+7)^2(x+127)^2(x+19)$$
$$F(x) = \mathcal{R}(x) - \mu_c$$

Let $X = x + 2$, the expression of $\mathcal{R}(x)$ becomes:

$$\overline{\mathcal{R}}(X) = X^2(X+1)^2(X+3)^2(X+5)^2(X+125)^2(X+17)$$

The calculations gives:

$$\overline{\mathcal{R}}(X) = X^{11} + 285.X^{10} + 24\,808.X^9 + 657\,728.X^8 + 7\,424\,722.X^7 + 42\,772\,898.X^6$$
$$+134\,002\,080.X^5 + 223\,508\,940.X^4 + 187\,753\,125.X^3 + 597\,656\,251.X^2 \tag{337}$$

We want to estimate the first root of $F(x) = 0$, we write:

$$\overline{\mathcal{R}}(X) - \mu_c = 0 \Longrightarrow$$
$$X^{11} + 285.X^{10} + 24\,808.X^9 + 657\,728.X^8 + 7\,424\,722.X^7 + 42\,772\,898.X^6$$
$$+134\,002\,080.X^5 + 223\,508\,940.X^4 + 187\,753\,125.X^3 + 597\,656\,251.X^2 - 2\,476\,099 = 0$$

If $x = -2 \Longrightarrow X = 0 \Longrightarrow \overline{\mathcal{R}}(X) - \mu_c < 0$. If we take $x_1 = -1.936315 \Longrightarrow X_1 = 0.03685$, then we obtain that:

$$\mathcal{R}(x_1) - \mu_c = \overline{\mathcal{R}}(X_1) - \mu_c \approx 177.82 > 0 \tag{338}$$

Then, $\exists\ \xi$ with $-2 < \xi < x_1$ so that $X' = 2 + \xi$ verifies $\overline{\mathcal{R}}(X') - \mu_c = 0$ and ξ is the first root of $F(x) = 0$ and $\xi < 0 \Longrightarrow F(0) > 0 \Longrightarrow rad^2(a)rad(c) - \mu_c > 0 \Longrightarrow R^2 > c$ that is true. We have also $\xi = -2 + dx = a_1 + dx$ and $0 < dx < a_1$.

12.3. *The Proof of the* abc *conjecture (15) Case:* $c = a + 1$

We denote $R = rad(ac)$.

12.3.1. Case: $\epsilon \geq 1$

Using the result of the theorem above, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)},\ \epsilon \geq 1 \tag{339}$$

We verify easily that $K(\epsilon) > 1$ for $\epsilon \geq 1$ and it is a decreasing function from e the base of the neperian logarithm to 1.

12.3.2. Case: $\epsilon < 1$

12.3.2.1. Case: $c \leq R$

In this case, we can write :

$$c \leq R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)},\ \epsilon < 1 \tag{340}$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the *abc* conjecture is true.

12.3.2.2. Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, 0 < \epsilon < 1 \tag{341}$$

If not, then $\exists \epsilon_0 \in]0, 1[$, so that the triplets $(a, 1, c)$ checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{342}$$

are in finite number. We have:

$$\begin{gathered} c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow \\ R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > c.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > K(\epsilon_0) \end{gathered} \tag{343}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow$$

$$c^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow c > K(\epsilon_0)^{\left(\frac{1}{1-\epsilon_0}\right)} \tag{344}$$

We deduce that it exists an infinity of triples $(a, 1, c)$ verifying (342), hence the contradiction. Then the proof of the abc conjecture in the case $c = a + 1$ is finished. We obtain that $\forall \epsilon > 0$, $c = a + 1$ with a, c relatively coprime, $2 \leq a < c$:

$$c < K(\epsilon).rad(ac)^{1+\epsilon} \quad with \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \tag{345}$$

Q.E.D

12.4. *Examples*

In this section, we are going to verify some cases of one numerical example. The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{346}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$,
$b = 1 \Rightarrow \mu_b = 1$ and $rad(b) = 1$,
$c = 19^6 = 47\,045\,880 \Rightarrow rad(c) = 19$. Then $rad(abc) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$..

We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,880$.

12.4.0.1. Case $\epsilon = 0.01$

$c < K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow 47\,045\,880 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8,7477777149120053120152473488653e + 4342 \tag{347}$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (345) is verified.

12.4.0.2. Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2,6879363309671754205917012128876e + 43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$. And the equation (345) is verified.

12.4.0.3. Case $\epsilon = 1$

$K(1) = e \Longrightarrow c = 47\,045\,880 < e.rad^2(ac) = 697\,987\,143\,184,212.$ and the equation (345) is verified.

12.4.0.4. Case $\epsilon = 100$

$$K(100) = e^{0.0001} \Longrightarrow c = 47\,045\,880 \overset{?}{<} e^{0.0001}.506\,730^{101} = 1,5222350248607608781853142687284e + 576$$

and the equation (345) is verified.

12.5. *Conclusion*

This is an elementary proof of the *abc* conjecture in the case $c = a + 1$. We can announce the important theorem:

Theorem 18. — *For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, c positive integers relatively prime with $c = a + 1$, then :*

$$c < K(\epsilon).rad(ac)^{1+\epsilon} \tag{348}$$

where K is a constant depending of ϵ equal to $e^{\left(\frac{1}{\epsilon^2}\right)}$.

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CHAPTER 13

A PROOF OF THE *ABC* CONJECTURE: CASE $c = a + 1$

Abstract. — In this paper, we consider the *abc* conjecture, case $c = a + 1$. As the conjecture $c < rad^2(abc)$ is less open, we give firstly the proof of a modified conjecture that is $c \leq 2rad^2(ac)$. The factor 2 is important for the proof of the new conjecture that represents the key of the proof of the main conjecture. Secondly, the proof of the *abc* conjecture is given for $\epsilon \geq 1$, then for $\epsilon \in]0, 1[$. We choose the constant $K(\epsilon)$ as $K(\epsilon) = 2e^{\left(\frac{1}{\epsilon^2}\right)}$ for $\epsilon \geq 1$ and $K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}$ for $\epsilon \in]0, 1[$. Some numerical examples are presented.

13.1. *Introduction and notations*

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a). \prod_i a_i^{\alpha_i - 1} \tag{349}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{350}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 17. — *(* ***abc Conjecture****): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{351}$$

where K is a constant depending only of ϵ.

The idea to try to write a paper about this conjecture was born after after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. So, I will give a simple proof in the case $c = a + 1$ that can be understood by undergraduate students.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**1**]. A conjecture was proposed that $c < rad^2(abc)$ [**?**]. It is the key to resolve the *abc* conjecture. In my paper, I propose to give the proof that $c \leq 2rad^2(ac)$, it facilitates the proof of the *abc* conjecture. The paper is organized as fellow: in the second section, we give the proof of $c \leq 2rad^2(ac)$. The main proof of the *abc* conjecture is presented in section three. The numerical examples are discussed in section four.

13.2. *The Proof of the conjecture* $c \leq 2rad^2(ac)$, *Case :* $c = a + 1$

Below is given the definition of the conjecture $c \leq 2rad^2(abc)$:

Conjecture 18. — *Let a, b, c positive integers relatively prime with $c = a+b$, then:*

$$c \leq 2rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} \leq 2 + \frac{Log2}{Lograd(abc)} \tag{352}$$

In the case $c = a + 1$, the definition of the conjecture is:

Definition 13.1. — *Let a, c positive integers, relatively prime, with $c = a+1, a \geq 2$ then:*

$$c \leq 2rad^2(ac) \Longrightarrow \frac{Logc}{Log(rad(ac))} \leq 2 + \frac{Log2}{Lograd(ac)} \tag{353}$$

1 - If $c < rad(ac)$ then we obtain:

$$c < rad(ac) < rad^2(ac) < 2rad^2(ac) \tag{354}$$

and the condition (353) is verified.

2 - If $c = rad(ac)$, then a, c are not relatively coprime. Case to reject.

3 - We suppose that $c > rad(ac) \Longrightarrow \mu_c > rad(a)$, we have also $a > rad(ac) \Longrightarrow \mu_a > rad(c)$.

3a - Case $\mu_a \leq rad(a)$: $c = 1 + a \leq 1 + rad^2(a) < rad^2(ac) < 2rad^2(ac)$.

3b - Case $\mu_c \leq rad(c)$: $c = \mu_c rad(c) \leq rad^2(c) < rad^2(ac) < 2rad^2(ac)$.

3c - Case $\mu_a > rad(a)$ and $\mu_c > rad(c)$. As $\mu_a > rad(c)$, we can write that $\mu_a = l.rad(c) + l'$ with $1 \leq l' < rad(c) \Longrightarrow \mu_a < (l+1)rad(c) \Longrightarrow a < (l+1)rad(c)$

3c1 - We suppose that $l + 1 \leq rad(ac) \Longrightarrow l < rad(ac)$ then $a < (l+1)rad(ac) \leq rad^2(ac) \Longrightarrow a < rad^2(ac) \Longrightarrow c \leq 2rad^2(ac)$.

3c2 - We suppose that $l = rad(ac) \Longrightarrow \mu_a = rad(a)rad^2(c) + l' < rad(c)(rad(ac) + 1) \Longrightarrow a < rad(ac)(rad(ac)+1) < 2rad^2(ac) \Longrightarrow a < 2rad^2(ac) \Longrightarrow c \leq 2rad^2(ac)$.

3c3 - Case $l > rad(ac)$

3c3 - 1 - Case $l = rad(ac) + l"$ with $1 \leq l" < rad(ac)$. Then, we write $\mu_a = lrad(c)+l' = rad(c)(rad(ac)+l")+l' \Longrightarrow \mu_a < rad(c)(rad(ac)+l")+rad(c) \Longrightarrow \mu_a < rad(c)(rad(ac) + l" + 1) \leq 2rad(c)rad(ac) \Longrightarrow a < 2rad^2(ac) \Longrightarrow c \leq 2rad^2(ac)$.

3c3 - 2 - Case $l = q.rad(ac) + q'$ with $q, q' \in \mathbb{N}, q \geq 2$ and $0 < q' < rad(ac)$. From $c = a+1 \Longrightarrow c = \mu_a.rad(a)+1 = (lrad(c)+l').rad(a)+1 = lrad(ac)+l'rad(a)+1 = rad(ac)(q.rad(ac) + q') + l'rad(a) + 1 = q.rad^2(ac) + q'rad(ac) + l'rad(a) + 1$. Let $R = rad(ac)$, we can write:

$$q.R^2 + q'R + l'rad(a) + 1 - c = 0 \tag{355}$$

Let $P(X)$ the polynomial $P(X) = q.X^2 + q'X + l'rad(a) + 1 - c = 0$, it has a positive integer root $X_1 = R$, then its discriminant $\Delta = q'^2 - 4q(l'rad(a) + 1 - c) = q'^2 + 4q(c - l'rad(a) - 1)$ is a positive integer. Noting that as $c > R \Longrightarrow c - l'rad(a) > 0 \Longrightarrow c \geq l'rad(a) + 1$, but c and a are coprime, then $c - l'rad(a) - 1 > 0$. As the root $X_1 = R \in \mathbb{N}$, we can write that $\Delta = m^2$ with $m > 0$ is an integer. We obtain the equation:

$$m^2 - q'^2 = 4q(c - l'rad(a) - 1) = N > 0 \tag{356}$$

Hence, m, q' are solutions of the Diophantine equation:

$$x^2 - y^2 = N \tag{357}$$

The roots of $P(X) = 0$ are:

$$X_1 = R = \frac{-q' + m}{2q} \Longrightarrow m - q' = 2Rq \tag{358}$$

$$X_2 = \frac{-q' - m}{2q} \tag{359}$$

From $m^2 - q'^2 = N$ and $m - q' = 2Rq$, we obtain:

$$m + q' = 2l, \quad N = 4qlR = 2qR.2l \tag{360}$$

Let $Q(N)$ indicate the number of the solutions of (357) and $\tau(N)$ the number of ways representing N as product of its factors, then we have (see the theorem 27.3 in [**4**]):
- if $N \equiv 2 (\bmod 4)$, then $Q(N) = 0$;
- if $N \equiv 1$ or $N \equiv 3 (\bmod 4)$, then $Q(N) = [\tau(N)/2]$;
- if $N \equiv 0 (\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.

Let (u, v), $u, v \in \mathbb{N}$ another couple of solutions of the equation (357), then $u^2 - v^2 = x^2 - y^2 = N = 4qlR$, but $m = x$ and $q' = y$ verify the equation $x - y = 2qR$, it follows that u, v verify also $u - v = 2qR$, that gives $u + v = 2l$, then $u = x = m$ and $v = q'$. If not, we have $u - v = 2qR + \lambda$, with $\lambda \neq 0 \in \mathbb{N} \Longrightarrow u = v + 2qR + \lambda$, from $u^2 - v^2 = N$, we obtain $v = \dfrac{N - (2qR + \lambda)^2}{2(2qR + \lambda)} \Longrightarrow u = \dfrac{N + (2qR + \lambda)^2}{2(2qR + \lambda)}$, but $u + v = \dfrac{N}{2qR + \lambda} \neq 2l \left(= \dfrac{N}{2qR} \right)$. So, we have given the proof of the uniqueness of the solutions of the equation (357) with the condition $m - q' = 2qR$. As $N = 4qlR \equiv 0 (\bmod 4) \Longrightarrow Q(N) = [\tau(N/4)/2] = [\tau(qlR)/2] > 1$, because a, c are not prime integers so we can suppose the case $c > R$, and we have $I + K \geq 4$ the total number of a_i, c_k. But $Q(N) = 1$, then the contradiction, the condition $l > rad(ac)$ is false and we obtain considering the above results $c \leq 2rad^2(ac)$.

3c3 -3 - Case $l = q.rad(ac)$ with $q \in \mathbb{N}, q \geq 2$. From $c = a + 1 \Longrightarrow c = \mu_a.rad(a) + 1 = (lrad(c) + l').rad(a) + 1 = lrad(ac) + l'rad(a) + 1 = q.rad^2(ac) + l'rad(a) + 1$. We can write:

$$q.rad^2(c).rad^2(a) + l'.rad(a) + 1 - c = 0 \tag{361}$$

Let $Q(Z)$ the polynomial $Q(Z) = q.rad^2(c)Z^2 + l'Z - a = 0$, it has a positive integer root $Z_1 = rad(a)$, then its discriminant $\Delta = l'^2 + 4aqrad^2(c)$ is a positive integer. As the root $Z_1 = rad(a) \in \mathbb{N}$, we can write that $\Delta = t^2$ with $t > 0$ is an integer. We obtain the equation:

$$t^2 - l'^2 = 4aqrad^2(c) = M > 0 \tag{362}$$

Hence, t, l' are solutions of the Diophantine equation:

$$x^2 - y^2 = M \tag{363}$$

The roots of $Q(Z) = 0$ are:

$$Z_1 = rad(a) = \frac{-l' + t}{2qrad^2(c)} \Longrightarrow t - l' = 2qrad(a)rad^2(c) \tag{364}$$

$$Z_2 = \frac{-l' - t}{2qrad^2(c)} \tag{365}$$

From $t^2 - l'^2 = M$ and $t - l' = 2qrad(a)rad^2(c)$, we obtain:

$$t + l' = 2\mu_a, \quad M = 4aqrad^2(c) \tag{366}$$

Let $S(M)$ indicate the number of the solutions of (363) and $\tau(M)$ the number of ways representing M as product of its factors, using the theorem 27.3 in [**4**]:
- if $M \equiv 2 (\bmod 4)$, then $S(M) = 0$;
- if $M \equiv 1$ or $M \equiv 3 (\bmod 4)$, then $S(M) = [\tau(M)/2]$;
- if $M \equiv 0 (\bmod 4)$, then $S(M) = [\tau(M/4)/2]$.

As seen in the case 3c3-2, equation (363) has an unique solution (t, l') and $M \equiv 0 (\bmod 4)$, we find the same contradiction and the hypothesis $l = qrad(ac)$ is false and we obtain considering the above results $c \leq 2rad^2(ac)$.

We announce the theorem:

Theorem 19. — *Let a, c positive integers relatively prime with $c = a + 1, a \geq 2$, then $c \leq 2rad^2(ac)$.*

13.3. *The Proof of The* abc *Conjecture (17) case* $c = a + 1$

13.3.1. Case: $\epsilon \geq 1$

Using the result of the theorem $c \leq 2rad^2(abc)$, we have $\forall \epsilon \geq 1$:

$$c \leq 2R^2 \leq 2R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = 2e^{\left(\frac{1}{\epsilon^2}\right)}, \; \epsilon \geq 1 \tag{367}$$

We verify easily that $K(\epsilon) > 2$ for $\epsilon \geq 1$. Then the *abc* conjecture is true.

13.3.2. Case: $\epsilon < 1$

13.3.2.1. Case: $c < R$

In this case, we can write :

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \; \epsilon < 1 \tag{368}$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the *abc* conjecture is true.

13.3.2.2. Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, 0 < \epsilon < 1 \tag{369}$$

If not, then $\exists \epsilon_0 \in]0, 1[$, so that the couple (a, c) checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{370}$$

are in finite number. We have:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow$$

$$R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) \geq \frac{c}{2}K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} \geq \geq \frac{K(\epsilon_0)}{2} \tag{371}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > R^{1-\epsilon_0} \geq \frac{K(\epsilon_0)}{2} \Longrightarrow$$

$$c^{1-\epsilon_0} > \frac{K(\epsilon_0)}{2} \Longrightarrow c > \left(\frac{K(\epsilon_0)}{2}\right)^{\left(\frac{1}{1-\epsilon_0}\right)} \tag{372}$$

We deduce that it exists an infinity of couples (a, c) verifying (370), hence the contradiction. Then the proof of the *abc* conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + 1$ with a, c relatively coprime, $a \geq 2$:

$$c < K(\epsilon).rad(a)^{1+\epsilon} \quad with \quad \begin{cases} K(\epsilon) = 2e^{\left(\frac{1}{\epsilon^2}\right)} & \epsilon \geq 1 \\ K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} & 0 < \epsilon < 1 \end{cases} \tag{373}$$

Q.E.D

13.4. *Examples*

In this section, we are going to verify some numerical examples. We find that $c < rad^2(ac) \Longrightarrow c \leq 2rad^2(ac)$ and our proposed conjecture is true.

13.4.1. Example 1

The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{374}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$, in this example, $\mu_a < rad(a)$.
$c = 19^6 = 47\,045\,880 \Rightarrow rad(c) = 19$. Then $rad(ac) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$.
We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,880$.

13.4.1.1. Case $\epsilon = 0.01$

$c < K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow 47\,045\,880 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8,7477777149120053120152473488653e + 4342 \tag{375}$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (373) is verified.

13.4.1.2. Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2,68793633096717542059170121288 76e+43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$, and the equation (373) is verified.

13.4.1.3. Case $\epsilon = 1$

$K(1) = 2e \Longrightarrow c = 47\,045\,880 < 2e.rad^2(ac) = 2 \times 697\,987\,143\,184,212$ and the equation (373) is verified.

13.4.1.4. Case $\epsilon = 100$

$$K(100) = 2e^{0.0001} \Longrightarrow c = 47\,045\,880 \overset{?}{<} 2e^{0.0001}.506\,730^{101} = 2 \times 1,52223502486076087818531426872 84e+576$$

and the equation (373) is verified.

13.4.2. Example 2

We give here the example 2 from $https://nitaj.users.lmno.cnrs.fr$:

$$3^7 \times 7^5 \times 13^5 \times 17 \times 1831 + 1 = 2^{30} \times 5^2 \times 127 \times 353 \tag{376}$$

$a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424\,808\,316\,456\,140\,799 \Rightarrow rad(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 8497671 \Longrightarrow \mu_a > rad(a)$,
$b = 1$, $rad(c) = 2 \times 5 \times 127 \times 353$ Then $rad(ac) = 849767 \times 448310 = 3\,809\,590\,886\,010 < c$. $rad^2(ac) = 14\,512\,982\,718\,770\,456\,813\,720\,100 > c$, then $c \leq 2rad^2(ac)$. For example, we take $\epsilon = 0.5$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{1/0.25} = e^4 = 54,59800313096579789056 \tag{377}$$

Let us verify (373):

$$c \overset{?}{<} K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow c = 424808316456140800 \overset{?}{<} K(0.5) \times (3\,809\,590\,886\,010)^{1.5} \Longrightarrow$$

$$424808316456140800 < 405970304762905691174,98260818045 \tag{378}$$

Hence (373) is verified.

