On a differential equation of Lienard type with strong nonlinearities

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Abstract

In this paper we present a remarkable Lienard equation with strong and high order nonlinear terms. The equation is explicitly integrable in terms of periodic tangent function. The related quadratic Lienard type equations may also exhibit the tangent function profile. The presented equation includes for example, the cubic and quintic Duffing equations with cubic singularity as special cases.

Keywords: Lienard equations, strong and high order nonlinearities, periodic general solutions, cubic Duffing equation, tangent function.

Introduction

The identification of differential equations susceptible to represent nonlinear conservative oscillators is the object of an intensive study in the literature. One of these, widely investigated in the literature is the cubic Duffing equation

$$\ddot{x} + \alpha x + \beta x^3 = 0 \tag{1}$$

where α and β are arbitrary constants and overdot denotes a differentiation with respect to time. This equation has been considered for a long time as a conservative oscillator having only periodic solutions. These solutions are the Jacobi elliptic functions [1-3]. The only way to such periodic solutions to exhibit trigonometric functions behavior is when the elliptic modulus is equal to zero. To exhibit non-periodic behavior, the hyperbolic behavior for example, as well known, the elliptic modulus should be equal to one. However, recently, Adjaï et al. [4] have shown that the Duffing equation (1) may exhibit tangent function behavior, but also a tanh behavior, that is to say, non-periodic solutions, following the sign of α and β parameters, without the necessity to have the

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elliptic modulus to be zero or one. More recently we have shown in [5] that the cubic Duffing equation (1) may exhibit, of course, its general solution in the complex domain, when $\alpha = \beta$. In such a situation the cubic Duffing equation (1) is not a conservative oscillator having only periodic solutions. The literature shows also another family of conservative systems called truly or purely nonlinear oscillators. We have recently investigated some of these nonlinear differential equations. In this case we have examined the purely nonlinear differential equation

$$\ddot{x} + \frac{(m+1)c}{8} x^m = \frac{(m+1)c}{8} \frac{1}{x^{m+2}} \tag{2}$$

where $m > 0$, and $c > 0$, presented in [6] as a purely nonlinear isotonic oscillator. No exact or explicit general solution is presented by the authors [6]. In fact only the general solution may provide a complete knowledge of the analytical properties of a differential equation or oscillator. However, the authors in [6] claimed to calculate the positive time period of the equation (2). In [7], we have shown, to be brief, that for $m = 1$, the equation (2) which turns into the Ermakov-Pinney equation, may exhibit complex-valued general solution while the displacement x is a real quantity. For the same selected model or design parameters, the equation (2) for $m = 1$, may exhibit real periodic or complex-valued solutions with no integrability criteria to cancel the complex-valued solution. Contrarily to the equation (2), the study performed in [7] shows clearly that all the general solutions of the Ermakov-Pinney equation of the form

$$\ddot{x} + \mu x + \frac{\mu}{x^3} = 0 \tag{3}$$

where $\mu > 0$, are periodic. In [5, 8] we have shown that the general solution of the Mickens truly nonlinear oscillator equation

$$\ddot{x} + \gamma x - \lambda x^{\frac{1}{3}} = 0 \tag{4}$$

where γ and λ are positive parameters, is periodic. Also the general solution of the Mickens truly nonlinear oscillator equation of the form [5]

$$\ddot{x} + c_0 x - \frac{c_0}{9} x^{-1/3} = 0 \tag{5}$$

where c_0 is a positive parameter, is calculated and found to be periodic. The equations (4) for $\lambda = \frac{\gamma}{3}$, and (5), belong to the exceptional equation

$$\ddot{x} + k_1 x + \frac{k_1 s(s+2)}{8} x^{-s-1} = 0 \tag{6}$$

presented in [5]. For $s > -2$, the general solution of (6) is periodic so that this equation may represent a nonlinear conservative oscillator. The corresponding potential may read $V(x) = \frac{1}{2} k_1 x^2 - \frac{k_1(s+2)}{8} \frac{1}{x^s}$. This anharmonic potential presents a singularity at $x = 0$, for $s > 0$, and has not been investigated in the literature. It differs from the well-known spiked harmonic oscillator potential studied first by Harrell [9] as one may see easily. For $s = 0$, one may recover the equation of the harmonic oscillator. From the equation (6), we have secured under the point transformation, the exceptional nonlinear oscillator equation

