THE MASTER FUNCTION AND APPLICATIONS

THEOPHILUS AGAMA

[1].

Abstract. In this paper we introduce a function that is neither additive nor multiplicative, and is somewhat akin to the Von Mangoldt function. As an application we show that

$$\sum_{p\leq x/2}\frac{\pi(p)}{p}\geq (1+o(1))\log\log x$$

as $x\longrightarrow \infty$, and

$$\sum_{p\leq x/2}\theta(x/p)\left(\frac{\log x}{\log p}-1\right)^{-1}\ll x\log\log x$$

where p runs over the primes.

1. Introduction

In analytic number theory, It is common practice to estimate the partial sums of the form

$$\sum_{n\leq x}f(n)g(n)$$

where $f(n)$ and $g(n)$ are arithmetic functions, which may or may not be additive or multiplicative. There are vast array of tools in the literature designed for handling such sums, especially when none of the function is trivial. In the sense that $f\not\equiv 1$ or $g\not\equiv 1$, for in such case the sum becomes somewhat easy to estimate. These sums can be hard to estimate mostly if the sum is restricted to a certain subsequence of the integers, like the primes for instance. There are, however, exceptional cases where most of the tools do not work. To make a head-way it suffices to know estimates for functions $f(n)$ or $g(n)$. In this short paper we prove an asymptotic inequality for the sum given by

$$\sum_{p\leq x/2}\frac{\pi(p)}{p}\geq (1+o(1))\log\log x.$$

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We begin by studying a function, which we have coined the master function $\Upsilon(n)$, defined by

$$\Upsilon(n) := \begin{cases} \log p & \text{if} \quad n = p^2 \\ \log pq & \text{if} \quad n = pq, \;\; p \neq q \\ 0 & \text{otherwise.} \end{cases}$$

It is basically an indicator function weighted on integers having exactly two prime factors.

2. Preliminary results

Lemma 2.1. *Let* $\pi_2 := \#\{n \leq x : \Omega(n) = 2\}$, *then we have*

$$\pi_2(x) = (1 + o(1))\frac{x \log\log x}{\log x}$$

as $x \longrightarrow \infty$.

Proof. For a proof see for instance [3]. □

Lemma 2.2. *Let* $\theta(x) = \sum\limits_{p \leq x} \log p$, *then*

$$\theta(x) = (1 + o(1))x$$

as $x \longrightarrow \infty$.

Proof. For a proof see for instance [2]. □

Remark 2.3. It needs to be said that, Lemma2.2 is not quite the best estimate but is good for the application we need.

3. The master function

In this section we launch the master function. In a similar framework as the Von mangoldt function - which is an incredibly important function over the primes - the master function can be seen as the performing the role of the Von mangoldt function on integers having exactly two primes factors.

Definition 3.1. Let $n \geq 2$, then we set

$$\Upsilon(n) := \begin{cases} \log p & \text{if} \quad n = p^2 \\ \log pq & \text{if} \quad n = pq, \;\; p \neq q \\ 0 & \text{otherwise.} \end{cases}$$

Next we examine some natural properties of the master function. As is usual of most arithmetic function, it makes sense to examine their partial sums. The following properties are in that context. The first property follows very easily from definition 3.2.

Theorem 3.2. *For all* $x \geq 2$

$$\sum_{n \leq x} \Upsilon(n) = (1 + o(1))x \log\log x$$

as $x \longrightarrow \infty$.

Proof. Applying definition 3.2, we can write

$$\begin{aligned}\sum_{n\leq x}\Upsilon(n) &= \sum_{\substack{pq\leq x\\ p\neq q}}\log pq + \sum_{p^2\leq x}\log p\\ &= \sum_{\substack{pq\leq x\\ p\neq q}}\log pq + \sum_{p^2\leq x}\log p^2 - \theta(\sqrt{x})\\ &= \sum_{\substack{n\leq x\\ \Omega(n)=2}}\log n - \theta(\sqrt{x}).\end{aligned}$$

Now applying the Riemann-Stielges integral, we can write

$$\begin{aligned}\sum_{\substack{n\leq x\\ \Omega(n)=2}}\log n &= \int_2^x \log t d\pi_2(t)\\ &= \pi_2(x)\log x - \int_2^x \frac{\pi_2(t)}{t}dt.\end{aligned}$$

Applying Lemma 2.1, we can write

$$\begin{aligned}\sum_{n\leq x}\log n &= (1+o(1))x\log\log x - \int_2^x \frac{\log\log t}{\log t}dt\\ &= (1+o(1))x\log\log x \quad (x\longrightarrow\infty).\end{aligned}$$

Using Lemma 2.2 completes the proof. ☐

Lemma 3.3. *For all* $x\geq 2$

$$\sum_{n\leq x}\Upsilon(n) = \sum_{p\leq x/2}\pi(x/p)\log p.$$

4. Applications

In this section we presents some immediate consequences of the master function.