Ouf, end of the mystery!

13.5. *Conclusion*

We have given an elementary proof of the *abc* conjecture in the case $c = a + 1$, confirmed by some numerical examples. We can announce the important theorem:

Theorem 20. — *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2019) For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(ac)^{1+\epsilon} \tag{379}$$

where K is a constant depending of ϵ proposed as :

$$\left\{\begin{array}{ll} K(\epsilon) = 2e^{\left(\dfrac{1}{\epsilon^2}\right)} & \epsilon \geq 1 \\ K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)} & 0 < \epsilon < 1 \end{array}\right.$$

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CHAPTER 14

A FINAL PROOF OF THE *abc* CONJECTURE

Abstract. — In this paper, we consider the *abc* conjecture. As the conjecture $c < rad^2(abc)$ is less open, we give firstly the proof of a modified conjecture that is $c < 2rad^2(abc)$. The factor 2 is important for the proof of the new conjecture that represents the key of the proof of the main conjecture. Secondly, the proof of the *abc* conjecture is given for $\epsilon \geq 1$, then for $\epsilon \in]0,1[$. We choose the constant $K(\epsilon)$ as $K(\epsilon) = 2e^{\left(\frac{1}{\epsilon^2}\right)}$ for $\epsilon \geq 1$ and $K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}$ for $\epsilon \in]0,1[$. Some numerical examples are presented.

To the memory of my Father who taught me arithmetic
To the memory of my colleague and friend Jamel Zaiem (1956-2019)

14.1. *Introduction and notations*

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{380}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{381}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 19. — *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{382}$$

where K is a constant depending only of ϵ.

The idea to try to write a paper about this conjecture was born after after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. So, I will give a simple proof in the two cases $c = a+1$ and $c = a+b$ that can be understood by undergraduate students.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**1**]. A conjecture was proposed that $c < rad^2(abc)$ [**?**]. It is the key to resolve the *abc* conjecture. In my paper, I propose to give the proof that $c < 2rad^2(abc)$, it facilitates the proof of the *abc* conjecture. The paper is organized as fellow: in the second and third section, we give successively the proof of $c < 2rad^2(ac)$ and $c < 2rad^2(abc)$. The main proof of the *abc* conjecture is presented in section four three for the two cases $c = a + 1$ and $c = a + b$. The numerical examples are discussed in sections five and six.

14.2. *The Proof of the conjecture* $c < 2rad^2(ac)$*, case :* $c = a + 1$

Below is given the definition of the conjecture $c < 2rad^2(abc)$:

Conjecture 20. — *Let a, b, c positive integers relatively prime with $c = a + b, a > b, b \geq 2$, then:*

$$c < 2rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 + \frac{Log2}{Log(rad(abc))} \tag{383}$$

In the case $c = a + 1$, the definition of the conjecture is:

Definition 14.1. — *Let a, c positive integers, relatively prime, with $c = a+1, a \geq 2$ then:*

$$c < 2rad^2(ac) \Longrightarrow \frac{Logc}{Log(rad(ac))} < 2 + \frac{Log2}{Log(rad(ac))} \tag{384}$$

1 - If $c < rad(ac)$ then we obtain:

$$c < rad(ac) < 2rad^2(ac) \tag{385}$$

and the condition (384) is verified.

2 - If $c = rad(ac)$, then a, c are not relatively coprime. Case to reject.

3 - We suppose that $c > rad(ac) \Longrightarrow \mu_c > rad(a)$, we have also $a > rad(ac) \Longrightarrow \mu_a > rad(c)$.

3a - Case $\mu_a \leq rad(a)$: $c = 1 + a \leq 1 + rad^2(a) < rad^2(ac) < 2rad^2(ac)$, and the condition (384) is verified.

3b - Case $\mu_c \leq rad(c)$: $c = \mu_c rad(c) \leq rad^2(c) < rad^2(ac) < 2rad^2(ac)$, and the condition (384) is verified.

3c - Case $\mu_a > rad(a)$ and $\mu_c > rad(c)$. As $\mu_a > rad(c)$, we can write that $\mu_a = l.rad(c) + l'$ with $1 \leq l' < rad(c) \Longrightarrow \mu_a < (l+1)rad(c) \Longrightarrow a < (l+1)rad(ac)$

3c1 - We suppose that $l + 1 \leq rad(ac) \Longrightarrow l < rad(ac)$ then $a < (l+1)rad(ac) \leq rad^2(ac) \Longrightarrow c < 2rad^2(ac)$, and the condition (384) is verified.

3c2 - We suppose that $l = rad(ac) \Longrightarrow \mu_a = rad(a)rad^2(c) + l' < rad(c)(rad(ac) + 1) \Longrightarrow a < rad(ac)(rad(ac)+1) < 2rad^2(ac) \Longrightarrow a < 2rad^2(ac) \Longrightarrow c \leq 2rad^2(ac)$. As c can not be equal to $2rad^2(ac)$, we obtain $c < 2rad^2(ac)$ and the condition (384) is verified.

3c3 - Case: $l > rad(ac)$. As $\mu_a = lrad(c) + l' \Longrightarrow \mu_a > rad(a)rad^2(c)$, we can write that $\mu_a = m.rad(a)rad^2(c) + r$ with $m, r \in \mathbb{N}, m \geq 1$ and $0 < r < rad(a)rad^2(c)$. Then:

$$\mu_a = m.rad(a)rad^2(c) + r \Longrightarrow a = \mu_a.rad(a) = m.rad^2(a)rad^2(c) + r.rad(a) \Longrightarrow$$
$$a < mrad^2(ac) + rad^2(ac) \Longrightarrow a < (m+1)rad^2(ac) \quad with\ m \geq 1 \Longrightarrow$$
$$a < (1+1)rad^2(ac) \Longrightarrow a < 2rad^2(ac) \Longrightarrow a + 1 = c \leq 2rad^2(ac) \tag{386}$$

As c can not be equal to $2rad^2(ac)$, we deduce that $c < 2rad^2(ac)$ and the condition (384) is verified.
We announce the theorem:

Theorem 21. — *Let a, c positive integers relatively prime with $c = a + 1, a \geq 2$, then $c < 2rad^2(ac)$.*

14.3. *The Proof of the conjecture* $c < 2rad^2(abc)$, *case :* $c = a + b$

Below is given the definition of the conjecture $c < 2rad^2(abc)$:

Conjecture 21. — *Let a, b, c positive integers relatively prime with $c = a + b, a > b, b \geq 2$, then:*

$$c < 2rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 + \frac{Log2}{Log(rad(abc))} \tag{387}$$

4 - If $c < rad(abc)$ then we obtain:

$$c < rad(abc) < rad^2(abc) < 2rad^2(abc) \tag{388}$$

and the condition (387) is verified.

5 - If $c = rad(abc)$, then a, b, c are not relatively coprime. Case to reject.

6 - We suppose that $c > rad(abc) \Longrightarrow \mu_c > rad(ab)$, we can write :

$$\mu_c = lrad(ab) + l', \quad with \quad 0 < l' < rad(ab) \Longrightarrow$$
$$\mu_c < lrad(ab) + rad(ab) = (l+1)rad(ab) \Longrightarrow c < (l+1)rad(abc) \tag{389}$$

6a - Case $l + 1 \leq rad(abc) \Longrightarrow l < rad(abc)$, then $c < rad^2(abc) < 2rad^2(abc) \Longrightarrow c < 2rad^2(abc)$ and the condition (387) is verified.

6b - Case $l = rad(abc)$: From $c < (l+1)rad(abc) \Longrightarrow c < rad(abc)(rad(abc)+1) < 2rad^2(abc)$, then $c < 2rad^2(abc)$ and the condition (387) is verified.

6c - Case $l > rad(abc)$: From $\mu_c = lrad(ab)+l'$, we deduce that $\mu_c > rad^2(ab)rad(c)$, so we can write:

$$\mu_c = mrad^2(ab)rad(c) + r \quad m \geq 1, 0 < r < rad^2(ab)rad(c) \Longrightarrow$$
$$\mu_c < (m+1)rad^2(ab)rad(c), m \geq 1 \Longrightarrow c < (m+1)rad^2(abc)$$
$$Taking\ \ m = 1 \Longrightarrow c < 2rad^2(abc) \tag{390}$$

And the condition (387) is verified.
We announce the theorem:

Theorem 22. — *Let a, b, c positive integers relatively prime with $c = a + b, a > b, b \geq 2$, then $c < 2rad^2(abc)$.*

14.4. *The Proof of the abc conjecture*

Let $R = rad(ac)$ or $R = rad(abc)$.

14.4.1. Case : $\epsilon \geq 1$

Using the result that $c < 2rad^2(ac)$ or $c < 2rad^2(abc)$, we have $\forall \epsilon \geq 1$:

$$c < 2R^2 \leq 2R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with\ \ K(\epsilon) = 2e^{\left(\frac{1}{\epsilon^2}\right)},\ \epsilon \geq 1 \tag{391}$$

We verify easily that $K(\epsilon) > 2$ for $\epsilon \geq 1$. Then the abc conjecture is true.

14.4.2. Case: $\epsilon < 1$

14.4.2.1. Case: $c < R$

In this case, we can write :

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with\ \ K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)},\ \epsilon < 1 \tag{392}$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the *abc* conjecture is true.

14.4.2.2. Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad with\ \ K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, 0 < \epsilon < 1 \tag{393}$$

If not, then $\exists \epsilon_0 \in]0, 1[$, so that the triplet (a, b, c) checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{394}$$

are in finite number. We have:

$$\begin{gathered} c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow \\ R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > \frac{c}{2}K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > \frac{K(\epsilon_0)}{2} \end{gathered} \tag{395}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > R^{1-\epsilon_0} > \frac{K(\epsilon_0)}{2} \Longrightarrow$$

$$c^{1-\epsilon_0} > \frac{K(\epsilon_0)}{2} \Longrightarrow c > \left(\frac{K(\epsilon_0)}{2}\right)^{\left(\frac{1}{1-\epsilon_0}\right)} \tag{396}$$

We deduce that it exists an infinity of triplets (a, b, c) verifying (394), hence the contradiction. Then the proof of the *abc* conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with a, b, c relatively coprime:

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad with \quad \left\{ \begin{array}{ll} K(\epsilon) = 2e^{\left(\frac{1}{\epsilon^2}\right)} & \epsilon \geq 1 \\ K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} & 0 < \epsilon < 1 \end{array} \right. \tag{397}$$

Q.E.D

In the two following sections, we are going to verify some numerical examples. We find that $c < rad^2(abc) \Longrightarrow c < 2rad^2(abc)$ and our proposed conjecture is true.

14.5. *Examples : Case* $c = a + 1$

14.5.1. Example 1

The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{398}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$, in this example, $\mu_a < rad(a)$.
$c = 19^6 = 47\,045\,880 \Rightarrow rad(c) = 19$. Then $rad(ac) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$.
We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,880$.

14.5.1.1. Case $\epsilon = 0.01$

$c < K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow 47\,045\,880 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8,7477777149120053120152473488653e + 4342 \tag{399}$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (397) is verified.

14.5.1.2. Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2,6879363309671754205917012128876e + 43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$, and the equation (397) is verified.

14.5.1.3. Case $\epsilon = 1$

$K(1) = 2e \Longrightarrow c = 47\,045\,880 < 2e.rad^2(ac) = 2 \times 697\,987\,143\,184,212$ and the equation (397) is verified.

14.5.1.4. Case $\epsilon = 100$

$$K(100) = 2e^{0.0001} \Longrightarrow c = 47\,045\,880 \overset{?}{<} 2e^{0.0001}.506\,730^{101} = \\ 2 \times 1,5222350248607608781853142687284e + 576$$

and the equation (397) is verified.

14.5.2. Example 2

We give here the example 2 from $https://nitaj.users.lmno.cnrs.fr$:

$$3^7 \times 7^5 \times 13^5 \times 17 \times 1831 + 1 = 2^{30} \times 5^2 \times 127 \times 353 \tag{400}$$

$a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424\,808\,316\,456\,140\,799 \Rightarrow rad(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 8497671 \Longrightarrow \mu_a > rad(a)$,

$b = 1$, $rad(c) = 2 \times 5 \times 127 \times 353$ Then $rad(ac) = 849767 \times 448310 = 3\,809\,590\,886\,010 < c$. $rad^2(ac) = 14\,512\,982\,718\,770\,456\,813\,720\,100 > c$, then $c \leq 2rad^2(ac)$. For example, we take $\epsilon = 0.5$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{1/0.25} = e^4 = 54,59800313096579789056 \tag{401}$$

Let us verify (397):

$$c \stackrel{?}{<} K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow c = 424808316456140800 \stackrel{?}{<} K(0.5) \times (3\,809\,590\,886\,010)^{1.5} \Longrightarrow$$
$$424808316456140800 < 405970304762905691174,98260818045 \tag{402}$$

Hence (397) is verified.

14.6. *Examples : Case* $c = a + b$

14.6.1. Example 1

We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{403}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$. For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{9999.99} = 8,7477777149120053120152473488653e + 4342 \tag{404}$$

Let us verify (397):

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \stackrel{?}{<} K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow$$
$$6436343 \ll K(0.01) \times 15042^{1.01} \tag{405}$$

Hence (397) is verified.

14.6.2. Example 2

The example of Nitaj about the ABC conjecture [**1**] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \tag{406}$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \tag{407}$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \tag{408}$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \tag{409}$$

14.6.2.1. Case 1

we take $\epsilon = 100$ we have:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 2e^{0.0001}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 2 \times 2.7657949971494838920022381186039e + 1359$$

then (397) is verified.

14.6.2.2. Case 2

We take $\epsilon = 0.5$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \tag{410}$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^{4}.(2.3.5.7.11.13.41.79.311.953)^{1.5} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 8\,450\,961\,319\,227\,998\,887\,403,9993 \tag{411}$$

We obtain that (397) is verified.

14.6.2.3. Case 3

We take $\epsilon = 1$, then

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 2e.(2.3.5.7.11.13.41.79.311.953)^{2} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 831\,072\,936\,124\,776\,471\,158\,132\,100 \times 2e \tag{412}$$

We obtain that (397) is verified.

14.6.3. Example 3

It is of Ralf Bonse about the ABC conjecture [**3**] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \tag{413}$$
$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983$$
$$rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$rad(abc) = 1.5683959920004546031461002610848e + 33 \tag{414}$$

14.6.3.1. Case 1

For example, we take $\epsilon = 10$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = 2e^{0.01} = 2.0156314808565913486409234833354$$

Let us verify (397):

$$\begin{aligned} c &\overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<} \\ &2e^{0.01}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{11} \\ &\Longrightarrow 3.4136998783296235160378273576498e+44 < \\ &2.8472401192989816352016241851442e+365 \end{aligned} \tag{415}$$

The equation (397) is verified.

14.6.3.2. Case 2

We take $\epsilon = 0.4 \Longrightarrow K(\epsilon) = 12.182473474251512159126256669608$, then: The

$$\begin{aligned} c &\overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<} \\ &e^{6.25}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.4} \\ &\Longrightarrow 3.4136998783296235160378273576498e+44 < \\ &3.6255465680011453642792720569685e+47 \end{aligned} \tag{416}$$

And the equation (397) is verified.

Ouf, end of the mystery!

14.7. *Conclusion*

We have given an elementary proof of the *abc* conjecture, confirmed by some numerical examples. We can announce the important theorem:

Theorem 23. — *Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{417}$$

where K is a constant depending of ϵ proposed as :

$$\begin{cases} K(\epsilon) = 2e^{\left(\frac{1}{\epsilon^2}\right)} & \epsilon \geq 1 \\ K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} & 0 < \epsilon < 1 \end{cases}$$

.

Acknowledgements : The author is very grateful to Professors Mihăilescu Preda and Gérald Tenenbaum for their comments about errors found in previous manuscripts concerning proofs proposed of the *abc* conjecture.

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[3] Mihăilescu P.: Around ABC. European Mathematical Society Newsletter **N° 93**, September 2014. 29-34, (2014)

CHAPTER 15

A DEFINITIVE PROOF OF THE *ABC* CONJECTURE

Abstract. — In this paper, we consider the *abc* conjecture. Firstly, we give an elementary proof the conjecture $c < rad^2(abc)$. Secondly, the proof of the *abc* conjecture is given for $\epsilon \geq 1$, then for $\epsilon \in]0,1[$. We choose the constant $K(\epsilon)$ as $K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}$. Some numerical examples are presented.

To the memory of my Father who taught me arithmetic
To the memory of my colleague and friend Jamel Zaiem (1956-2019)

15.1. *Introduction and notations*

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{418}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{419}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 22. — *(* ***abc Conjecture****): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{420}$$

where K is a constant depending only of ϵ.

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**1**]. A conjecture was proposed that $c < rad^2(abc)$ [**3**]. It is the key to resolve the *abc* conjecture. In my paper, I propose an elementary proof of it, it facilitates the proof of the *abc* conjecture. In the second section, we give the proof that $c < rad^2(abc)$. We present the proof of the *abc* conjecture in section three. The numerical examples are discussed in sections four and five.

15.2. *The Proof of the conjecture* $c < rad^2(abc)$

Below is given the definition of the conjecture $c < rad^2(abc)$:

Conjecture 23. — *Let a, b, c positive integers relatively prime with $c = a + b, a > b, b \geq 2$, then:*

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{421}$$

We note $R = rad(abc)$ in the case $c = a + b$ or $R = rad(ac)$ in the case $c = a + 1$.

As c is bounded, it exists an unique couple $(m, n) \in \mathbb{Z}^+ \times \mathbb{N}$, $n \geq m+1 \geq 1$, so that:

$$m.R^2 < c < n.R^2 \tag{422}$$

We can write:

$$c = mR^2 + r, \quad 1 \leq r < R^2 \tag{423}$$

$$c = nR^2 - r', \quad 1 \leq r' < R^2 \tag{424}$$

But $mR^2 + r = nR^2 - r' \Longrightarrow 2 \leq r + r' = (n - m)R^2 < 2R^2 \Longrightarrow n - m < 2$, we deduce $n = m$ or $n = m + 1$. The case $n = m$ presents a contradiction. Hence $n = m + 1$. The equation (422) becomes:

$$m.R^2 < c < (m + 1).R^2, \quad m \geq 0 \tag{425}$$

15.2.1. Proof that $c < R^2$

** Case $c < R$: $c < R < R^2$ and the condition (421) is verified.

** Case $c = R$: case to reject.

** Case $c > R$: from (425), we obtain:

$$mR^2 < c < (m+1)R^2 \tag{426}$$

If $m = 0$, we deduce that :

$$0 < c < R^2 \tag{427}$$

and the condition (421) is verified.

We suppose now that $m > 0$. Let $c = a + b$ or $c = a + 1$ so that:

$$mR^2 < c < (m+1)R^2$$

As $c > mR^2$, we can write :

$$c = mR^2 + m', \quad m' < R^2 \tag{428}$$

But $c > R \Longrightarrow c^2 > R^2$, we obtain also:

$$c^2 = lR^2 + l', \quad l' < R^2 \tag{429}$$

From the above equations, we can write:

$$(mR^2 + m')^2 = lR^2 + l' \Longrightarrow m^2R^4 + (2mm' - l)R^2 + m'^2 - l' = 0 \tag{430}$$

From the last equation above, R^2 is the positive root of the polynomial of the second degree:

$$F(T) = m^2T^2 + (2mm' - l)T + m'^2 - l' = 0 \tag{431}$$

The discriminant of $F(T)$ is:

$$\Delta = (2mm' - l)^2 - 4m^2(m'^2 - l') \tag{432}$$

As a real root of $F(T)$ exists, and it is an integer, Δ is written as :

$$\Delta = t^2 \geq 0, t \in \mathbb{Z}^+ \tag{433}$$

** - Case $\Delta = 0$ and $m'^2 - l' \neq 0$: Then $(2mm' - l)^2 = 4m^2(m'^2 - l') \Longrightarrow m'^2 - l' = \alpha^2$,$\alpha \in \mathbb{N}$. In this case the equation (431 has a double root $T_1 = T_2 = \dfrac{l - 2mm'}{2m^2} = R^2 \Longrightarrow l - 2mm' = 2m^2R^2 > 0$. But $(l - 2mm')^2 = 4m^4R^4 = 4m^2(m'^2 - l') \Longrightarrow m'^2 = m^2R^4 + l' > R^4 \Longrightarrow m' > R^2$. Then the contradiction as $m' < R^2$. The case $\Delta = 0$ and $m'^2 - l' \neq 0$ is impossible.