$$\ddot{x} + \frac{1-n}{n} \frac{\dot{x}^2}{x} + n\omega x + \frac{n^2(n-1)\omega}{2} \frac{1}{x} = 0 \tag{7}$$

where n is a positive parameter. The general solution of (7) is periodic and exhibit harmonic oscillations. As it is well known, a limited number of Lienard nonlinear differential equations have general solution expressed in terms of trigonometric functions. It is also very rare to find in the literature Lienard nonlinear differential equations having general solution expressed as a power law of trigonometric functions. In this respect, we consider in this paper the Lienard equation

$$\ddot{x} + c_1 x + c_2 q x^{-2q-1} + c_3 x^{2q+3} = 0 \tag{8}$$

where c_1, c_2, c_3, and q are arbitrary parameters. For $q = 0$, the equation (8) reduces to the well-known cubic Duffing equation

$$\ddot{x} + c_1 x + c_3 x^3 = 0 \tag{9}$$

For $q = 1$, the equation (8) becomes

$$\ddot{x} + c_1 x + \frac{c_2}{x^3} + c_3 x^5 = 0 \tag{10}$$

The equation (10) is the quintic Duffing equation with a cubic singularity. It may also be defined as the Ermakov-Pinney equation with quintic nonlinearity.

As can be seen, the equation (8) may include several nonlinear differential equations as special cases. Due to the presence of strong and high order nonlinear terms, no one can assure that the equation (8) is explicitly integrable. No one can also ensure that its general solution may be expressed in terms of elementary function such as the trigonometric functions. Such equations are very less investigated in the literature from the explicit integration point of view, since it is very complicated to find the parameters scope that may ensure this integrability [10, 11]. Even if such parameters are found, the solution is usually a complicated formula of special functions [10]. It is rare to find in the literature the explicit general solution of such equations expressed in terms of elementary functions [5]. Therefore, it is natural to ask whether the equation (8) may be explicitly solved in terms of elementary functions to ensure a periodic general solution. The existence of such a general solution may allow one to express the exact and explicit general solution for several nonlinear differential equations of Lienard type like the equations (9) and (10) in terms of elementary functions, and may be of physical and engineering importance. The objective in this work is to integrate explicitly the equation (8) to secure the periodic general solution in terms of trigonometric functions and to study the implications. To do this, we establish and solve explicitly the equation (8) (section 2) and discuss its implications (section 3). Finally a conclusion is formulated for the work.

2- General theory

This section is devoted to the statement of the type of equation (8) and the calculation of its explicit general solution.

2.1 Statement

To formulate the type of equation (8), let us consider the differential equation of Lienard type stated in [5,7, 8]

$$\ddot{x}+\frac{q}{\ell x}\left(b x^{-q}-a x^{\alpha-q}\right)^{\frac{2}{\ell}}+\frac{a\alpha}{\ell}x^{\alpha-q-1}\left(b x^{-q}-a x^{\alpha-q}\right)^{\frac{2-\ell}{\ell}}=0 \tag{11}$$

where a, b, $\ell \neq 0$, q and α, are arbitrary parameters. The application of $\ell=1$, leads to

$$\ddot{x}+(q-\alpha)a^2 x^{2(\alpha-q)-1}-(2q-\alpha)ab\,x^{\alpha-2q-1}+b^2 q\,x^{-2q-1}=0 \tag{12}$$

Substituting $b=-a(q+1)^2$, and $\alpha=2(q+1)$, into (12) yields

$$\ddot{x} - 2a^2(q+1)^2 x + a^2 q(q+1)^4 x^{-2q-1} - a^2(q+2)x^{2q+3} = 0 \tag{13}$$

The equation (13) is the desired Lienard equation to integrate explicitly. This equation is identical to (8) for $c_1 = -2a^2(q+1)^2$, $c_2 = a^2(q+1)^4$, and $c_3 = -a^2(q+2)$.