Corollary 1. The inequality

$$\sum_{p\leq x/2}\frac{\pi(p)}{p} \geq (1+o(1))\log\log x$$

as $x\longrightarrow\infty$ is valid.

Proof. By Lemma 3.3 and by interchanging the order of summation, we can write

$$\begin{aligned}\sum_{n\leq x}\Upsilon(n) &= \sum_{p\leq x/2}\pi(x/p)\log p\\ &= \sum_{\substack{p\leq \frac{x}{2}\\ pq\leq x\\ \pi(q)=\pi(x/p)}}\pi(p)\log q\\ &= \sum_{p\leq x/2}\pi(p)\sum_{\substack{q\leq \frac{x}{p}\\ \pi(q)=\pi(x/p)}}\log q\\ &\leq \sum_{p\leq x/2}\pi(p)\sum_{q\leq \frac{x}{p}}\log q\\ &= \sum_{p\leq x/2}\pi(p)\theta(x/p)\end{aligned}$$

Applying Lemma 2.2, we can write

$$\sum_{n\leq x}\Upsilon(n)\leq (1+o(1))x\sum_{p\leq x/2}\frac{\pi(p)}{p}$$

and by using the estimate in Theorem 3.2, the result follows immediately. □

Corollary 2. The estimate

$$\sum_{p\leq x/2}\theta(x/p)\left(\frac{\log x}{\log p}-1\right)^{-1}\ll x\log\log x$$

is valid.

Proof. Using the well-known identity

$$\pi(x)=\frac{\theta(x)}{\log x}+\int_2^x\frac{\theta(t)}{t\log^2 t}dt$$

[1], we can write

$$\begin{aligned}\sum_{n\leq x}\Upsilon(n) &= \sum_{p\leq x/2}\pi(x/p)\log p\\ &= \sum_{p\leq x/2}\left(\frac{\theta(x/p)}{\log(x/p)}+\int_2^{x/p}\frac{\theta(t)}{t\log^2 t}dt\right)\log p\\ &= \sum_{p\leq x/2}\theta(x/p)\left(\frac{\log x}{\log p}-1\right)^{-1}+\sum_{p\leq x/2}\log p\int_2^{x/p}\frac{\theta(t)}{t\log^2 t}dt.\end{aligned}$$

Next, we observe that

$$\begin{aligned}\sum_{p\leq x/2}\log p\int_2^{x/p}\frac{\theta(t)}{t\log^2 t}dt &\ll x\sum_{p\leq x/2}\frac{\log p}{p\log^2(x/p)}\\ &\ll x.\end{aligned}$$

Applying Theorem 3.2, we see that the result follows immediately. □

Corollary 3. Let $\mathrm{Li}(x) := \int_2^x 1/\log t dt$, then

$$\sum_{p\le x/2} \mathrm{Li}(x/p) = (1+o(1))\frac{x\log\log x}{\log x}.$$

Proof. Using the stronger form of the prime number theorem

$$\pi(x) = (1+o(1))\mathrm{Li}(x)$$

we can write

$$\begin{aligned}\sum_{n\le x}\Upsilon(n) &= \sum_{p\le x/2}\pi(x/p)\log p\\ &= (1+o(1))\sum_{p\le x/2}\mathrm{Li}(x/p)\log p.\end{aligned}$$

By applying partial summation and using Theorem 3.2, the result follows immediately. □

5. Conclusion

In this paper we have established an inequality for the sum

$$\sum_{p\le x/2}\frac{\pi(p)}{p} \ge (1+o(1))\log\log x.$$

Another quest might be generalize sums of the form to those of the form

$$\sum_{p\le x/2}\frac{\pi^m(p)}{p},$$

for $m > 1$ and where p ranges over the primes. Aside this, the master function which has been introduced could be exploited in many ways to obtain other estimates, given the connection to the Chebyshev theta and the Psi function.

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Department of Mathematics, African Institute for Mathematical science, Ghana
Email address: theophilus@aims.edu.gh/emperordagama@yahoo.com