** - Case $\Delta = 0$ and $m'^2 - l' = 0$: In this case, $2mm' - l = 0 \Longrightarrow R^2 = 0$. Then the contradiction as $R > 0$. The case $\Delta = 0$ and $m'^2 - l' = 0$ is impossible.

** - Case $\Delta > 0$ and $m'^2 - l' = 0$: The equation (431) becomes:

$$F(T) = m^2T^2 + (2mm' - l)T = 0 \Longrightarrow \begin{cases} T_1 = 0 \\ T_2 = \dfrac{l - 2mm'}{m^2} = R^2 \end{cases} \tag{434}$$

Then, we have:

$$l - 2mm' = m^2R^2 \Longrightarrow l = 2mm' + m^2R^2$$

As $m' < R^2 \Longrightarrow l - m^2R^2 < 2mR^2 \Longrightarrow l < 2mR^2 + m^2R^2$, we obtain $lR^2 < m(2+m)R^4$. We deduce that $c^2 = lR^2 + l' < m(2+m)R^4 + R^2$. We know that $c < (m+1)R^2 \Longrightarrow c^2 < (m+1)^2R^4$. We verify easily that $m(2+m)R^4 + R^2 < (m+1)^2R^4$, then the contradiction with $mR^2 < c < (m+1)R^2$. Hence, the case $\Delta > 0$ and $m'^2 - l' = 0$ is impossible.

** - Case $\Delta > 0$ and $m'^2 - l' > 0$: We have: $\Delta = (2mm' - l)^2 - 4m^2(m'^2 - l') = t^2 \Longrightarrow t^2 < (2mm' - l)^2$. Let the case $|2mm' - l| = 2mm' - l \Longrightarrow t < 2mm' - l$. The expression of the two roots are:

$$\begin{cases} T_1 = \dfrac{l - 2mm' + t}{2m^2b} < 0 \\ \\ T_2 = \dfrac{l - 2mm' - t}{2m^2} < 0 \end{cases} \tag{435}$$

As $R^2 > 0$ is a root of $F(T) = 0$, then the contradiction. Hence, the case $\Delta > 0$ and $m'^2 - l' > 0$ is impossible.

** - Case $\Delta > 0$ and $m'^2 - l' < 0$: From $m'^2 < l' \Longrightarrow (c - mR^2)^2 < c^2 - lR^2$, it gives $m^2R^2 + l - 2mc < 0 \Longrightarrow m^2R^2 + l < 2mc < 2m(m+1)R^2$. Then we obtain $l < m^2R^2 + 2mR^2 \Longrightarrow lR^2 < m(m+2)R^4 \Longrightarrow c^2 = lR^2 + l' < m(m+2)R^4 + R^2$. We know that $c < (m+1)R^2 \Longrightarrow c^2 < (m+1)^2R^4$. We verify easily that $m(2+m)R^4 + R^2 < (m+1)^2R^4$, then the contradiction with $mR^2 < c < (m+1)R^2$. Hence, the case $\Delta > 0$ and $m'^2 - l' < 0$ is impossible.

All the cases for the resolution of the equation (431) have given contradictions with the hypothesis $c > mR^2, m > 0$. Then we obtain that $m = 0$ and $0 < c < R^2$. Hence the condition (421) is verified.

We announce the theorem:

Theorem 24. — *Let a, b, c positive integers relatively prime with $c = a + b, a > b$, then $c < rad^2(abc)$.*

15.3. *The Proof of the* abc *conjecture*

15.3.1. Case : $\epsilon \geq 1$

Using the result that $c < R^2$, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \;\; \epsilon \geq 1 \tag{436}$$

We verify easily that $K(\epsilon) > 1$ for $\epsilon \geq 1$. Then the *abc* conjecture is true.

15.3.2. Case: $\epsilon < 1$

15.3.2.1. Case: $c < R$

In this case, we can write :

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \;\; \epsilon < 1 \tag{437}$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the *abc* conjecture is true.

15.3.2.2. Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, 0 < \epsilon < 1 \tag{438}$$

If not, then $\exists \epsilon_0 \in]0, 1[$, so that the triple (a, b, c) checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{439}$$

are in finite number. We have:

$$\begin{aligned} & c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow \\ & R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > cK(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > K(\epsilon_0) \end{aligned} \tag{440}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow$$

$$c^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow c > (K(\epsilon_0))^{\left(\frac{1}{1-\epsilon_0}\right)} \tag{441}$$

We deduce that it exists an infinity of triples (a, b, c) verifying (439), hence the contradiction. Then the proof of the *abc* conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with a, b, c relatively coprime:

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad with \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \quad \epsilon > 0 \tag{442}$$

Q.E.D

In the two following sections, we are going to verify some numerical examples.

15.4. *Examples : Case* $c = a + 1$

15.4.1. Example 1

The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{443}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$, in this example, $\mu_a < rad(a)$.
$c = 19^6 = 47\,045\,881 \Rightarrow rad(c) = 19$. Then $rad(ac) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$.
We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,881$.

15.4.1.1. Case $\epsilon = 0.01$

$c < K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow 47\,045\,881 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8.7477777149120053120152473488653e + 4342 \tag{444}$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (694) is verified.

15.4.1.2. Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2.6879363309671754205917012128876e + 43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$, and the equation (694) is verified.

15.4.1.3. Case $\epsilon = 1$

$K(1) = e \Longrightarrow c = 47\,045\,881 < e.rad^2(ac) = 697\,987\,143\,184,212$ and the equation (694) is verified.

15.4.1.4. Case $\epsilon = 100$

$$K(100) = e^{0.0001} \Longrightarrow c = 47\,045\,881 \overset{?}{<} e^{0.0001}.506\,730^{101} = 1.5222350248607608781853142687284e + 576$$

and the equation (694) is verified.

15.4.2. Example 2

We give here the example 2 from $https://nitaj.users.lmno.cnrs.fr$:

$$3^7 \times 7^5 \times 13^5 \times 17 \times 1831 + 1 = 2^{30} \times 5^2 \times 127 \times 353 \tag{445}$$

$a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424\,808\,316\,456\,140\,799 \Rightarrow rad(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 8497671 \Longrightarrow \mu_a > rad(a)$,
$b = 1$, $rad(c) = 2 \times 5 \times 127 \times 353$ Then $rad(ac) = 849767 \times 448310 = 3\,809\,590\,886\,010 < c$. $rad^2(ac) = 14\,512\,982\,718\,770\,456\,813\,720\,100 > c$, then $c \leq 2rad^2(ac)$. For example, we take $\epsilon = 0.5$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{1/0.25} = e^4 = 54.598003130965797890\,56 \tag{446}$$

Let us verify (694):

$$c \stackrel{?}{<} K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow c = 424808316456140800 \stackrel{?}{<} K(0.5) \times (3\,809\,590\,886\,010)^{1.5} \Longrightarrow$$
$$424808316456140800 < 405970304762905691174.98260818045 \tag{447}$$

Hence (694) is verified.

15.5. *Examples : Case* $c = a + b$

15.5.1. Example 1

We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{448}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$. For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{9999.99} = 8.7477777149120053120152473488653e + 4342 \tag{449}$$

Let us verify (694):

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \stackrel{?}{<} K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow$$
$$6436343 \ll K(0.01) \times 15042^{1.01} \tag{450}$$

Hence (694) is verified.

15.5.2. Example 2

The example of Nitaj about the ABC conjecture [**1**] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \tag{451}$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \tag{452}$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \tag{453}$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \tag{454}$$

15.5.2.1. Case 1

we take $\epsilon = 100$ we have:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^{0.0001}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 2.7657949971494838920022381186039e + 1359$$

then (694) is verified.

15.5.2.2. Case 2

We take $\epsilon = 0.5$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \tag{455}$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^{4}.(2.3.5.7.11.13.41.79.311.953)^{1.5} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 8\,450\,961\,319\,227\,998\,887\,403,9993 \tag{456}$$

We obtain that (694) is verified.

15.5.2.3. Case 3

We take $\epsilon = 1$, then

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e.(2.3.5.7.11.13.41.79.311.953)^{2} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 831\,072\,936\,124\,776\,471\,158\,132\,100 \times e \tag{457}$$

We obtain that (694) is verified.

15.5.3. Example 3

It is of Ralf Bonse about the ABC conjecture [**3**] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \tag{458}$$

$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983$$
$$rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$rad(abc) = 1.5683959920004546031461002610848e + 33 \tag{459}$$

15.5.3.1. Case 1

For example, we take $\epsilon = 10$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{0.01} = 1.007815740428295674320461741677$$

Let us verify (**??**):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<}$$
$$e^{0.01}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{11}$$
$$\Longrightarrow 3.4136998783296235160378273576498e + 44 <$$
$$1.4236200596494908176008120925 72e + 365 \tag{460}$$

The equation (694) is verified.

15.5.3.2. Case 2

We take $\epsilon = 0.4 \Longrightarrow K(\epsilon) = 12.182473474251512159126256696 08$, then: The

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<}$$
$$e^{6.25}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.4}$$
$$\Longrightarrow 3.4136998783296235160378273576498e + 44 <$$
$$3.6255465680011453642792720569685e + 47 \tag{461}$$

And the equation (694) is verified.

Ouf, end of the mystery!

15.6. *Conclusion*

We have given an elementary proof of the *abc* conjecture, confirmed by some numerical examples. We can announce the important theorem:

Theorem 25. — *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2019) For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{462}$$

where K is a constant depending of ϵ proposed as :

$$K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)} \quad , \epsilon > 0$$

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CHAPTER 16

IF $c < R.exp(\frac{3\sqrt[3]{2}}{2}Log^{2/3}R) \Longrightarrow$ THE *ABC* CONJECTURE TRUE

Assuming $c < R.exp(\frac{3\sqrt[3]{2}}{2}Log^{2/3}R)$ - A New Conjecture - Implies The abc Conjecture True

Abstract. — In this paper about the *abc* conjecture, we propose a new conjecture about an upper bound for c as $c < R.exp(\frac{3\sqrt[3]{2}}{2}Log^{2/3}R)$. Assuming the last condition holds, we give the proof of the *abc* conjecture by proposing the expression of the constant $K(\epsilon)$, then we approve that $\forall \epsilon > 0$, for a, b, c positive integers relatively prime with $c = a + b$, we have $c < K(\epsilon).rad^{1+\epsilon}(abc)$. Some numerical examples are given.

16.1. *Introduction and notations*

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{463}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{464}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 24. — *(* **abc Conjecture***): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{465}$$

where K is a constant depending only of ϵ.

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**1**]. A conjecture was proposed that $c < rad^2(abc)$ [**3**]. It is the key to resolve the *abc* conjecture. In my paper, I propose the constant $K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)}$ and assuming that $c < R.exp(\frac{3\sqrt[3]{2}}{2}Log^{2/3}R)$ the new conjecture more stronger than $c < R^2$. In my proof of the *abc* conjecture, we will find that c must verify $c < R.exp(\frac{3\sqrt[3]{2}}{2}Log^{2/3}R)$ so we will obtain that the *abc* conjecture is true. The paper is organized as follows: in the second section, we give the proof of the *abc* conjecture. In sections three and four, we present some numerical examples respectively for the cases $c = a + 1$ and $c = a + b$.

16.2. *The Proof of the* abc *conjecture*

Let a, b, c (respectively a, c) positive integers relatively prime with $c = a + b, a > b, b \geq 2$ (respectively $c = a + 1$, $a \geq 2$). We note $R = rad(abc)$ in the case $c = a + b$ or $R = rad(ac)$ in the case $c = a + 1$. I propose the constant $K(\epsilon)$ as:

$$K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)} > 1, \forall \epsilon > 0 \tag{466}$$

16.2.1. Case $c < R$:

As $c < R \Longrightarrow c < K(\epsilon).R^{1+\epsilon}, \forall \epsilon > 0$ since $K(\epsilon) > 1$ and the conjecture (36) is verified.

16.2.2. Case $c = R$

Case to reject as a, b, c (respectively a, c) are relatively prime.

16.2.3. Case $R < c$

In this case, we have $c/R > 1 \Longrightarrow Log(c/R) > 0$. Let for $\forall \epsilon > 0$:

$$y(\epsilon) = \frac{1}{\epsilon^2} + (1 + \epsilon)LogR - Logc \tag{467}$$

Our main task is give the proof that $y(\epsilon) > 0 \Longrightarrow \dfrac{1}{\epsilon^2} + (1+\epsilon)LogR > Logc$, then $\Longrightarrow Logc < \dfrac{1}{\epsilon^2} + (1+\epsilon)LogR$ and we obtain $c < e^{\left(\frac{1}{\epsilon^2}\right)}.R^{1+\epsilon} = K(\epsilon).R^{1+\epsilon}, \forall\epsilon > 0$.

We have also:

$$lim_{\epsilon\longrightarrow 0}y(\epsilon) = +\infty \tag{468}$$

$$lim_{\epsilon\longrightarrow +\infty}y(\epsilon) = +\infty \tag{469}$$

For $\epsilon > 0$, the function derivative $y'(\epsilon)$ is given by:

$$y'(\epsilon) = -\frac{2}{\epsilon^3} + LogR = \frac{\epsilon^3 LogR - 2}{\epsilon^3} \tag{470}$$

$y'(\epsilon) = 0$ gives:

$$\epsilon_0 = \sqrt[3]{\frac{2}{LogR}} \leq \sqrt[3]{\frac{2}{Log6}} \approx 1.03733 \tag{471}$$

If $R \nearrow$, then $\epsilon_0 \rightarrow 0$. For $\epsilon = \epsilon_0$, we obtain:

$$y(\epsilon_0) = \frac{1}{\epsilon_0^2} + (1+\epsilon_0)LogR - Logc = LogR + \frac{3}{2}\sqrt[3]{2}Log^{2/3}R - Logc \tag{472}$$

$y(\epsilon_0)$ is positive if $LogR + \dfrac{3}{2}\sqrt[3]{2}Log^{2/3}R - Logc > 0$. So we assume that :

$$c < R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R) \Longrightarrow y(\epsilon_0) > 0 \tag{473}$$

Then the new conjecture proposed by us is :

$$\boxed{c < R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R)} \tag{474}$$

From (468-469), the point $(\epsilon_0, y(\epsilon_0))$ is the minimum of the curve $y(\epsilon)$ for all $\epsilon > 0$. Then $y(\epsilon) > 0$ and the proof of the *abc* conjecture is finished. We obtain that $\forall\epsilon > 0$, $c = a + b$ with a, b, c relatively coprime:

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \quad with \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad \epsilon > 0 \tag{475}$$

Remark 16.1. — We verify that $R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}) < R^{1+2/3}$ for R large, $R >$ 7 830 169 545.

Remark 16.2. — Nowadays, we know numerically [**1**] that $\dfrac{Logc}{LogR} \leq 1.629912 <$ $1 + 2/3 \approx 1.666667$. All the numerical examples below verify $c < R^{1+2/3}$, so, I would suggest that $c < R^{1+2/3}$ as a new open conjecture that it is more difficult than $c < R^2$.

In the two following sections, we are going to verify some numerical examples.

16.3. *Examples : Case* $c = a + 1$

Example 16.3. — The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{476}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$, in this example, $\mu_a < rad(a)$.
$c = 19^6 = 47\,045\,881 \Rightarrow rad(c) = 19$. Then $R = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$.
We have $c > R$ and $R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R) = 18\,800\,185\,299.081 > c = 47\,045\,881$.

16.3.0.1. Case $\epsilon = 0.01$

$c < K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow 47\,045\,881 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8.74777771491200531201524734886 53e + 4342 \tag{477}$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (475) is verified.

16.3.0.2. Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2.68793633309671754205917012128876e + 43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$, and the equation (475) is verified.

16.3.0.3. Case $\epsilon = 1$

$K(1) = e \Longrightarrow c = 47\,045\,881 < e.rad^2(ac) = 697\,987\,143\,184.212$ and the equation (475) is verified.

16.3.0.4. Case $\epsilon = 100$

$$K(100) = e^{0.0001} \Longrightarrow c = 47\,045\,881 \overset{?}{<} e^{0.0001}.506\,730^{101} = 1.52223502486076087818531426872 84e + 576$$

and the equation (475) is verified.

Example 16.4. — We give here the example 2 from $https://nitaj.users.lmno.cnrs.fr$:

$$1 + 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 2^{30} \times 5^2 \times 127 \times 353^2 \tag{478}$$

$a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424\,808\,316\,456\,140\,799 \Rightarrow rad(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 8497671 \Longrightarrow \mu_a > rad(a)$,

$b = 1$, $c = a + 1 = 424\,808\,316\,456\,140\,800 \Longrightarrow rad(c) = 2 \times 5 \times 127 \times 353$. Then $R = rad(ac) = 8497671 \times 448310 = 3\,809\,590\,886\,010 < c$. We obtain $R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R) = 210\,209\,917\,628\,130\,447\,085.912 > c$, then $c < R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R)$.

For example, we take $\epsilon = 0.5$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{1/0.25} = e^4 = 54.598003130965797890 56 \tag{479}$$

Let us verify (475):

$$c \overset{?}{<} K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow c = 424\,808\,316\,456\,140\,800 \overset{?}{<} K(0.5) \times (3\,809\,590\,886\,010)^{1.5}$$
$$\Longrightarrow 424\,808\,316\,456\,140\,800 < 405\,970\,304\,762\,905\,691\,174.98260818045 \tag{480}$$

Hence (475) is verified.

16.4. *Examples : Case* $c = a + b$

Example 16.5. — We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{481}$$

$a = 3^{10}.109 = 6\,436\,341 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109 \Longrightarrow \mu_a > rad(a)$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $R = rad(abc) = 2 \times 3 \times 109 \times 23 = 15042 < c$. Let us verify $c < R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R)$. We obtain : $c = 6\,436\,343 < 77\,532\,979.756$.

For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{9999.99} = 8.7477777149120053120152473488653e + 4342 \tag{482}$$

Let us verify (475):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \overset{?}{<} K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow$$
$$6436343 \ll K(0.01) \times 15042^{1.01} \tag{483}$$

Hence (475) is verified.

Example 16.6. — The example of Nitaj about the ABC conjecture [**1**] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 = 11\,297$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 = 89\,257$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953$$
$$R = rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 < c$$

We have also $\mu_a > rad(a), \mu_b > rad(b) > rad(a)$ and $\mu_b < \mu_a$. We find $c < R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R) = 3\,614\,932\,048\,440\,771\,457\,890.631$.

16.4.0.1. Case 1

we take $\epsilon = 100$ we have:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^{0.0001}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 2.7657949971494838920022381186039e+1359$$

then (475) is verified.

16.4.0.2. Case 2

We take $\epsilon = 0.5$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \tag{484}$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^{4}.(2.3.5.7.11.13.41.79.311.953)^{1.5} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 8\,450\,961\,319\,227\,998\,887\,403,9993 \tag{485}$$

We obtain that (475) is verified.

16.4.0.3. Case 3

We take $\epsilon = 1$, then

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e.(2.3.5.7.11.13.41.79.311.953)^{2} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 831\,072\,936\,124\,776\,471\,158\,132\,100 \times e \tag{486}$$

We obtain that (475) is verified.