2.2 Periodic solution of (13)

Using the corresponding first order differential equation [5,7, 8]

$$\dot{x}x^q + a x^{2q+2} = -a(q+1)^2 \tag{14}$$

one may obtain

$$\int \frac{x^q}{1+\left(\frac{x^{q+1}}{q+1}\right)^2} dx = -a(q+1)^2(t+K) \tag{15}$$

where K is an integration constant. From (15) one may immediately secure the exact and periodic general solution of (13) in the form

$$x(t) = \left\{(q+1)\tan\left[-a(q+1)^2(t+K)\right]\right\}^{\frac{1}{q+1}} \tag{16}$$

where $q \neq -1$. The solution (16) is periodic, but not bounded. However, it may be usefull for bounded solution applications on the closed interval [-1, 1], interval on which the tangent function is bounded. So we may discuss of implications of this solution (16).

3- Discussion

Illustrative special cases are considered in this section and discussed in connection with the periodic general solution. We discuss also of the related quadratic Lienard type equations to (13).

3.1 Illustrative special cases

The cubic Duffing equation (9) obtained for $q = 0$, takes the form

$$\ddot{x} - 2a^2 x - 2a^2 x^3 = 0 \tag{17}$$

where its general solution is

$$x(t) = \tan[-a(t+K)] \tag{18}$$

For $2a^2 = -\omega^2 < 0$, that is $a = \pm\frac{i\omega\sqrt{2}}{2}$, the solution (18) becomes complex solution

$$x(t) = i\tanh\left[\pm\frac{\omega\sqrt{2}}{2}(t+K)\right] \tag{19}$$

The result (19) confirms the fact that the cubic Duffing equation (1) may exhibit complex-valued solutions when $\alpha = \beta$. In other words, for the same selected model parameters such that $\alpha = \beta$, the cubic Duffing equation may exhibit periodic and complex-valued solutions. The Ermakov-Pinney equation with quintic nonlinearity (10) obtained for $q = 1$, becomes

$$\ddot{x} - 8a^2 x + \frac{16a^2}{x^3} - 3a^2 x^5 = 0 \tag{20}$$

with the general solution

$$x(t) = \{2\tan[-4a(t+K)]\}^{\frac{1}{2}} \tag{21}$$

The cubic Duffing equation

$$\ddot{x} - 2a^2 x + a^2 x^3 = 0 \tag{22}$$

is obtained for $q = -2$, so that its general solution may read

$$x(t) = [\tan[a(t+K)]]^{-1} \tag{23}$$

For $a^2 < 0$, the solution becomes complex. Now we may investigate the quadratic Lienard type equations related to the equation (13).

3.2 Quadratic Lienard type equations

By application of the point transformation

$$u = x^p \tag{24}$$

where p is an arbitrary parameter, the equation (13) transforms into the quadratic Lienard type equation

$$\ddot{u} + \left(\frac{1}{p} - 1\right)\frac{\dot{u}^2}{u} - 2a^2 p(q+1)^2 u + a^2 pq(q+1)^4 u^{\frac{p-2(q+1)}{p}} - a^2 p(q+2)u^{\frac{p+2(q+1)}{p}} = 0 \tag{25}$$

having the general solution

$$u(t) = \left\{(q+1)\tan\left[-a(q+1)^2(t+K)\right]\right\}^{\frac{p}{q+1}} \tag{26}$$

The solution (26) becomes very interesting since it may exhibit tangent behavior when $\frac{p}{q+1} = 1$, that is to say, $p = q+1$, so that the equation (25) takes the form

$$\ddot{u} + \left(\frac{1}{p} - 1\right)\frac{\dot{u}^2}{u} - 2a^2 p^3 u + a^2 p^5 (p-1) u^{-1} - a^2 p(p+1)u^3 = 0 \tag{27}$$

with the general solution

$$u(t) = p\tan\left(-a\,p^2(t+K)\right) \tag{28}$$

So with that a conclusion of the work may be performed.

Conclusion

A remarkable Lienard differential equation with strong and high order nonlinear terms is presented. The equation has an explicit general solution as a power law of trigonometric functions. The equation contains also some well-known nonlinear differential equations as special cases. The related quadratic Lienard type equations may exhibit periodic tangent function solutions behavior.

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