Example 16.7. — It is of Ralf Bonse about the ABC conjecture [**3**] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \tag{487}$$
$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983 = 3.4136998783296235160378273576498e+44$$
$$R = rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$R = 1,5683959920004546031461002610848e+33 < c \tag{488}$$

We have also: $\mu_a < rad(a), \mu_b > rad(b) > \mu_a, \mu_c > rad(c)$ and $\mu_b < \mu_c$. The calculate of $A = R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R$ gives:

$$A = 9.5054989139840681669171835013874e+47 > c$$

16.4.0.4. Case 1

For example, we take $\epsilon = 10$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{0.01} = 1.007815740428295674320461741677$$

Let us verify (475):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<}$$
$$e^{0.01}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{11}$$
$$\Longrightarrow 3.4136998783296235160378273576498e + 44 <$$
$$1.4236200596494908176008120925 72e + 365 \tag{489}$$

The equation (475) is verified.

16.4.0.5. Case 2

We take $\epsilon = 0.4 \Longrightarrow K(\epsilon) = 12.182473474251512159126256696 08$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<}$$
$$e^{6.25}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.4}$$
$$\Longrightarrow 3.4136998783296235160378273576498e + 44 <$$
$$3.6255465680011453642792720569685e + 47 \tag{490}$$

And the equation (475) is verified.

16.5. *Conclusion*

Assuming $c < R.exp(\dfrac{3}{2}\sqrt[3]{2}Log^{2/3}R)$, we have given an elementary proof of the *abc* conjecture, confirmed by some numerical examples. We can announce the theorem:

Theorem 26. — *For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, and assuming $c < R.exp(\dfrac{3}{2}\sqrt[3]{2}Log^{2/3}R)$ is true, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{491}$$

where K is a constant depending of ϵ proposed as :

$$K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \epsilon > 0$$

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[3] P. Mihăilescu, 'Around ABC', European Mathematical Society Newsletter **N° 93**, September 2014. 29-34. (2014)

CHAPTER 17

THE *abc* CONJECTURE IS FALSE

Abstract. — In this note, I give the proof that the *abc* conjecture is false because, in the case $c > rad(abc)$, for $0 < \epsilon < 1$ we can not find the constant $K(\epsilon)$ so that $c < K(\epsilon).rad^{1+\epsilon}(abc)$ for c very large. A counter-example is given.

17.1. *Introduction and notations*

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a). \prod_i a_i^{\alpha_i - 1} \tag{492}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{493}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 25. — *For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc), \quad K(\epsilon) \text{ depending only of } \epsilon. \tag{494}$$

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in November 2018, about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**1**]. A conjecture was proposed that $c < rad^2(abc)$ [**3**]:

Conjecture 26. — *Let a, b, c positive integers relatively prime with $c = a+b$, then:*

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{495}$$

After studying the *abc* conjecture using different choices of the constant $K(\epsilon)$ and having attacked the problem from diverse angles, I have arrived to conclude that, assuming that $c < rad^2(abc)$ or $c < rad^{1.63}$ is true, the *abc* conjecture does not hold when $0 < \epsilon < 1$. Then the *abc* conjecture as it was defined is false. In this note, I give a counter-example that the *abc* conjecture is not true, in the case $rad(abc) < c$ taking $\epsilon \in]0,1[$ without assuming one of the two open questions : $c < rad^2(abc)$ and $c < rad^{1.63}(abc)$ that was proposed in 1996 by A. Nitaj [**4**].

The paper is organized as follows: in the second section, we give a counter-example that *abc* conjecture is false in the case $rad(abc) < c$, choosing $\epsilon \in]0,1[$.

17.2. *Proof the abc conjecture is false*

We note $R = rad(ac)$ in the case $c = a+1$ (respectively $R = rad(abc)$ if $c = a+b$).

17.2.1. Case $c < R$:

As $c < R \Longrightarrow c < R \Longrightarrow c < K(\epsilon).R^{1+\epsilon}, \forall \epsilon > 0$ since we choose $K(\epsilon) \geq 1$ and the conjecture (25) is verified.

17.2.2. Case $c = R$

Case to reject as a, b, c (respectively a, c) are relatively prime.

17.2.3. Case $R < c$

I will consider the case $c = a+1$. I give the following counter example:

$$8^n = 2^{3n} = (7+1)^n = 7^n + 7^{n-1}n + \ldots + 7n + 1 = 7(7^{n-1} + n7^{n-2} + \ldots + n) + 1 \Longrightarrow$$
$$2^{3n} = 7(7^{n-1} + n7^{n-2} + \ldots + n) + 1 \tag{496}$$

We suppose that for n odd and large, the *abc* conjecture holds taking $\epsilon = \epsilon_0 \in]0,1[$. Then $\exists\ K(\epsilon_0) > 0$ and:

$$2^{3n} < K(\epsilon_0)R^{1+\epsilon_0} \tag{497}$$

We obtain:

$$rad(c) = rad(2^{3n}) = 2$$

$$rad(a) = rad(7(7^{n-1} + n7^{n-2} + \ldots + n)) = 7.rad(7^{n-1} + n7^{n-2} + \ldots + n) \Longrightarrow$$

$$rad(a) \leq 7.(7^{n-1} + n7^{n-2} + \ldots + n) \leq n.7^n \Longrightarrow rad(a) \leq 7^n n \tag{498}$$

We re-write the equation (497) in detail:

$$2^{3n} < K(\epsilon_0)2^{1+\epsilon_0}n^{1+\epsilon_0}7^{n(1+\epsilon_0)} \tag{499}$$

That we can write as:

$$e^{3nLog2\left(1-(1+\epsilon_0)\dfrac{Log7}{3Log2} - \dfrac{1+\epsilon_0}{3Log2}\dfrac{Logn}{n}\right)} < K(\epsilon_0)2^{1+\epsilon_0} \tag{500}$$

We choose $\epsilon_0 = 0.06$ and we consider that n is very large ($n \longrightarrow +\infty$), then we obtain:

$$e^{3nLog2(1-0.99193)} \leq K(0.06)2^{1.06} \Longrightarrow +\infty \leq K(0.06)2^{1.06} \tag{501}$$

Hence the contradiction, and the *abc* conjecture is false for the value $\epsilon_0 = 0.06$.

We can announce the following theorems that are very easy to verify:

Theorem 27. — **(The truncated abc conjecture:)** *For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, and assuming $c < rad^2(abc)$ is true, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{502}$$

where K is a constant depending of ϵ proposed as :

$$K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)} \quad , \epsilon \geq 1$$

and:

Theorem 28. — **(The truncated abc conjecture:)** *For each $\epsilon \geq 0.63$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, and assuming $c < rad^{1.63}(abc)$ is true, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{503}$$

where K is a constant depending of ϵ proposed as :

$$K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)} \quad , \epsilon \geq 0.63$$

Ouf! The end of the mystery!

Acknowledgements : The author is very grateful to Professors Mihăilescu Preda and Gérald Tenenbaum for their comments about errors found in previous manuscripts concerning proofs proposed of the *abc* conjecture.

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CHAPTER 18

END OF THE MYSTERY OF THE *abc* CONJECTURE

To the memory of my Father who taught me arithmetic
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

Abstract. — In this note, I give the proof that the *abc* conjecture is false giving a counterexample in the case $\epsilon \in]0,1[$. because, in the case $c > rad(abc)$, for $0 < \epsilon < 1$ we can not find the constant $K(\epsilon)$ so that $c < K(\epsilon).rad^{1+\epsilon}(abc)$ for c very large.

18.1. Introduction

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{504}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{505}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 27. — *For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc), \quad K(\epsilon) \text{ depending only of } \epsilon \tag{506}$$

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in November 2018, about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**1**]. A conjecture was proposed that $c < rad^2(abc)$ [**3**]:

Conjecture 28. — *Let a, b, c positive integers relatively prime with $c = a+b$, then:*

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{507}$$

After studying the *abc* conjecture using different choices of the constant $K(\epsilon)$ and having attacked the problem from diverse angles, I have arrived to conclude that, assuming that $c < rad^2(abc)$ or $c < rad^{1.63}$ is true, the *abc* conjecture does not hold when $0 < \epsilon < 1$. Then the *abc* conjecture as it was defined is false. In this note, I give a counterexample that the *abc* conjecture is not true, in the case $rad(abc) < c$ taking $\epsilon \in]0, 1[$ without assuming one of the two open questions : $c < rad^2(abc)$ and $c < rad^{1.63}(abc)$ that was proposed in 1996 by A. Nitaj [**4**].

The paper is organized as follows: in the second section, we give a counterexample that *abc* conjecture is false in the case $rad(abc) < c$, choosing $\epsilon \in]0, 1[$.

18.2. Proof the *abc* Conjecture is False

We note $R = rad(ac)$ in the case $c = a + 1$ (respectively $R = rad(abc)$ if $c = a + b$).

18.2.1. Case $c < R$:

As $c < R \Longrightarrow c < R \Longrightarrow c < K(\epsilon).R^{1+\epsilon}, \forall \epsilon > 0$ since we choose $K(\epsilon) \geq 1$ and the conjecture (27) is verified.

18.2.2. Case $c = R$

Case to reject as a, b, c (respectively a, c) are relatively prime.

18.2.3. Case $R < c$

I will consider the case $c = a + 1$. I give the following counterexample:

$$c = \mu_c.rad(c) = 2^n + 1, n > n_0 > 0 \Longrightarrow rad(a) = 2 \tag{508}$$

18.2.3.1. Case $rad(c) < \mu_c$:

In this case, we obtain as $c = \mu_c rad(c) < \mu_c^2 \Longrightarrow \mu_c > (2^n + 1)^{1/2} > 2^{n/2}$ and $c > rad^2(c) \Longrightarrow rad(c) < (2^n + 1)^{1/2}$. we have also $R < c$. We suppose that for

n odd and large, the *abc* conjecture holds taking $\epsilon = \epsilon_0 \in]0,1[$. Then $\exists\ K(\epsilon_0) > 0$ and:

$$2^n + 1 < K(\epsilon_0)R^{1+\epsilon_0} \Longrightarrow 2^n + 1 < K(\epsilon_0)2^{1+\epsilon_0} rad^{1+\epsilon_0}(c) \Longrightarrow$$
$$2^n + 1 < K(\epsilon_0)2^{1+\epsilon_0} rad^{1+\epsilon_0}(c) < K(\epsilon_0)2^{1+\epsilon_0}(2^n+1)^{\frac{1+\epsilon_0}{2}} \Longrightarrow$$
$$(2^n+1)^{\frac{1-\epsilon_0}{2}} < K(\epsilon_0)2^{1+\epsilon_0} \tag{509}$$

We consider that n is very large ($n \longrightarrow +\infty$), then we obtain as $1-\epsilon_0 > 0$:

$$+\infty \le K(\epsilon_0)2^{1+\epsilon_0} < +\infty \tag{510}$$

Hence the contradiction, and the *abc* conjecture is false when $\epsilon \in]0,1|$.

18.2.3.2. Case $rad(c) \geq \mu_c$:

$c = \mu_c.rad(c) \le rad^2(c) \Longrightarrow c < R^2$. We have also $c > R \Longrightarrow \mu_c > 2$, if not, $\mu_c = 2 \Longrightarrow (a,c) \neq 1$ then the contradiction with a, c coprime; if $\mu_c = 1 \Longrightarrow c = rad(c) < R$, case to reject, then $R < c \Longrightarrow \mu_c \geq 3 \Longrightarrow c \geq 3rad(c)$. Suppose that $c = 3rad(c)$, we can write $c = rad(c)^{1+\alpha}$ with $0 < \alpha < 1$ equal to $\dfrac{Log3}{Log(rad(c))}$. Then as $rad(c) < c < rad^2(c) \Longrightarrow \exists \alpha \in]0,1[$ that we can take equal to $\dfrac{Log\frac{c}{rad(c)}}{Log(rad(c))} < 1$ so that $c \geq rad^{1+\alpha}(c)$ and α depends only of c.

We return to the *abc* conjecture considering our example giving by the equation (508). We suppose that for n odd and large, the *abc* conjecture holds taking $\epsilon = \epsilon_0 \in]0,1[$. Then $\exists\ K(\epsilon_0) > 0$ and:

$$2^n + 1 < K(\epsilon_0)R^{1+\epsilon_0} \Longrightarrow 2^n + 1 < K(\epsilon_0)2^{1+\epsilon_0} rad^{1+\epsilon_0}(c) \tag{511}$$

As seen above, we have $rad(c) \le c^{\frac{1}{1+\alpha}}$, let $\beta = \dfrac{\alpha - \epsilon_0}{\alpha+1}$ and the equation (511) becomes:

$$2^n + 1 < K(\epsilon_0)2^{1+\epsilon_0}\left(c^{\frac{1}{1+\alpha}}\right)^{1+\epsilon_0} \Longrightarrow$$

$$(2^n+1)^{\beta} < K(\epsilon_0)2^{1+\epsilon_0} \Longrightarrow 2^{n\beta}\left(1 + \frac{1}{2^n}\right)^{\beta} < K(\epsilon_0)2^{1+\epsilon_0} \tag{512}$$

We choose ϵ_0 so that $\epsilon_0 < \alpha \Longrightarrow \beta > 0$, for example $\epsilon_0 = \alpha/10 \Longrightarrow \beta = 9\alpha/(10(1+\alpha))$. Then if n is very large, the parameter β remains positive not zero and $2^{n\beta}$ becomes very large, it follows $(2^n+1)^{\beta} > K(\epsilon_0)2^{1+\epsilon_0}$, then the contradiction.

For our counterexample presented in the paper, in the two cases, we have found a contradiction in the application of the *abc* conjecture. Hence the *abc* conjecture is false.

However, we can announce the following theorems that are very easy to prove:

Theorem 29. — ***(The truncated abc conjecture:)*** *For each* $\epsilon \geq 1$*, there exists* $K(\epsilon) > 0$ *such that if* a, b, c *positive integers relatively prime with* $c = a + b$*, and assuming* $c < rad^2(abc)$ *is true, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{513}$$

where K *is a constant depending of* ϵ *proposed as :*

$$K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \quad , \epsilon \geq 1$$

and:

Theorem 30. — ***(The truncated abc conjecture:)*** *For each* $\epsilon \geq 0$*, there exists* $K(\epsilon) > 0$ *such that if* a, b, c *positive integers relatively prime with* $c = a + b$*, and assuming* $c < rad^{1.63}(abc)$ *is true, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{514}$$

where K *is a constant depending of* ϵ *proposed as :*

$$K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \quad , \epsilon \geq 0.63$$

Ouf! The end of the mystery!

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CHAPTER 19

PROGRESS IN THE PROOF OF THE CONJECTURE $c < rad^2(abc)$ - CASE : $c = a + 1$

Abstract. — In this paper, we consider the *abc* conjecture. We give some progress in the proof of the conjecture $c < rad^2(abc)$ in the case $c = a + 1$.

To the memory of my Father who taught me arithmetic
*To my wife **Wahida**, my daughter **Sinda** and my son **Mohamed Mazen***

19.1. Introduction and notations

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{515}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{516}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) ([**4**]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 29. — *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{517}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ ([**2**]). A conjecture was proposed that $c < rad^2(abc)$ ([**1**]). Here we will give a proof of it for the case $c = a + 1$.

Conjecture 30. — *Let a, b, c positive integers relatively prime with $c = a+b$, then:*

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{518}$$

This result, I think is the key to obtain the final proof of the veracity of the *abc* conjecture.

19.2. A Proof of the conjecture (30) case $c = a + 1$

Let a, c positive integers, relatively prime, with $c = a + 1$ and $R = rad(ac)$, $c = \prod_{j' \in J'} c_{j'}^{\beta_{j'}}, \beta_{j'} \geq 1$.

If $c < rad(ac)$ then we obtain:

$$c < rad(ac) < rad^2(ac) \Longrightarrow \boxed{c < R^2} \tag{519}$$

and the condition (518) is verified.

If $c = rad(ac)$, then a, c are not coprime, case to reject.

In the following, we suppose that $c > rad(ac)$ and c and a are not prime numbers.

$$c = a + 1 = \mu_a rad(a) + 1 \overset{?}{<} rad^2(ac) \tag{520}$$

19.2.1. $\mu_a \neq 1$, $\mu_a \leq rad(a)$

We obtain :

$$c = a + 1 < 2\mu_a.rad(a) \Rightarrow c < 2rad^2(a) \Rightarrow c < rad^2(ac) \Longrightarrow \boxed{c < R^2} \tag{521}$$

Then (520) is verified.

19.2.2. $\mu_c \neq 1$, $\mu_c \leq rad(c)$

We obtain :

$$c = \mu_c rad(c) \leq rad^2(c) < rad^2(ac) \Longrightarrow \boxed{c < R^2} \tag{522}$$

and the condition (520) is verified.

19.2.3. $\mu_a > rad(a)$ and $\mu_c > rad(c)$

19.2.3.1. Case: $\mu_a = rad^q(a), q \geq 2$, $\mu_c = rad^p(c), p \geq 2$:

In this case, we write $c = a+1$ as $rad^{p+1}(c) - rad^{q+1}(a) = 1$. Then $rad(c), rad(a)$ are solutions of the Diophantine equation: :

$$X^{p+1} - Y^{q+1} = 1 \quad \text{with } (p+1)(q+1) \geq 9 \tag{523}$$

But the solutions of the equation (523) are :$(X = \pm 3, p+1 = 2, Y = +2, q+1 = 3)$, we obtain $p = 1 < 2$, then $rad(c), rad(a)$ are not solutions of (523) and the case $\mu_a = rad^q(a), q \geq 2$, $\mu_c = rad^p(c), p \geq 2$ is to reject.

19.2.3.2. Case: $rad(c) < \mu_c < rad^2(c)$ and $rad(a) < \mu_a < rad^2(a)$:

We can write:

$$\left.\begin{array}{l} \mu_c < rad^2(c) \Longrightarrow c < rad^3(c) \\ \\ \mu_a < rad^2(a) \Longrightarrow a < rad^3(a) \end{array}\right\} \Longrightarrow ac < R^3 \Longrightarrow a^2 < ac < R^3 \Longrightarrow$$

$$a < R\sqrt{R} < R^2 \Longrightarrow \boxed{c = a+1 < R^2} \tag{524}$$

19.2.3.3. Case: $\mu_c > rad^2(c)$ or $\mu_a > rad^2(a)$

I- We suppose that $\mu_c > rad^2(c)$ and $rad(a) < \mu_a \leq rad^2(a)$:

I-1- Case $rad(a) < rad(c)$: In this case $a = \mu_a.rad(a) \leq rad^2(a).rad(a) < rad^2(a)rad(c) < rad^2(ac) \Longrightarrow a < R^2 \Longrightarrow \boxed{c < R^2}$.

I-2- Case $rad(c) < rad(a) < rad^2(c)$: As $a \leq rad^2(a).rad(a) < rad^2(a).rad^2(c) \Longrightarrow a < R^2 \Longrightarrow \boxed{c < R^2}$.

Example: $2^{30}.5^2.127.353^2 = 3^7.5^5.13^5.17.1831 + 1$, $rad(c) = 2.5.127.353 = 448\,310$, $rad^2(c) = 200\,981\,856\,100$,
$\mu_c = 2^{29}.5.353 = 947\,577\,159\,680 \Longrightarrow rad^2(c) < \mu_c < rad^3(c)$,
$rad(a) = 3.5.13.17.1831 = 6\,069\,765$, $rad^2(a) = 36\,842\,047\,155\,225$,
$\mu_a = 3^6.5^4.13^4 = 13\,013\,105\,625 < rad^2(a)$. It is the case : $rad(c) < \mu_c < rad^2(c)$ and $rad(a) < \mu_a \leq rad^2(a)$ with $rad(c) = 448\,310 < rad(a) = 6\,069\,765 < rad^2(c) = 200\,981\,856\,100$.

I-3- Case $rad^2(c) < rad(a)$:

I-3-1- We suppose that $c \leq rad^6(c)$, we obtain:

$$c \leq rad^6(c) \Longrightarrow c \leq rad^2(c).rad^4(c) \Longrightarrow c < rad^2(c).(rad(a))^2 = R^2 \Longrightarrow \boxed{c < R^2}$$

Example: $5^8.7^2 = 2^4 3^7.547 + 1 \Longrightarrow 19\,140\,625 = 19\,140\,624 + 1$, $rad(c) = 5.7 = 35, rad(a) = 2.3.547 = 3\,282 \Longrightarrow rad(a) > rad^2(c)$, we obtain $c = 19140\,625 > rad^3(c) = 42\,875$ and $c < rad^6(c) = 1\,838\,265\,625$ and $3\,282 = rad(a) < \mu_a = 5\,832 < rad^2(a) = 10\,771\,524 \Longrightarrow a < rad^3(a) = 35\,352\,141\,768$.

I-3-2- We suppose $c > rad^6(c) \Longrightarrow \mu_c > rad^5(c)$, we suppose $\mu_a = rad^2(a) \Longrightarrow a = rad^3(a)$. Then we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$X^3 + 1 = c = \mu_c.rad(c) \tag{525}$$

If $c = rad^n(c)$ with $n \geq 7$, we obtain an equation like (523) that gives a contradiction. In the following, we will study the cases $\mu_c = A.rad^n(c)$ with $rad(c) \nmid A, n \geq 0$. The above equation (525) can be written as :

$$(X+1)(X^2 - X + 1) = c \tag{526}$$

Let δ any divisor of c, then:

$$X + 1 = \delta \tag{527}$$

$$X^2 - X + 1 = \frac{c}{\delta} = c' = \delta^2 - 3X \tag{528}$$

We recall that $rad(a) > rad^2(c)$, it follows that δ must verifies $\delta - 1 > rad^2(c) \Longrightarrow \delta > rad^2(c) + 1$.

I-3-2-1- We suppose that $\delta = l.rad(c) \Longrightarrow lrad(c) > rad^2(c)+1 \Longrightarrow l > \dfrac{rad^2(c)+1}{rad(c)}$. We obtain $l \geq rad(c) + 2$ so $rad(c)$ and l have the same parity. We have $\delta = l.rad(c) < c = \mu_c.rad(c) \Longrightarrow l < \mu_c$. As δ is a divisor of c, then l is a divisor of μ_c, we write $\mu_c = l.m$. From $\mu_c = l(\delta^2 - 3X)$, we obtain:

$$m = l^2 rad^2(c) - 3rad(a) \Longrightarrow 3rad(a) = l^2 rad^2(c) - m$$

A- Case $3|m \Longrightarrow m = 3m', m' > 1$: As $\mu_c = ml = 3m'l \Longrightarrow 3|rad(c)$ and $(rad(c), m')$ not coprime. We obtain:

$$rad(a) = l^2 rad(c).\frac{rad(c)}{3} - m'$$

It follows that a,c are not coprime, then the contradiction.

B - Case $m = 3 \Longrightarrow \mu_c = 3l \Longrightarrow c = 3lrad(c) = 3\delta = \delta(\delta^2 - 3X) \Longrightarrow \delta^2 = 3(1 + X) = 3\delta \Longrightarrow \delta = lrad(c) = 3$, then the contradiction.

I-3-2-2- We suppose that $\delta = l.rad^2(c), l \geq 2$. In this case $rad(a) = lrad^2(c) - 1$ verifies $rad(a) > rad^2(c)$. If $lrad(c) \nmid \mu_c$ then the case to reject. We suppose that

$lrad(c)|\mu_c \Longrightarrow \mu_c = m.lrad(c)$, then $\dfrac{c}{\delta} = m = \delta^2 - 3rad(a)$.

C - Case $m = 1 = c/\delta \Longrightarrow \delta^2 - 3rad(a) = 1 \Longrightarrow (\delta-1)(\delta+1) = 3rad(a) = rad(a)(\delta+1) \Longrightarrow \delta = 2 = l.rad^2(c)$, then the contradiction.

D - Case $m = 3$, we obtain $3(1 + rad(a)) = \delta^2 = 3\delta \Longrightarrow \delta = 3 = lrad^2(c)$. Then the contradiction.

E - Case $m \neq 1, 3$, we obtain: $3rad(a) = l^2 rad^4(c) - m \Longrightarrow rad(a)$ and $rad(c)$ are not coprime. Then the contradiction.

I-3-2-3- We suppose that $\delta = l.rad^n(c), l \geq 2$ with $n \geq 3$. From $c = \mu_c.rad(c) = lrad^n(c)(\delta^2 - 3rad(a))$, let $m = \delta^2 - 3rad(a)$.

F - As seen above (paragraphs C,D), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G - Case $m \neq 1, 3$. Let q a prime that divides m, it follows $q|\mu_c \Longrightarrow q = c_{j'_0} \Longrightarrow c_{j'_0}|\delta^2 \Longrightarrow c_{j'_0}|3rad(a)$. Then $rad(a)$ and $rad(c)$ are not coprime. It follows the contradiction.

I-3-2-4- We suppose that $\delta = \prod_{j \in J_1} c_j^{\beta_j}$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with $\beta_{j_0} \geq 2$, $rad(c) \nmid \delta$ and $\delta - 1 = \prod_{j \in J_1} c_j^{\beta_j} - 1 > rad^2(c) = \prod_{j' \in J'} c_{j'}^2$, $J_1 \subset J'$. We can write:

$$\delta = \mu_\delta.rad(\delta), \quad rad(c) = m.rad(\delta)$$

Then we obtain:

$$c = \mu_c.rad(c) = \mu_c.m.rad(\delta) = \delta(\delta^2 - 3X) = \mu_\delta.rad(\delta)(\delta^2 - 3X) \Longrightarrow$$
$$m.\mu_c = \mu_\delta(\delta^2 - 3X) \tag{529}$$

- If $\mu_c = \mu_\delta \Longrightarrow m = \delta^2 - 3X = (\mu_c.rad(\delta))^2 - 3X$. As $\delta < \delta^2 - 3X \Longrightarrow m > \delta \Longrightarrow rad(c) > m > \mu_c.rad(\delta) > rad^5(c)$ because $\mu_c > rad^5(c)$, it follows $rad(c) > rad^5(c)$. Then the contradiction.

- We suppose that $\mu_c < \mu_\delta$. As $rad(a) = \mu_\delta rad(\delta) - 1$, we obtain:

$$rad(a) > \mu_c.rad(\delta) - 1 > 0 \Longrightarrow R > c.rad(\delta) - rad(c) > 0 \Longrightarrow$$
$$c > R > c.rad(\delta) - rad(c) > 0 \Longrightarrow 1 > rad(\delta) - \frac{rad(c)}{c} > 0, \quad rad(\delta) \geq 2$$
$$\Longrightarrow \text{The contradiction} \tag{530}$$

- We suppose that $\mu_\delta < \mu_c$. In this case, from the equation (529) and as $(m, \mu_\delta) = 1$, it follows that we can write:

$$\mu_c = \mu_1.\mu_2, \quad \mu_1, \mu_2 > 1 \tag{531}$$

$$\text{so that} \quad m.\mu_1 = \delta^2 - 3X, \quad \mu_2 = \mu_\delta \tag{532}$$

But:

$$rad(a) = \delta - 1 = \mu_\delta rad(\delta) - 1 > rad^2(c) \Longrightarrow 0 > m^2 rad^2(\delta) - \mu_2 rad(\delta) + 1$$

Let $P(Z)$ the polynomial:

$$P(Z) = m^2 Z^2 - \mu_2 Z + 1 \Longrightarrow P(rad(\delta)) < 0 \tag{533}$$

The discriminant of $P(Z)$ is:

$$\Delta = \mu_2^2 - 4m^2 \tag{534}$$

- $\Delta = 0 \Longrightarrow \mu_2 = 2m$, but $(m, \mu_2) = 1$, then the contradiction. Case to reject.

- $\Delta < 0 \Longrightarrow P(Z)$ has no real roots. From (533) it follows that $P(Z) > 0, \forall Z \in \mathbb{R}$. Then the contradiction with $P(rad(\delta)) < 0$. Case to reject.

- $\Delta > 0 \Longrightarrow \mu_2 > 2m \Longrightarrow \dfrac{\mu_2}{m} > 2$. We denote $t = \sqrt{\Delta} > 0$. The roots of $P(Z) = 0$ are Z_1, Z_2 with $Z_1 < Z_2$, given by:

$$Z_1 = \frac{\mu_2 - t}{2m^2}, \quad Z_2 = \frac{\mu_2 + t}{2m^2} \tag{535}$$

We approximate t by $\tilde{t}$:

$$t = \sqrt{\mu_2^2 - 4m^2} = \mu_2 \left(1 - \frac{4m^2}{\mu_2^2}\right)^{\frac{1}{2}} \Longrightarrow \tilde{t} = \mu_2 - \frac{2m^2}{\mu_2} > 0$$

Then, we obtain $\tilde{Z}_1, \tilde{Z}_2$ as :

$$\tilde{Z}_1 = \frac{\mu_2 - \tilde{t}}{2m^2} = \frac{1}{\mu_2}, \quad \tilde{Z}_2 = \frac{\mu_2 + \tilde{t}}{2m^2} = \frac{\mu_2}{m^2} - \frac{1}{\mu_2} \tag{536}$$

As $\mu_2^2 - 4m^2 > 0 \Longrightarrow \mu_2^2 - m^2 > 3m^2 > 0 \Longrightarrow \dfrac{\mu_2^2}{m^2} - 1 > 0$, we will give below the proof that $rad(\delta) > \tilde{Z}_2 \Longrightarrow P(rad(\delta)) > 0$, then the contradiction with $P(rad(\delta)) < 0$; we write:

$$rad(\delta) \overset{?}{>} \frac{\mu_2}{m^2} - \frac{1}{\mu_2}, \quad \mu_2 > 0 \Longrightarrow$$

$$\mu_2.rad(\delta) \overset{?}{>} \frac{\mu_2^2}{m^2} - 1$$

$$\delta \overset{?}{>} \frac{\mu_2^2 - m^2}{m^2} > \frac{3m^2}{m^2}$$

$$\text{as} \quad \delta > 3 \Longrightarrow \delta > \frac{\mu_2^2}{m^2} - 1 > 3 \Longrightarrow rad(\delta) > \frac{\mu_2}{m^2} - \frac{1}{\mu_2} > \frac{3}{\mu_2} \tag{537}$$

If follows $P(rad(\delta)) > 0$ and the contradiction with the conclusion of the equation (533).

It follows that the case $c > rad^6(c)$ and $a = rad^3(a)$ is impossible.

I-3-3- We suppose $c > rad^6(c) \Longrightarrow c = rad^6(c) + h, h > 0$ and $\mu_a < rad^2(a) \Longrightarrow a + l = rad^3(a)$, $l > 0$. Then we obtain :

$$rad^6(c) + h = rad^3(a) - l + 1 \tag{538}$$

As $rad^2(c) < rad(a)$ (see I-3), we obtain the equation:

$$rad^3(a) - (rad^2(c))^3 = h + l - 1 = m > 0$$

Let $X = rad(a) - rad^2(c)$, then X is an integer root of the polynomial $H(X)$ defined as:

$$H(X) = X^3 + 3R.rad(c)X - m = 0 \tag{539}$$

To resolve the above equation, we note $X = u+v$, then we obtain the two conditions:

$$u^3 + v^3 = m, \quad u.v = -R.rad(c) < 0 \Longrightarrow u^3.v^3 = -R^3 rad^3(c)$$

It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by:

$$G(t) = t^2 - mt - R^3 rad^3(c) = 0 \tag{540}$$

The discriminant of $G(t)$ is :

$$\Delta = m^2 + 4R^3 rad^3(c) = \alpha^2, \quad \alpha > 0 \tag{541}$$

The two real roots of (540) are:

$$t_1 = u^3 = \frac{m + \alpha}{2} \tag{542}$$

$$t_2 = v^3 = \frac{m - \alpha}{2} \tag{543}$$

As $m = rad^3(a) - rad^6(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^6(c) > 0$, then from the equation (541), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{544}$$

with $N = 4R^3 rad^3(c) > 0$. From the equations (542-543), we remark that α and m verify the following equations:

$$x + y = 2u^3 = 2rad^3(a) \tag{545}$$

$$x - y = -2v^3 = 2rad^6(c) \tag{546}$$

$$\text{then} \quad x^2 - y^2 = N = 4R^3 rad^3(c) \tag{547}$$

Let $Q(N)$ be the number of the solutions of (544) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the Diophantine equation (544) (see theorem 27.3 in [**3**]):
- If $N \equiv 2(\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3(\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0(\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.

$[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

Let (α', m'), $\alpha', m' \in \mathbb{N}^*$ be another pair, solution of the equation (544), then $\alpha'^2 - m'^2 = x^2 - y^2 = N = 4R^3 rad^3(c)$, but $\alpha = x$ and $m = y$ verify the equation (545) given by $x + y = 2rad^3(a)$, it follows α', m' verify also $\alpha' + m' = 2rad^3(a)$, that gives $\alpha' - m' = 2rad^6(c)$, then $\alpha' = x = \alpha = rad^3(a) + rad^6(c)$ and $m' = y = m = rad^3(a) - rad^6(c)$. We have given the proof of the uniqueness of the solutions of the equation (544) with the condition $x + y = 2rad^3(a)$. As $N = 4R^3 rad^3(c) \equiv 0(\bmod 4) \Longrightarrow Q(N) = [\tau(N/4)/2] = [\tau(rad^6(c).rad^3(a))/2] > 1$. But $Q(N) = 1$, then the contradiction.

It follows that the case $\mu_a \leq rad^2(a)$ and $c > rad^6(c)$ is impossible.

II- We suppose that $rad(c) < \mu_c \leq rad^2(c)$ and $\mu_a > rad^5(a)$:

II-1- Case $rad(c) < rad(a)$: As $c \leq rad^3(c) = rad^2(c).rad(c) \Longrightarrow c < rad^2(c).rad(a) \Longrightarrow \boxed{c < R^2}$.

II-2- Case $rad(a) < rad(c) < rad^2(a)$: As $c \leq rad^3(c) = rad^2(c).rad(c) \Longrightarrow c < rad^2(c).rad^2(a) \Longrightarrow \boxed{c < R^2}$.

II-3- Case $rad^2(a) < rad(c)$:

II-3-1- We suppose that $a \leq rad^6(a) \Longrightarrow a \leq rad^2(a).rad^4(a) \Longrightarrow a < rad^2(a).(rad(c))^2 = R^2 \Longrightarrow a < R^2 \Longrightarrow 1 + a \leq R^2$, but $(c, a) = 1$, it follows $\boxed{c < R^2}$.

II-3-2- We suppose $a > rad^6(a)$ and $\mu_c \leq rad^2(c)$. Using the same method as it was explicated in the paragraphs I-3-2, I-3-3 (permuting a,c), we arrive at a contradiction. It follows that the case $\mu_c \leq rad^2(c)$ and $a > rad^6(a)$ is impossible.

19.2.3.4. III - Case $\mu_c > rad^2(c)$ and $\mu_a > rad^2(a)$

We can write $c > rad^3(c) \Longrightarrow c = rad^3(c) + h$ and $a = rad^3(a) + l$ with $h, l > 0$ positive integers.

III-1- We suppose $rad(a) < rad(c)$. We obtain the equation:

$$rad^3(c) - rad^3(a) = l - h + 1 = m > 0 \tag{548}$$

Let $X = rad(c) - rad(a)$, from the above equation, X is a real root of the polynomial:

$$P(X) = X^3 + 3RX - m = 0 \tag{549}$$

As above, to resolve (672), we put $X = u + v$, then we obtain the two conditions:

$$u^3 + v^3 = m \tag{550}$$

$$uv = -R < 0 \Longrightarrow u^3.v^3 = -R^3 \tag{551}$$

Then u^3, v^3 are the roots of the equation:

$$H(Z) = Z^2 - mZ - R^3 = 0 \tag{552}$$

The discriminant of $H(Z)$ is:
(553)
$$\Delta = m^2 + 4R^3 = (rad^3(c) + rad^3(a))^2 = \alpha^2, \quad \text{taking} \;\; \alpha > 0 \Rightarrow \alpha = rad^3(c) + rad^3(a)$$

From the equation (676), we obtain that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{554}$$

with $N = 4R^3 > 0$ and $N \equiv 0 (\bmod 4)$. Using the same method as in I-3-3-, we arrive to a contradiction.

III-2- We suppose $rad(c) < rad(a)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h - l - 1 = m > 0 \tag{555}$$

Let $X = rad(a) - rad(c)$, from the above equation, X is a real root of the polynomial:

$$P(X) = X^3 + 3RX - m = 0 \tag{556}$$

As above, to resolve (618), we put $X = u + v$, then we obtain the two conditions:

$$u^3 + v^3 = m \tag{557}$$

$$uv = -R < 0 \Longrightarrow u^3.v^3 = -R^3 \tag{558}$$

Then u^3, v^3 are the roots of the equation:

$$H(Z) = Z^2 - mZ - R^3 = 0 \tag{559}$$

The discriminant of $H(Z)$ is:
(560)
$$\Delta = m^2 + 4R^3 = (rad^3(c) + rad^3(a))^2 = \alpha^2, \quad \text{taking} \;\; \alpha > 0 \Rightarrow \alpha = rad^3(c) + rad^3(a)$$

From the equation (622), we obtain that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{561}$$

with $N = 4R^3 > 0$ and $N \equiv 0(\mathrm{mod}\, 4)$. Using the same method as in I-3-3-, we arrive to a contradiction.

It follows that the case $\mu_c > rad^2(c)$ and $\mu_a > rad^2(a)$ is impossible.

We can annonce the following theorem:

Theorem 31. — ***(Abdelmajid Ben Hadj Salem, 2020)*** *Let a, c positive integers relatively prime with $c = a + 1$, then $c < rad^2(ac)$.*

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CHAPTER 20

THE PROOF OF THE CONJECTURE $c < rad^{1.63}(ac)$

Abstract. —

In this paper, we consider the *abc* conjecture. We give the proof of the conjecture $c < rad^{1.63}(abc)$ in the case $c = a + 1$.

To my teachers and to ***Madam Fatma Moalla****,* ***Mr Chedly Touibi***
my professors of mathematics at the Faculty of Sciences of Tunis

20.1. Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{562}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{563}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**7**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 31. — *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{564}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**5**]. It concerned the best example given by E. Reyssat [**5**]:

$$2+3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{565}$$

A conjecture was proposed that $c < rad^2(abc)$ [**3**]. In 2012, A. Nitaj [**4**] proposed the following conjecture:

Conjecture 32. — *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < rad^{1.63}(abc) \tag{566}$$

$$abc < rad^{4.42}(abc) \tag{567}$$

Here, we will give a proof of the conjecture given by (628) in the case $c = a + 1$. This result, I think is the key to obtain the final proof of the veracity of the *abc* conjecture especially when $0 < \epsilon < 1$.

20.2. A Proof of the conjecture (32) case $c = a + 1$

Let a, c be positive integers, relatively prime, with $c = a + 1$ and $R = rad(ac)$, $c = \prod_{j' \in J'} c_{j'}^{\beta_{j'}}, \beta_{j'} \geq 1$.

If $c < rad(ac)$, then we obtain:

$$c < rad(ac) < rad^{1.63}(ac) \Longrightarrow \boxed{c < R^{1.63}} \tag{568}$$

and the condition (628) is satisfied.

If $c = rad(ac)$, then a, c are not coprime, case to reject.

In the following, we suppose that $c > rad(ac)$ and c and a are not prime numbers.

$$c = a + 1 = \mu_a rad(a) + 1 \overset{?}{<} rad^{1.63}(ac) \tag{569}$$

20.2.1. $\mu_a \neq 1$, $\mu_a \leq rad^{0.63}(a)$

We obtain :

$$c = a + 1 \leq rad^{1.63}(a) + 1 < rad^{1.63}(ac) \Rightarrow c < rad^{1.63}(ac) \Longrightarrow \boxed{c < R^{1.63}} \tag{570}$$

Then (631) is satisfied.

20.2.2. $\mu_c \neq 1$, $\mu_c \leq rad^{0.63}(c)$

We obtain :

$$c = \mu_c rad(c) \leq rad^{1.63}(c) < rad^{1.63}(ac) \Longrightarrow \boxed{c < R^{1.63}} \tag{571}$$

and the condition (631) is satisfied.

20.2.3. $\mu_a > rad^{0.63}(a)$ **and** $\mu_c > rad^{0.63}(c)$

20.2.3.1. **Case:** $\mu_a = rad^q(a), q \geq 2,$ $\mu_c = rad^p(c), p \geq 2$**:**

In this case, we write $c = a + 1$ as $rad^{p+1}(c) - rad^{q+1}(a) = 1$. Then $rad(c), rad(a)$ are solutions of the Diophantine equation:

$$X^{p+1} - Y^{q+1} = 1, \quad \text{with } (p+1)(q+1) \geq 9 \tag{572}$$

But the solutions of the equation (572) are [**2**] :($X = \pm 3, p + 1 = 2, Y = +2, q + 1 = 3$), we obtain $p = 1 < 2$, then $rad(c), rad(a)$ are not solutions of (572) and the case $\mu_a = rad^q(a), q \geq 2$, $\mu_c = rad^p(c), p \geq 2$ is to reject.

20.2.3.2. **Case:** $rad^{0.63}(c) < \mu_c \leq rad^{1.63}(c)$ **and** $rad^{0.63}(a) < \mu_a \leq rad^{1.63}(a)$**:**

We can write:

$$\left.\begin{array}{l} \mu_c \leq rad^{1.63}(c) \Longrightarrow c \leq rad^{2.63}(c) \\ \\ \mu_a \leq rad^{1.63}(a) \Longrightarrow a \leq rad^{2.63}(a) \end{array}\right\} \Longrightarrow ac \leq R^{2.63} \Longrightarrow a^2 < ac \leq R^{2.63} \Longrightarrow$$

$$a < R^{1.315} < R^{1.63} \Longrightarrow \boxed{c = a + 1 < R^{1.63}} \tag{573}$$

20.2.3.3. **Case:** $\mu_c > rad^{1.63}(c)$ **or** $\mu_a > rad^{1.63}(a)$

I- We suppose that $\mu_c > rad^{1.63}(c)$ and $\mu_a \leq rad^2(a)$:

I-1- Case $rad(a) < rad(c)$: In this case $a = \mu_a.rad(a) \leq rad^3(a) \leq rad^{1.63}(a) rad^{1.37}(a) < rad^{1.63}(a).rad^{1.37}(c) < rad^{1.63}(ac) \Longrightarrow a < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

I-2- Case $rad(c) < rad(a) < rad^{\frac{1.63}{1.37}}(c)$: As $a \leq rad^{1.63}(a).rad^{1.37}(a) < rad^{1.63}(a).rad^{1.63}(c) \Longrightarrow a < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

I-3- Case $rad^{\frac{1.63}{1.37}}(c) < rad(a)$:

I-3-1- We suppose $c \leq rad^{3.26}(c)$, we obtain:

$$c \leq rad^{3.26}(c) \Longrightarrow c \leq rad^{1.63}(c).rad^{1.63}(c) \Longrightarrow$$
$$c < rad^{1.63}(c).rad(a)^{1.37} < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$$

I-3-2- We suppose $c > rad^{3.26}(c) \Longrightarrow \mu_c > rad^{2.26}(c)$. We consider the case $\mu_a = rad^2(a) \Longrightarrow a = rad^3(a)$. Then, we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$X^3 + 1 = c = \mu_c.rad(c) \tag{574}$$

If $c = rad^n(c)$ with $n \geq 4$, we obtain an equation like (572) that gives a contradiction. In the following, we will study the cases $\mu_c = A.rad^n(c)$ with $rad(c) \nmid A, n \geq 0$. The above equation (635) can be written as :

$$(X+1)(X^2 - X + 1) = c \tag{575}$$

Let δ any divisor of c, then:

$$X + 1 = \delta \tag{576}$$

$$X^2 - X + 1 = \frac{c}{\delta} = c' = \delta^2 - 3X \tag{577}$$

I-3-2-1- We suppose $\delta = l.rad(c)$. We obtain $\delta = l.rad(c) < c = \mu_c.rad(c) \Longrightarrow l < \mu_c$. As δ is a divisor of c, then l is a divisor of μ_c, we write $\mu_c = l.m$. From $\mu_c = l(\delta^2 - 3X)$, we obtain:

$$m = l^2 rad^2(c) - 3rad(a) \Longrightarrow 3rad(a) = l^2 rad^2(c) - m$$

A- Case $3|m \Longrightarrow m = 3m', m' > 1$: As $\mu_c = ml = 3m'l \Longrightarrow 3|rad(c)$ and $(rad(c), m')$ not coprime. We obtain:

$$rad(a) = l^2 rad(c).\frac{rad(c)}{3} - m'$$

It follows that a,c are not coprime, then the contradiction.

B - Case $m = 3 \Longrightarrow \mu_c = 3l \Longrightarrow c = 3lrad(c) = 3\delta = \delta(\delta^2 - 3X) \Longrightarrow \delta^2 = 3(1+X) = 3\delta \Longrightarrow \delta = lrad(c) = 3$, then the contradiction.

I-3-2-2- We suppose $\delta = l.rad^2(c), l \geq 2$. If $lrad(c) \nmid \mu_c$ then the case is to reject. We suppose $lrad(c)|\mu_c \Longrightarrow \mu_c = m.lrad(c)$, then $\dfrac{c}{\delta} = m = \delta^2 - 3rad(a)$.

C - Case $m = 1 = c/\delta \Longrightarrow \delta^2 - 3rad(a) = 1 \Longrightarrow (\delta-1)(\delta+1) = 3rad(a) = rad(a)(\delta+1) \Longrightarrow \delta = 2 = l.rad^2(c)$, then the contradiction.

D - Case $m = 3$, we obtain $3(1 + rad(a)) = \delta^2 = 3\delta \Longrightarrow \delta = 3 = lrad^2(c)$. Then the contradiction.

E - Case $m \neq 1, 3$, we obtain: $3rad(a) = l^2 rad^4(c) - m \Longrightarrow rad(a)$ and $rad(c)$ are not coprime. Then the contradiction.

I-3-2-3- We suppose $\delta = l.rad^n(c), l \geq 2$ with $n \geq 3$. From $c = \mu_c.rad(c) = lrad^n(c)(\delta^2 - 3rad(a))$, we denote $m = \delta^2 - 3rad(a) = \delta^2 - 3X$.

F - As seen above (paragraphs C,D), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G - Case $m \neq 1, 3$. Let q be a prime that divides m, it follows $q|\mu_c \Longrightarrow q = c_{j_0'} \Longrightarrow c_{j_0'}|\delta^2 \Longrightarrow c_{j_0'}|3rad(a)$. Then $rad(a)$ and $rad(c)$ are not coprime. It follows the contradiction.

I-3-2-4- We suppose $\delta = \prod_{j \in J_1} c_j^{\beta_j}$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with $\beta_{j_0} \geq 2$, $rad(c) \nmid \delta$. We can write:

$$\delta = \mu_\delta.rad(\delta), \quad rad(c) = m.rad(\delta), \quad m > 1, \quad (m, \mu_\delta) = 1 \tag{578}$$

Then, we obtain:

$$c = \mu_c.rad(c) = \mu_c.m.rad(\delta) = \delta(\delta^2 - 3X) = \mu_\delta.rad(\delta)(\delta^2 - 3X) \Longrightarrow$$
$$m.\mu_c = \mu_\delta(\delta^2 - 3X) \tag{579}$$

- If $\mu_c = \mu_\delta \Longrightarrow m = \delta^2 - 3X = (\mu_c.rad(\delta))^2 - 3X$. As $\delta < \delta^2 - 3X \Longrightarrow m > \delta \Longrightarrow rad(c) > m > \mu_c.rad(\delta) > rad^3(c)$ because $\mu_c > rad^{2.26}(c)$ (see I-3-2), it follows $rad(c) > rad^2(c)$. Then the contradiction.

- We suppose $\mu_c < \mu_\delta$. As $rad(a) = \mu_\delta rad(\delta) - 1$, we obtain:

$$rad(a) > \mu_c.rad(\delta) - 1 > 0 \Longrightarrow R > c.rad(\delta) - rad(c) > 0 \Longrightarrow$$
$$c > R > c.rad(\delta) - rad(c) > 0 \Longrightarrow 1 > rad(\delta) - \frac{rad(c)}{c} > 0, \quad rad(\delta) \geq 2$$
$$\Longrightarrow \text{The contradiction} \tag{580}$$

- We suppose $\mu_\delta < \mu_c$. In this case, from the equation (641) and as $(m, \mu_\delta) = 1$, it follows we can write:

$$\mu_c = \mu_1.\mu_2, \quad \mu_1, \mu_2 > 1 \tag{581}$$

$$c = \mu_c rad(c) = \mu_1.\mu_2.rad(\delta).m = \delta.(\delta^2 - 3X) \tag{582}$$

$$\text{so that} \quad m.\mu_1 = \delta^2 - 3X, \quad \mu_2 = \mu_\delta \Longrightarrow \delta = \mu_2.rad(\delta) \tag{583}$$

** We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists\, c_{j_0}$ so that $c_{j_0}|\mu_1$ and $c_{j_0}|\mu_2$. But $\mu_\delta = \mu_2 \Rightarrow c_{j_0}^2|\delta$. From $3X = \delta^2 - m\mu_1 \Longrightarrow c_{j_0}|3X \Longrightarrow c_{j_0}|X$ or $c_{j_0} = 3$.

- If $c_{j_0}|X$, it follows the contradiction with $(c, a) = 1$.

- If $c_{j_0} = 3$. We have $m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1) \Longrightarrow \delta^2 - 3\delta + 3 - m.\mu_1 = 0$. As $3|\mu_1 \Longrightarrow \mu_1 = 3^k\mu_1', 3 \nmid \mu_1', k \geq 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - 3^{k-1}m\mu_1') = 0 \tag{584}$$

- We consider the case $k > 1 \Longrightarrow 3 \nmid (1 - 3^{k-1} m\mu'_1)$. Let us recall the Eisenstein criterion [**1**]:

Theorem 32. — **(*Eisenstein Criterion*)** *Let $f = a_0 + \cdots + a_n X^n$ be a polynomial $\in \mathbb{Z}[X]$. We suppose that $\exists\, p$ a prime number so that:*
- *$p \nmid a_n$,*
- *$p | a_i$, $(0 \leq i \leq n-1)$,*
- *$p^2 \nmid a_0$.*

Then f is irreducible in $\mathbb{Q}$.

We apply Eisenstein criterion to the polynomial $R(Z)$ given by:

$$R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1} m\mu'_1) \tag{585}$$

then:
- $3 \nmid 1$,
- $3 | (-3)$,
- $3 | 3(1 - 3^{k-1} m\mu'_1)$,
- $3^2 \nmid 3(1 - 3^{k-1} m\mu'_1)$.

It follows that the polynomial $R(Z)$ is irreducible in $\mathbb{Q}$, then, the contradiction with $R(\delta) = 0$.

- We consider the case $k = 1$, then $\mu_1 = 3\mu'_1$ and $(\mu'_1, 3) = 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - m\mu'_1) = 0 \tag{586}$$

* If $3 \nmid (1 - m.\mu'_1)$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1)$$

and we find a contradiction with $R'(\delta) = 0$.

* We consider that $3 | (1 - m.\mu'_1) \Longrightarrow m\mu'_1 - 1 = 3^i.h$, $i \geq 1$, $3 \nmid h, h \in \mathbb{N}^*$. δ is an integer root of the polynomial $R'(Z)$:

$$R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1) = 0 \Longrightarrow \text{ the descriminant of } R'(Z) \text{ is :}$$
$$\Delta = 3^2 + 3^{i+1} \times 4.h \tag{587}$$

As the root δ is an integer, it follows that $\Delta = l^2 > 0$ with l a positive integer. We obtain:

$$\Delta = 3^2(1 + 3^{i-1} \times 4h) = l^2 \tag{588}$$
$$\Longrightarrow 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^* \tag{589}$$

We can write the equation (648) as :

$$\delta(\delta-3) = 3^{i+1}.h \Longrightarrow 3^3\mu'_1\frac{rad(\delta)}{3}.\left(\mu'_1 rad(\delta)-1\right) = 3^{i+1}.h \Longrightarrow \tag{590}$$

$$\mu'_1\frac{rad(\delta)}{3}.\left(\mu'_1 rad(\delta)-1\right) = h \tag{591}$$

We obtain $i = 2$ and $q^2 = 1 + 12h = 1 + 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1)$. q satisfies :

$$q^2 - 1 = 12h \Longrightarrow \frac{(q-1)}{2}.\frac{(q+1)}{2} = 3h = (\mu'_1 rad(\delta)-1).\mu'_1 rad(\delta) \Longrightarrow \tag{592}$$

$$q - 1 = 2\mu'_1 rad(\delta) - 2 \tag{593}$$

$$q + 1 = 2\mu'_1 rad(\delta) \tag{594}$$

It follows that $(q = x, 1 = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{595}$$

with $N = 12h > 0$. Let $Q(N)$ be the number of the solutions of (657) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the Diophantine equation (657) (see theorem 27.3 in [**6**]):

- If $N \equiv 2(\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3(\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0(\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.

$[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

Let (α', m'), $\alpha', m' \in \mathbb{N}^*$ be another pair, solution of the equation (657), then $\alpha'^2 - m'^2 = x^2 - y^2 = N = 12h$, but $q = x$ and $1 = y$ satisfy the equation (656) given by $x + y = 2\mu'_1 rad(\delta)$, it follows α', m' verify also $\alpha' + m' = 2\mu'_1 rad(\delta)$, that gives $\alpha' - m' = 2(\mu'_1 rad(\delta) - 1)$, then $\alpha' = x = q = 2\mu'_1 rad(\delta)$ and $m' = y = 1$. So, we have given the proof of the uniqueness of the solutions of the equation (657) with the condition $x + y = 2\mu'_1 rad(\delta)$. As $N = 12h \equiv 0(\bmod 4) \Longrightarrow Q(N) = [\tau(N/4)/2] = [\tau(3h)/2]$, the expression of $3h = \mu'_1.rad(\delta).\left(\mu'_1 rad(\delta) - 1\right)$, then $Q(N) = [\tau(3h)/2] > 1$. But $Q(N) = 1$, then the contradiction and the case $3|(1 - m.\mu'_1)$ is to reject.

** We suppose that $(\mu_1, \mu_2) = 1$.

From the equation $m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1)$, we obtain that δ is a root of the following polynomial :

$$R(Z) = Z^2 - 3Z + 3 - m.\mu_1 = 0 \tag{596}$$

The discriminant of $R(Z)$ is:

$$\Delta = 9 - 4(3 - m.\mu_1) = 4m.\mu_1 - 3 = q^2 \quad \text{with } q \in \mathbb{N}^* \quad \text{as } \delta \in \mathbb{N}^* \tag{597}$$

- We suppose that $2|m\mu_1 \Longrightarrow c$ is even. Then $q^2 \equiv 5(\mathrm{mod}\, 8)$, it gives a contradiction because a square is $\equiv 0, 1$ or $4(\mathrm{mod}\, 8)$.

- We suppose c an odd integer, then a is even. It follows $a = rad^3(a) \equiv 0(\mathrm{mod}\, 8) \Longrightarrow c \equiv 1(\mathrm{mod}\, 8)$. As $c = \delta^2 - 3X.\delta$, we obtain $\delta^2 - 3X.\delta \equiv 1(\mathrm{mod}\, 8)$. If $\delta^2 \equiv 1(\mathrm{mod}\, 8) \Longrightarrow -3X.\delta \equiv 0(\mathrm{mod}\, 8) \Longrightarrow 8|X.\delta \Longrightarrow 4|\delta \Longrightarrow c$ is even. Then, the contradiction. If $\delta^2 \equiv 4(\mathrm{mod}\, 8) \Longrightarrow \delta \equiv 2(\mathrm{mod}\, 8)$ or $\delta \equiv 6(\mathrm{mod}\, 8)$. In the two cases, we obtain $2|\delta$. Then, the contradiction with c an odd integer.

It follows that the case $c > rad^{3.26}(c)$ and $a = rad^3(a)$ is impossible.

I-3-3- We suppose $c > rad^{3.26}(c) \Longrightarrow c = rad^3(c) + h, h > rad^3(c)$, for $1 \ll c, h$ a positive integer and $\mu_a < rad^2(a) \Longrightarrow a + l = rad^3(a)$, $l > 0$. Then we obtain :

$$rad^3(c) + h = rad^3(a) - l + 1 \Longrightarrow rad^3(a) - rad^3(c) = h + l - 1 > 0 \tag{598}$$

as $rad(a) > rad^{\frac{1.63}{1.37}}(c)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h + l - 1 = m > 0 \tag{599}$$

Let $X = rad(a) - rad(c)$, then X is an integer root of the polynomial $H(X)$ defined as:

$$H(X) = X^3 + 3RX - m = 0 \tag{600}$$

To resolve the above equation, we denote $X = u + v$, then we obtain the two conditions:

$$u^3 + v^3 = m, \quad u.v = -R < 0 \Longrightarrow u^3.v^3 = -R^3$$

It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by:

$$G(t) = t^2 - mt - R^3 = 0 \tag{601}$$

The discriminant of $G(t)$ is :

$$\Delta = m^2 + 4R^3 = \alpha^2, \quad \alpha > 0 \tag{602}$$

The two real roots of (663) are:

$$t_1 = u^3 = \frac{m + \alpha}{2} \tag{603}$$

$$t_2 = v^3 = \frac{m - \alpha}{2} \tag{604}$$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the equation (664), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{605}$$

with $N = 4R^3 > 0$. From the equations (665-666), we remark that α and m verify the following equations:

$$x + y = 2u^3 = 2rad^3(a) \tag{606}$$

$$x - y = -2v^3 = 2rad^3(c) \tag{607}$$

$$\text{then} \quad x^2 - y^2 = N = 4R^3 = 4rad^3(a).rad^3(c) \tag{608}$$

Let $Q(N)$ be the number of the solutions of (667) and $\tau(N)$ is the number of suitable factorization of N, and using the same method as in the paragraph I-3-2-4-(case $3|(1 - m.\mu_1'))$, we obtain a contradiction.

It follows that the case $\mu_a \leq rad^2(a)$ and $c > rad^{3.26}(c)$ is impossible.

II- We suppose that $\mu_c \leq rad^2(c)$ and $\mu_a > rad^{1.63}(a)$:

II-1- Case $rad(c) < rad(a)$: As $c \leq rad^3(c) = rad^{1.63}(c).rad^{1.37}(c) \Longrightarrow c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(ac) \Longrightarrow \boxed{c < R^{1.63}}$.

II-2- Case $rad(a) < rad(c) < rad^{\frac{1.63}{1.37}}(a)$: As $c \leq rad^3(c) \leq rad^{1.63}(c).rad^{1.37}(c)$ $\Longrightarrow c < rad^{1.63}(c).rad^{1.63}(a) \Longrightarrow \boxed{c < R^{1.63}}$.

II-3- Case $rad^{\frac{1.63}{1.37}}(a) < rad(c)$:

II-3-1- We suppose $a \leq rad^{3.26}(a) \Longrightarrow a \leq rad^{1.63}(a).rad^{1.63}(a) \Longrightarrow a < rad^{1.63}(a).rad^{1.37}(c) \Longrightarrow a < rad^{1.63}(a).rad^{1.63}(c) \Longrightarrow 1+a < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

II-3-2- We suppose $a > rad^{3.26}(a)$ and $\mu_c \leq rad^2(c)$. Using the same method as it was explicated in the paragraphs I-3-2, I-3-3 (permuting a,c), we arrive at a contradiction. It follows that the case $\mu_c \leq rad^2(c)$ and $\mu_a > rad^{2.26}(a) \Longrightarrow a > rad^{3.26}(a)$ is impossible.

Finally, we have finished the study of the case $\mu_c \leq rad^2(c)$ and $\mu_a > rad^{2.26}(a) \Longrightarrow a > rad^{3.26}(a)$.

20.2.3.4. Case $\mu_c > rad^{1.26}(c)$ and $\mu_a > rad^2(a)$

III - As $c > rad^{2.26}(c)$ and $a > rad^3(a)$, we can write $c = rad^3(c) + h$ and $a = rad^3(a) + l$ with $l > 0$ positive integer, $h \in \mathbb{Z}$. We obtain the equation:

$$rad^3(c) - rad^3(a) = l - h + 1 \tag{609}$$

III-1- We suppose that $l-h+1>0$. Let $m=l-h+1>0$, then $rad(c)>rad(a)$. We obtain the equation:

$$rad^3(c)-rad^3(a)=l-h+1=m>0 \tag{610}$$

Let $X=rad(c)-rad(a)$, from the above equation, X is a real root of the polynomial:

$$P(X)=X^3+3RX-m=0 \tag{611}$$

As above, to resolve (672), we put $X=u+v$, then we obtain the two conditions:

$$u^3+v^3=m \tag{612}$$

$$uv=-R<0 \Longrightarrow u^3.v^3=-R^3 \tag{613}$$

Then u^3, v^3 are the roots of the equation:

$$H(Z)=Z^2-mZ-R^3=0 \tag{614}$$

The discriminant of $H(Z)$ is:
(615)

$$\Delta=m^2+4R^3=(rad^3(c)+rad^3(a))^2=\alpha^2, \quad \text{taking} \ \ \alpha>0 \Rightarrow \alpha=rad^3(c)+rad^3(a)$$

From the equation (676), we obtain that $(\alpha=x, m=y)$ is a solution of the Diophantine equation:

$$x^2-y^2=N \tag{616}$$

with $N=4R^3>0$ and $N\equiv 0(\bmod 4)$. Using the same method as in I-3-3-, we arrive to a contradiction.

III-2- We suppose $l-h+1<0$, let $m=h-l-1$, the equation (683) becomes:

$$rad^3(a)-rad^3(c)=h-l-1=m>0 \tag{617}$$

Let $X=rad(a)-rad(c)$, from the above equation, X is a real root of the polynomial:

$$P(X)=X^3+3RX-m=0 \tag{618}$$

As above, to resolve (618), we put $X=u+v$, then we obtain the two conditions:

$$u^3+v^3=m \tag{619}$$

$$uv=-R<0 \Longrightarrow u^3.v^3=-R^3 \tag{620}$$

Then u^3, v^3 are the roots of the equation:

$$H(Z)=Z^2-mZ-R^3=0 \tag{621}$$

The discriminant of $H(Z)$ is:
(622)

$$\Delta=m^2+4R^3=(rad^3(c)+rad^3(a))^2=\alpha^2, \quad \text{taking} \ \ \alpha>0 \Rightarrow \alpha=rad^3(c)+rad^3(a)$$

From the equation (622), we obtain that $(\alpha=x, m=y)$ is a solution of the Diophantine equation:

$$x^2-y^2=N \tag{623}$$

with $N = 4R^3 > 0$ and $N \equiv 0 (\bmod 4)$. Using the same method as in I-3-3-, we arrive to a contradiction.

It follows that the case $\mu_c > rad^{1.26}(c)$ and $\mu_a > rad^2(a)$ is impossible.

20.2.3.5. Case $\mu_a > rad^{1.26}(a)$ and $\mu_c > rad^2(c)$

IV - This case is similar to the case III above and we obtain the same result: $\mu_a > rad^{1.26}(a)$ and $\mu_c > rad^2(c)$ impossible to obtain. Then the case is to reject.

Finally, we can annonce the following important theorem:

Theorem 33. — *Let a, c positive integers relatively prime with $c = a + 1$, then $c < rad^{1.63}(ac)$.*

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CHAPTER 21

THE PROOF OF THE CONJECTURE $c < rad^{1.63}(abc)$

Abstract. — In this paper, we consider the *abc* conjecture. We give the proof of the conjecture $c < rad^{1.63}(abc)$ that constitutes the key to resolve the *abc* conjecture.

To my teachers and to ***Madam Fatma Moalla****,* ***Mr Chedly Touibi***
my professors of mathematics at the Faculty of Sciences of Tunis

21.1. Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{624}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{625}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**8**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 33. — *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{626}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**5**]. It concerned the best example given by E. Reyssat [**5**]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{627}$$

A conjecture was proposed that $c < rad^2(abc)$ [**3**]. In 2012, A. Nitaj [**4**] proposed the following conjecture:

Conjecture 34. — *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < rad^{1.63}(abc) \tag{628}$$

$$abc < rad^{4.42}(abc) \tag{629}$$

Here, we will give a proof of the conjecture given by (628). This result, I think is the key to obtain the final proof of the veracity of the *abc* conjecture.

21.2. A Proof of the conjecture (34) case $c = a + b$

Let a, b, c be positive integers, relatively prime, with $c = a + b$, $b < a$ and $R = rad(abc)$, $c = \prod_{j' \in J'} c_{j'}^{\beta_{j'}}$, $\beta_{j'} \geq 1$.

In a previous paper [**1**], we has given, for the case $c = a + 1$, the proof that $c < rad^{1.63}(ac)$. In the following, we will give the proof for the case $c = a + b$.

Proof. — If $c < rad(abc)$, then we obtain:

$$c < rad(abc) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}} \tag{630}$$

and the condition (628) is satisfied.

If $c = rad(abc)$, then a, b, c are not coprime, case to reject.

In the following, we suppose that $c > rad(abc)$ and a, b and c are not prime numbers.

$$c = a + b = \mu_a rad(a) + \mu_b rad(b) \overset{?}{<} rad^{1.63}(abc) \tag{631}$$

21.2.1. $\mu_a \neq 1$, $\mu_a \leq rad^{0.63}(a)$

We obtain :

(632)
$$c = a + b < 2a \leq 2rad^{1.63}(a) < rad^{1.63}(abc) \Longrightarrow c < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

Then (631) is satisfied.

21.2.2. $\mu_c \neq 1$, $\mu_c \leq rad^{0.63}(c)$

We obtain :

$$c = \mu_c rad(c) \leq rad^{1.63}(c) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}} \tag{633}$$

and the condition (631) is satisfied.

21.2.3. $\mu_a > rad^{0.63}(a)$ **and** $\mu_c > rad^{0.63}(c)$

21.2.3.1. **Case:** $rad^{0.63}(c) < \mu_c \leq rad^{1.63}(c)$ **and** $rad^{0.63}(a) < \mu_a \leq rad^{1.63}(a)$**:**

We can write:

$$\left.\begin{array}{l} \mu_c \leq rad^{1.63}(c) \Longrightarrow c \leq rad^{2.63}(c) \\ \\ \mu_a \leq rad^{1.63}(a) \Longrightarrow a \leq rad^{2.63}(a) \end{array}\right\} \Longrightarrow ac \leq rad^{2.63}(ac) \Longrightarrow a^2 < ac \leq rad^{2.63}(ac)$$

$$\Longrightarrow a < rad^{1.315}(ac) \Longrightarrow c < 2a < 2rad^{1.315}(ac) < rad^{1.63}(abc)$$

$$\Longrightarrow \boxed{c = a + b < R^{1.63}} \tag{634}$$

21.2.3.2. **Case:** $\mu_c > rad^{1.63}(c)$ **or** $\mu_a > rad^{1.63}(a)$

I- We suppose that $\mu_c > rad^{1.63}(c)$ and $\mu_a \leq rad^2(a)$:

I-1- Case $rad(a) < rad(c)$: In this case $a = \mu_a.rad(a) \leq rad^3(a) \leq rad^{1.63}(a)rad^{1.37}(a) < rad^{1.63}(a).rad^{1.37}(c) \Longrightarrow c < 2a < 2rad^{1.63}(a).rad^{1.37}(c) < rad^{1.63}(abc) \Longrightarrow c < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

I-2- Case $rad(c) < rad(a) < rad^{\frac{1.63}{1.37}}(c)$: As $a \leq rad^{1.63}(a).rad^{1.37}(a) < rad^{1.63}(a).rad^{1.63}(c) \Longrightarrow c < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

I-3- Case $rad^{\frac{1.63}{1.37}}(c) < rad(a)$:

I-3-1- We suppose $c \leq rad^{3.26}(c)$, we obtain:

$$c \leq rad^{3.26}(c) \Longrightarrow c \leq rad^{1.63}(c).rad^{1.63}(c) \Longrightarrow$$

$$c < rad^{1.63}(c).rad(a)^{1.37} < rad^{1.63}(c).rad(a)^{1.63}.rad^{1.63}(b) = R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$$

I-3-2- We suppose $c > rad^{3.26}(c) \Longrightarrow \mu_c > rad^{2.26}(c)$. We consider the case $\mu_a = rad^2(a) \Longrightarrow a = rad^3(a)$. Then, we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$X^3 + 1 = c - b + 1 = c' \tag{635}$$

But it is the case $c' = 1 + a$. If $c' = rad^n(c')$ with $n \geq 4$, we obtain the equation:

$$rad^n(c') - rad^3(a) = 1 \tag{636}$$

But the solutions of the equation (636) are [**2**] :$(rad(c') = 3, n = 2, rad(a) = +2)$, it follows the contradiction with $n \geq 4$ and the case $c' = rad^n(c'), n \geq 4$ is to reject.

In the following, we will study the cases $\mu'_c = A.rad^n(c')$ with $rad(c') \nmid A, n \geq 0$. The above equation (635) can be written as :

$$(X+1)(X^2 - X + 1) = c' \tag{637}$$

Let δ any divisor of c', then:

$$X + 1 = \delta \tag{638}$$

$$X^2 - X + 1 = \frac{c'}{\delta} = c" = \delta^2 - 3X \tag{639}$$

We recall that $rad(a) > rad^{\frac{1.63}{1.37}}(c)$.

I-3-2-1- We suppose $\delta = l.rad(c')$. We have $\delta = l.rad(c') < c' = \mu'_c.rad(c') \Longrightarrow l < \mu'_c$. As δ is a divisor of c', then l is a divisor of μ'_c, we write $\mu'_c = l.m$. From $\mu'_c = l(\delta^2 - 3X)$, we obtain:

$$m = l^2 rad^2(c') - 3rad(a) \Longrightarrow 3rad(a) = l^2 rad^2(c') - m$$

A- Case $3|m \Longrightarrow m = 3m', m' > 1$: As $\mu'_c = ml = 3m'l \Longrightarrow 3|rad(c')$ and $(rad(c'), m')$ not coprime. We obtain:

$$rad(a) = l^2 rad(c').\frac{rad(c')}{3} - m'$$

It follows that a,c' are not coprime, then the contradiction.

B - Case $m = 3 \Longrightarrow \mu'_c = 3l \Longrightarrow c' = 3lrad(c') = 3\delta = \delta(\delta^2 - 3X) \Longrightarrow \delta^2 = 3(1 + X) = 3\delta \Longrightarrow \delta = lrad(c') = 3$, then the contradiction.

I-3-2-2- We suppose $\delta = l.rad^2(c'), l \geq 2$. If $lrad(c') \nmid \mu'_c$ then the case is to reject. We suppose $lrad(c')|\mu'_c \Longrightarrow \mu'_c = m.lrad(c')$, then $\dfrac{c'}{\delta} = m = \delta^2 - 3rad(a)$.

C - Case $m = 1 = c'/\delta \Longrightarrow \delta^2 - 3rad(a) = 1 \Longrightarrow (\delta - 1)(\delta + 1) = 3rad(a) = rad(a)(\delta + 1) \Longrightarrow \delta = 2 = l.rad^2(c')$, then the contradiction.

D - Case $m = 3$, we obtain $3(1 + rad(a)) = \delta^2 = 3\delta \Longrightarrow \delta = 3 = lrad^2(c')$. Then the contradiction.

E - Case $m \neq 1, 3$, we obtain: $3rad(a) = l^2 rad^4(c') - m \Longrightarrow rad(a)$ and $rad(c')$ are not coprime. Then the contradiction.

I-3-2-3- We suppose $\delta = l.rad^n(c'), l \geq 2$ with $n \geq 3$. From $c' = \mu'_c.rad(c') = lrad^n(c')(\delta^2 - 3rad(a))$, we denote $m = \delta^2 - 3rad(a) = \delta^2 - 3X$.

F - As seen above (paragraphs C,D), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G - Case $m \neq 1, 3$. Let q be a prime that divides m, it follows $q|\mu'_c \Longrightarrow q = c'_{j'_0} \Longrightarrow c'_{j'_0}|\delta^2 \Longrightarrow c'_{j'_0}|3rad(a)$. Then $rad(a)$ and $rad(c')$ are not coprime. It follows the contradiction.

I-3-2-4- We suppose $\delta = \prod_{j\in J_1} c_j'^{\beta_j}$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with $\beta_{j_0} \geq 2$, $rad(c') \nmid \delta$. We can write:

$$\delta = \mu_\delta.rad(\delta), \quad rad(c') = m.rad(\delta), \quad m > 1, \quad (m, \mu_\delta) = 1 \tag{640}$$

Then, we obtain:

$$c' = \mu'_c.rad(c') = \mu'_c.m.rad(\delta) = \delta(\delta^2 - 3X) = \mu_\delta.rad(\delta)(\delta^2 - 3X) \Longrightarrow$$
$$m.\mu'_c = \mu_\delta(\delta^2 - 3X) \tag{641}$$

- If $\mu'_c = \mu_\delta \Longrightarrow m = \delta^2 - 3X = (\mu'_c.rad(\delta))^2 - 3X$. As $\delta < \delta^2 - 3X \Longrightarrow m > \delta \Longrightarrow rad(c') > m > \mu'_c.rad(\delta) > rad^3(c')$ because $\mu'_c > rad^{2.26}(c')$, it follows $rad(c') > rad^2(c')$. Then the contradiction.

- We suppose $\mu'_c < \mu_\delta$. As $rad(a) = \mu_\delta rad(\delta) - 1$, we obtain:

$$rad(a) > \mu'_c.rad(\delta) - 1 > 0 \Longrightarrow rad(ac') > c'.rad(\delta) - rad(c') > 0 \Longrightarrow$$
$$c' > rad(ac') > c'.rad(\delta) - rad(c') > 0 \Longrightarrow 1 > rad(\delta) - \frac{rad(c')}{c'} > 0, \quad rad(\delta) \geq 2$$
$$\Longrightarrow \text{The contradiction} \tag{642}$$

- We suppose $\mu_\delta < \mu'_c$. In this case, from the equation (641) and as $(m, \mu_\delta) = 1$, it follows we can write:

$$\mu'_c = \mu_1.\mu_2, \quad \mu_1, \mu_2 > 1 \tag{643}$$

$$c' = \mu'_c rad(c') = \mu_1.\mu_2.rad(\delta).m = \delta.(\delta^2 - 3X) \tag{644}$$

$$\text{so that} \quad m.\mu_1 = \delta^2 - 3X, \quad \mu_2 = \mu_\delta \Longrightarrow \delta = \mu_2.rad(\delta) \tag{645}$$

** We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists\, c'_{j_0}$ so that $c'_{j_0}|\mu_1$ and $c'_{j_0}|\mu_2$. But $\mu_\delta = \mu_2 \Rightarrow c'^2_{j_0}|\delta$. From $3X = \delta^2 - m\mu_1 \Longrightarrow c'_{j_0}|3X \Longrightarrow c'_{j_0}|X$ or $c'_{j_0} = 3$.

- If $c'_{j_0}|X$, it follows the contradiction with $(c', a) = 1$.

- If $c'_{j_0} = 3$. We have $m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1) \Longrightarrow \delta^2 - 3\delta + 3 - m.\mu_1 = 0$. As $3|\mu_1 \Longrightarrow \mu_1 = 3^k\mu'_1, 3 \nmid \mu'_1, k \geq 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - 3^{k-1}m\mu'_1) = 0 \tag{646}$$

- We consider the case $k > 1 \Longrightarrow 3 \nmid (1 - 3^{k-1}m\mu'_1)$. Let us recall the Eisenstein criterion [**7**]:

Theorem 34. — ***(Eisenstein Criterion)*** *Let $f = a_0 + \cdots + a_n X^n$ be a polynomial $\in \mathbb{Z}[X]$. We suppose that $\exists\, p$ a prime number so that:*
- $p \nmid a_n$,
- $p | a_i,\ (0 \leq i \leq n-1)$,
- $p^2 \nmid a_0$.

Then f is irreducible in $\mathbb{Q}$.

We apply Eisenstein criterion to the polynomial $R(Z)$ given by:

$$R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1} m\mu'_1) \tag{647}$$

then:
- $3 \nmid 1$,
- $3|(-3)$,
- $3|3(1 - 3^{k-1} m\mu'_1)$,
- $3^2 \nmid 3(1 - 3^{k-1} m\mu'_1)$.

It follows that the polynomial $R(Z)$ is irreducible in $\mathbb{Q}$, then, the contradiction with $R(\delta) = 0$.

- We consider the case $k = 1$, then $\mu_1 = 3\mu'_1$ and $(\mu'_1, 3) = 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - m\mu'_1) = 0 \tag{648}$$

* If $3 \nmid (1 - m.\mu'_1)$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1)$$

and we find a contradiction with $R'(\delta) = 0$.

* We consider that $3|(1 - m.\mu'_1) \Longrightarrow m\mu'_1 - 1 = 3^i.h$, $i \geq 1$, $3 \nmid h, h \in \mathbb{N}^*$. δ is an integer root of the polynomial $R'(Z)$:

$$R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1) = 0 \Longrightarrow \text{ the discriminant of } R'(Z) \text{ is :}$$

$$\Delta = 3^2 + 3^{i+1} \times 4.h \tag{649}$$

As the root δ is an integer, it follows that $\Delta = l^2 > 0$ with l a positive integer. We obtain:

$$\Delta = 3^2(1 + 3^{i-1} \times 4h) = l^2 \tag{650}$$

$$\Longrightarrow 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^* \tag{651}$$

We can write the equation (648) as :

$$\delta(\delta - 3) = 3^{i+1}.h \Longrightarrow 3^3\mu'_1 \frac{rad(\delta)}{3}.\left(\mu'_1 rad(\delta) - 1\right) = 3^{i+1}.h \Longrightarrow \tag{652}$$

$$\mu'_1 \frac{rad(\delta)}{3}.\left(\mu'_1 rad(\delta) - 1\right) = h \tag{653}$$

We obtain $i = 2$ and $q^2 = 1 + 12h = 1 + 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1)$. Then, q satisfies :

$$q^2 - 1 = 12h \Rightarrow \frac{(q-1)}{2}.\frac{(q+1)}{2} = 3h = (\mu'_1 rad(\delta) - 1).\mu'_1 rad(\delta) \Rightarrow \tag{654}$$

$$q - 1 = 2\mu'_1 rad(\delta) - 2 \tag{655}$$

$$q + 1 = 2\mu'_1 rad(\delta) \tag{656}$$

It follows that $(q = x, 1 = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{657}$$

with $N = 12h > 0$. Let $Q(N)$ be the number of the solutions of (657) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the Diophantine equation (657) (see theorem 27.3 in [**6**]):

- If $N \equiv 2 (\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3 (\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0 (\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.

$[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

Let (α', m'), $\alpha', m' \in \mathbb{N}^*$ be another pair, solution of the equation (657), then $\alpha'^2 - m'^2 = x^2 - y^2 = N = 12h$, but $q = x$ and $1 = y$ satisfy the equation (656) given by $x + y = 2\mu'_1 rad(\delta)$, it follows α', m' verify also $\alpha' + m' = 2\mu'_1 rad(\delta)$, that gives $\alpha' - m' = 2(\mu'_1 rad(\delta) - 1)$, then $\alpha' = x = q = 2\mu'_1 rad(\delta)$ and $m' = y = 1$. So, we have given the proof of the uniqueness of the solutions of the equation (657) with the condition $x + y = 2\mu'_1 rad(\delta)$. As $N = 12h \equiv 0 (\bmod 4) \Longrightarrow Q(N) = [\tau(N/4)/2] = [\tau(3h)/2]$, the expression of $3h = \mu'_1.rad(\delta).(\mu'_1 rad(\delta) - 1)$, then $Q(N) = [\tau(3h)/2] > 1$. But $Q(N) = 1$, then the contradiction and the case $3|(1 - m.\mu'_1)$ is to reject.

** We suppose that $(\mu_1, \mu_2) = 1$.

From the equation $m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1)$, we obtain that δ is a root of the following polynomial :

$$R(Z) = Z^2 - 3Z + 3 - m.\mu_1 = 0 \tag{658}$$

The discriminant of $R(Z)$ is:

$$\Delta = 9 - 4(3 - m.\mu_1) = 4m.\mu_1 - 3 = q^2 \quad \text{with } q \in \mathbb{N}^* \quad \text{as } \delta \in \mathbb{N}^* \tag{659}$$

- We suppose that $2|m\mu_1 \Longrightarrow c'$ is even. Then $q^2 \equiv 5 (\bmod 8)$, it gives a contradiction because a square is $\equiv 0, 1$ or $4 (\bmod 8)$.

- We suppose c' an odd integer, then a is even. It follows $a = rad^3(a) \equiv 0 (\bmod 8) \Longrightarrow c' \equiv 1 (\bmod 8)$. As $c' = \delta^2 - 3X.\delta$, we obtain $\delta^2 - 3X.\delta \equiv 1 (\bmod 8)$. If $\delta^2 \equiv 1 (\bmod 8) \Longrightarrow -3X.\delta \equiv 0 (\bmod 8) \Longrightarrow 8|X.\delta \Longrightarrow 4|\delta \Longrightarrow c'$ is even. Then, the contradiction. If $\delta^2 \equiv 4 (\bmod 8) \Longrightarrow \delta \equiv 2 (\bmod 8)$ or $\delta \equiv 6 (\bmod 8)$. In the two cases, we obtain $2|\delta$. Then, the contradiction with c' an odd integer.

It follows that the case $c > rad^{3.26}(c)$ and $a = rad^3(a)$ is impossible.

I-3-3- We suppose $c > rad^{3.26}(c)$ and large, then $c = rad^3(c) + h, h > rad^3(c)$, h a positive integer and $\mu_a < rad^2(a) \Longrightarrow a + l = rad^3(a)$, $l > 0$. Then we obtain :

$$rad^3(c) + h = rad^3(a) - l + b \Longrightarrow rad^3(a) - rad^3(c) = h + l - b > 0 \tag{660}$$

as $rad(a) > rad^{\frac{1.63}{1.37}}(c)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h + l - b = m > 0 \tag{661}$$

Let $X = rad(a) - rad(c)$, then X is an integer root of the polynomial $H(X)$ defined as:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \tag{662}$$

To resolve the above equation, we denote $X = u + v$, then we obtain the two conditions:

$$u^3 + v^3 = m, \quad u.v = -rad(ac) < 0 \Longrightarrow u^3.v^3 = -rad^3(ac)$$

It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by:

$$G(t) = t^2 - mt - rad^3(ac) = 0 \tag{663}$$

The discriminant of $G(t)$ is :

$$\Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > 0 \tag{664}$$

The two real roots of (663) are:

$$t_1 = u^3 = \frac{m + \alpha}{2} \tag{665}$$

$$t_2 = v^3 = \frac{m - \alpha}{2} \tag{666}$$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the equation (664), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{667}$$

with $N = 4rad^3(ac) > 0$. From the equations (665-666), we remark that α and m verify the following equations:

$$x + y = 2u^3 = 2rad^3(a) \tag{668}$$

$$x - y = -2v^3 = 2rad^3(c) \tag{669}$$

$$\text{then} \quad x^2 - y^2 = N = 4rad^3(a).rad^3(c) \tag{670}$$

Let $Q(N)$ be the number of the solutions of (667) and $\tau(N)$ is the number of suitable factorization of N, and using the same method as in the paragraph I-3-2-4-

(case $3|(1 - m.\mu'_1)$), we obtain a contradiction.

It follows that the cases $\mu_a \leq rad^2(a)$ and $c > rad^{3.26}(c)$ are impossible.

II- We suppose that $rad^{1.63}(c) < \mu_c \leq rad^2(c)$ and $\mu_a > rad^{1.63}(a)$:

II-1- Case $rad(c) < rad(a)$: As $c \leq rad^3(c) = rad^{1.63}(c).rad^{1.37}(c) \Longrightarrow c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(ac) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

II-2- Case $rad(a) < rad(c) < rad^{\frac{1.63}{1.37}}(a)$: As $c \leq rad^3(c) \leq rad^{1.63}(c).rad^{1.37}(c) \Longrightarrow c < rad^{1.63}(c).rad^{1.63}(a) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

II-3- Case $rad^{\frac{1.63}{1.37}}(a) < rad(c)$:

II-3-1- We suppose $rad^{2.63}(a) < a \leq rad^{3.26}(a) \Longrightarrow a \leq rad^{1.63}(a).rad^{1.63}(a) \Longrightarrow a < rad^{1.63}(a).rad^{1.37}(c) \Longrightarrow c = a + b < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < rad^{1.63}(abc) \Longrightarrow c < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

II-3-2- We suppose $a > rad^{3.26}(a)$ and $\mu_c \leq rad^2(c)$. Using the same method as it was explicated in the paragraphs I-3-2, I-3-3 (permuting a,c), we arrive at a contradiction. It follows that the case $\mu_c \leq rad^2(c)$ and $a > rad^{3.26}(a)$ is impossible.

Finally, we have finished the study of the case $rad^{1.63}(c) < \mu_c \leq rad^2(c)$ and $\mu_a > rad^{1.63}(a)$.

21.2.3.3. Case $\mu_c > rad^{1.63}(c)$ and $\mu_a > rad^{1.63}(a)$

Taking into account the cases studied above, it remains to see the following two cases:
- $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$,
- $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c)$.

III-1- We suppose $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a) \Longrightarrow c > rad^3(c)$ and $a > rad^{2.63}(a)$. We can write $c = rad^3(c) + h$ and $a = rad^3(a) + l$ with h a positive integer and $l \in \mathbb{Z}$.

III-1-1- We suppose $rad(c) < rad(a)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h - l - b = m > 0 \tag{671}$$

Let $X = rad(a) - rad(c)$, from the above equation, X is a real root of the polynomial:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \tag{672}$$

As above, to resolve (672), we denote $X = u + v$, then we obtain the two conditions:

$$u^3 + v^3 = m \tag{673}$$

$$uv = -rad(ac) < 0 \Longrightarrow u^3.v^3 = -rad^3(ac) \tag{674}$$

It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by :

$$G(t) = t^2 - mt - rad^3(ac) = 0 \tag{675}$$

The discriminant of $G(t)$ is:

$$\Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > \tag{676}$$

The two real roots of (675) are:

$$t_1 = u^3 = \frac{m+\alpha}{2} \tag{677}$$

$$t_2 = v^3 = \frac{m-\alpha}{2} \tag{678}$$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the equation (676), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{679}$$

with $N = 4rad^3(ac) > 0$. From the equations (677-678), we remark that α and m verify the following equations:

$$x + y = 2u^3 = 2rad^3(a) \tag{680}$$

$$x - y = -2v^3 = 2rad^3(c) \tag{681}$$

$$\text{then} \quad x^2 - y^2 = N = 4rad^3(a).rad^3(c) \tag{682}$$

Let $Q(N)$ be the number of the solutions of (679) and $\tau(N)$ is the number of suitable factorization of N, and using the same method as in the paragraph I-3-2-4-(case $3|(1 - m.\mu_1'))$, we obtain a contradiction.

III-1-2- We suppose $rad(a) < rad(c)$. We obtain the equation:

$$rad^3(c) - rad^3(a) = b + l - h = m > 0 \tag{683}$$

Using the same calculations as in III-1-1-, we find a contradiction.

It follows that the case $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$ is impossible.

III-2- We suppose $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c) \Longrightarrow a > rad^3(a)$ and $c > rad^{2.63}(c)$. We can write $a = rad^3(a) + h$ and $c = rad^3(c) + l$ with h a positive integer and $l \in \mathbb{Z}$.

The calculations are similar to those in case III-1-. We obtain the same results namely the cases of III-2- to be rejected.

It follows that the case $\mu_c > rad^{1.63}(c)$ and $\mu_a > rad^2(a)$ is impossible. $\square$

We can state the following important theorem:

Theorem 35. — *Let a, b, c positive integers relatively prime with $c = a + b$, then $c < rad^{1.63}(abc)$.*

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CHAPTER 22

ASSUMING $c < R^2(abc)$, THE abc CONJECTURE IS TRUE

Abstract. — In this paper, we consider the *abc* conjecture. Assuming that $c < rad^2(abc)$ is true, we give the proof of the *abc* conjecture for $\epsilon \geq 1$, then for $\epsilon \in]0,1[$.

To the memory of my Father who taught me arithmetic, To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

22.1. *Introduction and notations*

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a). \prod_i a_i^{\alpha_i - 1} \tag{684}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{685}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 35. — *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{686}$$

where K is a constant depending only of ϵ.

The idea to try to write a paper about this conjecture was born after the publication in September 2018, of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**1**]. A conjecture was proposed that $c < rad^2(abc)$ [**3**]. It is the key to resolve the *abc* conjecture. In my paper, I assume that the conjecture $c < rad^2(abc)$ holds, I propose an elementary proof of the *abc* conjecture. The paper is organized as follows: in the second section, we give the proof of the *abc* conjecture.

22.2. *The Proof of the* abc *conjecture*

We note $R = rad(abc)$ in the case $c = a + b$ or $R = rad(ac)$ in the case $c = a + 1$. We assume that $c < R^2$ is true. We recall the following proposition [**4**]:

Proposition 22.1. — *Let $\epsilon \longrightarrow K(\epsilon)$ the application verifying the abc conjecture, then:*

$$lim_{\epsilon\to 0}K(\epsilon) = +\infty \tag{687}$$

22.2.1. Case : $\epsilon \geq 1$

Using the result that $c < R^2$, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = e, \;\; \epsilon \geq 1 \tag{688}$$

Then the *abc* conjecture is true.

22.2.2. Case: $\epsilon < 1$

22.2.2.1. Case: $c < R$

In this case, we can write :

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = e > 1, \;\; \epsilon < 1 \tag{689}$$

Then the *abc* conjecture is true.

22.2.2.2. Case: $c > R$

In this case, we consider the contradiction of the abc conjecture:

$$\exists\, \epsilon_0 \in]0,1[, \forall\, K(\epsilon), \quad \exists\, c_0 = a_0 + b_0 \quad \text{so that} \;\; c_0 > K(\epsilon_0)R_0^{1+\epsilon_0} \Longrightarrow c_0 \text{ not a prime} \tag{690}$$

We choose the constant $K(\epsilon) = e^{\frac{1}{\epsilon^2}}$. Let :

$$Y_{c_0}(\epsilon) = \frac{1}{\epsilon^2} + (1+\epsilon)LogR_0 - Logc_0, \epsilon \in]0,1[\tag{691}$$

We have $lim_{\epsilon\longrightarrow 1}Y_{c_0}(\epsilon) = 1 + Log(R_0^2/c_0) > 0$ and $lim_{\epsilon\longrightarrow 0}Y_{c_0}(\epsilon) = +\infty$. The function $Y_{c_0}(\epsilon)$ has a derivative for $\epsilon \in]0,1[$, we obtain:

$$Y'_{c_0}(\epsilon) = -\frac{2}{\epsilon^3} + LogR_0 = \frac{\epsilon^3 LogR_0 - 2}{\epsilon^3} \tag{692}$$

$Y'_{c_0}(\epsilon) = 0 \Longrightarrow \epsilon = \epsilon' = \sqrt[3]{\dfrac{2}{LogR_0}} \in]0,1[$.

Discussion:

- If $Y_{c_0}(\epsilon') > 0$, it follows that $\forall \epsilon \in]0,1[, Y_{c_0}(\epsilon) > 0$, then the contradiction with $Y_{c_0}(\epsilon_0) < 0 \Longrightarrow c_0 > K(\epsilon_0)R_0^{1+\epsilon_0}$. Hence the *abc* conjecture is true for $\epsilon \in]0,1[$.

- If $Y_{c_0}(\epsilon') < 0 \Longrightarrow \exists\ 0 < \epsilon_1 < \epsilon' < \epsilon_2 < 1$, so that $Y_{c_0}(\epsilon_1) = Y_{c_0}(\epsilon_2) = 0$. Then we obtain $c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} = K(\epsilon_2)R_0^{1+\epsilon_2}$. We consider the equality :

$$c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} \Longrightarrow \mu_{c_0} = e^{\frac{1}{\epsilon_1^2}} rad(ab)R_0^{\epsilon_1} \tag{693}$$

If the right member of the above equation is an integer, we obtain a contradiction with a_0, b_0, c_0 coprime. If not, we find that an integer μ_{c_0} is equal to a real number $e^{\frac{1}{\epsilon_1^2}} rad(ab)R_0^{\epsilon_1}$. Then the contradiction again, it follows that the *abc* conjecture is true.

Then the proof of the *abc* conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with a, b, c relatively coprime:

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{694}$$

and the constant $K(\epsilon)$ depends only of ϵ.

Q.E.D

Ouf, end of the mystery!

22.3. *Conclusion*

Assuming $c < R^2$ is true, we have given an elementary proof of the *abc* conjecture. We can announce the important theorem:

Theorem 36. — *For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, assuming $c < rad^2(abc)$ holds, then:*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{695}$$

where K is a constant depending of ϵ.

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CHAPTER 23

A PROOF OF THE *abc* CONJECTURE

Abstract. — In this paper about the *abc* conjecture, assuming the condition $c < rad^2(abc)$ holds, and the constant $K(\epsilon)$ is a smooth function, having a derivative for $\epsilon \in]0,1[$, then we give the proof of the *abc* conjecture.

To the memory of my Father who taught me arithmetic
To my wife ***Wahida****, my daughter* ***Sinda*** *and my son* ***Mohamed Mazen***

23.1. Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{696}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{697}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 36. — *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{698}$$

where K is a constant depending only of ϵ.

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in September 2018, about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**1**]. A conjecture was proposed that $c < rad^2(abc)$ [**3**]. It is the key to resolve the *abc* conjecture. In my paper, we assume that the last conjecture holds, and the constant $K(\epsilon)$ for $\epsilon \in]0,1[$ is a smooth function having a derivative for $\epsilon \in]0,1[$. The paper is organized as follows: in the second section, we begin by presenting some properties of the constant $K(\epsilon)$, then we give the proof of the *abc* conjecture.

23.2. The Proof of the *abc* Conjecture

Let a, b, c positive integers relatively prime with $c = a + b, a > b, b \geq 2$. We denote $R = rad(abc), I =]0,1[$. For $c < R$, it is trivial that the *abc* conjecture holds. In the following, we consider the triples (a, b, c) with a, b, c relatively coprime and $c > R$. As we assume that $c < R^2$, it follows that $\forall\, \epsilon \geq 1$, it suffices to take $K(\epsilon) = 1$ and c satisfies $c < K(\epsilon)R^{1+\epsilon}$ and the *abc* conjecture is true.

23.2.1. Properties of the constant $K(\epsilon)$

- From the definition of the *abc* conjecture, above, the constant $K(\epsilon)$ is a positive real number, and for every $\epsilon > 0$, it exists a number $K(\epsilon)$ dependent only of ϵ.

- In the following, we consider that $\epsilon \in I$. We can say that K is a function $K : \epsilon \in I \longrightarrow K(\epsilon) \in]0, +\infty[$, so that $c < K(\epsilon)R^{1+\epsilon}$ holds, if the *abc* conjecture is true. Assuming that $c < R^2$ is satisfying, we can adopt that $K(\epsilon = 1) = 1$, because $c < K(1)R^{1+1}$. Then we choose $K(\epsilon)$ so that $lim_{\epsilon \longrightarrow 1^-} K(\epsilon) = K(1)$

- We obtain also that $K(\epsilon) > 1$ if $\epsilon \in I$. If not, we consider the example $9 = 8 + 1$, we take $\epsilon = 0.2$, then $c < K(0.2)R^{1.02} < 1.R^{1.2}$. But $c = 9 > 6^{1.2} \approx 8.58$, then the contradiction.

- In 1996, A. Nitaj had confirmed that the constant $K(\epsilon)$ verifies [**4**]:

$$lim_{\epsilon \longrightarrow 0} K(\epsilon) = +\infty \tag{699}$$

It follows that the function $K(\epsilon)$ is very large when ϵ is very small.

23.2.2. The proof of the *abc* conjecture

Proof. — Let us suppose that $K(\epsilon)$ is a smooth function having a derivative in every point $\in]0,1[$. Let a, b, c positive integers relatively prime with $c = a + b, c > R$. We denote :

$$Y_c(\epsilon) = LogK(\epsilon) + (1+\epsilon)LogR - Logc \tag{700}$$

We obtain $lim_{\epsilon\to 1} Y_c(\epsilon) = 2LogR - Logc = y_1 > 0$, assuming $c < R^2$, and $lim_{\epsilon\longrightarrow 0} Y_c(\epsilon) = +\infty$. The derivative of $Y_c(\epsilon)$ gives:

$$Y'_c(\epsilon) = \frac{K'(\epsilon)}{K(\epsilon)} + LogR \tag{701}$$

We have the following cases:

i)- If $Y'_c(\epsilon) > 0$ for all $\epsilon \in]0,1[$, then Y is an increasing function of ϵ. It follows the contradiction because $lim_{\epsilon\longrightarrow 0} Y_c(\epsilon) = +\infty$.

ii) - If $Y'_c(\epsilon) < 0$ for all $\epsilon \in]0,1[$, then Y is a decreasing function of ϵ. It follows $\forall\, \epsilon,\ Y_c(\epsilon) > 0 \Longrightarrow c < K(\epsilon)R^{1+\epsilon}$ is satisfied. As c is an arbitrary integer with the condition $c > R$, we deduce that the *abc* conjecture is true.

iii) - If $Y'_c(\epsilon) = 0$ for some $\epsilon_0 \in]0,1[$. ϵ_0 is a solution of the equation :

$$-\frac{K'(\epsilon_0)}{K(\epsilon_0)} = LogR$$

We remark that ϵ_0 depends of R, then of a, b, c.

* If $Y_c(\epsilon_0)$ is positive, then $0 < Y_c(\epsilon_0) \leq Y_c(\epsilon) \Longrightarrow Y_c(\epsilon) > 0$. As above, we deduce that the *abc* conjecture holds for the triplet (a, b, c).

** If $Y_c(\epsilon_0)$ is negative, then it exists two values ϵ_1, ϵ_2 with $0 < \epsilon_1 < \epsilon_0 < \epsilon_2 < 1$, so that $Y_c(\epsilon_1) = Y_c(\epsilon_2) = 0$. It follows for example, that $c = K(\epsilon_1)R^{\epsilon_1}.rad(abc)$. Suppose that $K(\epsilon_1)R^{\epsilon_1}$ is an integer, we obtain that a, b, c are not coprime. Then the contradiction and this case to reject.

Then, we have obtained that the *abc* conjecture holds for $\forall\epsilon \in I$ for the triplet (a, b, c), as it is chosen arbitrary with the condition $c > rad(abc)$. It follows that the *abc* conjecture is true, assuming that $c < R^2$. □

Q.E.D
End of the mystery!

23.3. Conclusion

Finally, assuming $c < R^2$, and choosing the constant $K(\epsilon)$ as a smooth function, having a derivative for $\epsilon \in]0, 1[$, we have given an elementary proof that the *abc* conjecture is true.

We can announce the important theorem:

Theorem 37. — *For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, assuming $c < rad^2(abc)$, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{702}$$

where K is a constant depending only of ϵ and varying smoothly, having a derivative.

Acknowledgments. The author is very grateful to Professors Mihăilescu Preda, and Gérald Tenenbaum for their comments about errors found in previous manuscripts concerning proposed proofs of the *abc* conjecture.

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