A RIGOROUS GEOMETRIC PROOF OF THE FORMULA $\sum\limits_{k=1}^{n}(2k-1)=n^2$ AND ITS CONNECTION TO THE BINARY GOLDBACH PROBLEM

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Abstract. In this paper we give a geometric proof of the fact that the sum of all odd numbers up to any given threshold is always a perfect square. We then relate this fact to the binary Goldbach conjecture, one of the oldest unsolved problem in mathematics.

1. Introduction

The Goldbach conjecture is one of the oldest unsolved problem in number theory. The origin of the problem probably dates back to the correspondence between the Swiss mathematician Leonard Euler and the German mathematician Christian Goldbach. The conjecture is in two folds; the binary Goldbach conjecture and the ternary Goldbach conjecture. The binary Goldbach problem ask if every even number can be written as a sum of two prime numbers, whereas the ternary ask if every odd number $n > 5$ can be written as a sum of three primes. The binary case, which is considered stronger than the ternary-since binary implies ternary-have been settled for almost all even numbers[2]. Unfortunately a complete proof that remains valid for all even numbers without any exception remains elusive. However, a complete proof for the ternary problem has already been found by the mathematician Herald Helfgot [1]. All attempts have been based on a rigorous quantitative proof. In this paper, however, using a well-known elementary formula we examine another equivalent formulation of the problem.

2. Geometric proof (Staircase approach)

Theorem 1. $\sum\limits_{j=1}^{2n-1}(2n-j)=n^2$, *for all natural number* n *and where* $(2,j)=1$.

Proof. Let us consider a right angled triangle $\angle ABC$. Let us chop the height of this triangle into n subintervals, by parallel lines joining the height and the hypothenus. Along the hypothenus, let us construct small pieces of triangle, each of base and height (a_i, b_i) $(i = 1, 2 \ldots, n)$ so that the trapezoid and the one triangle formed by partitioning becomes rectangles and squares. Now, we compute the area of the triangle in two different ways. By direct strategy, we have that the area of the triangle, denoted $\mathcal{A}$, is given by $\mathcal{A} = 1/2\left(\sum\limits_{i=1}^{n} a_i\right)\left(\sum\limits_{i=1}^{n} b_i\right)$. On the other hand, we compute the area of the triangle by computing the area of each trapezium and the one remaining triangle and sum them together. That is,

$$\mathcal{A} = b_n/2\left(\sum_{i=1}^{n} a_i + \sum_{i=2}^{n} a_i\right) + b_{n-1}/2\left(\sum_{i=2}^{n} a_i + \sum_{i=3}^{n} a_i\right) + \cdots + 1/2a_1b_1.$$

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By taking $a_1 = a_2 = \ldots = a_n = b_1 = b_2 \cdots = b_n$, we have that the total area of the trapezoid and the last remaining triangle is given by $\mathcal{A} = a_1^2/2\Big((2n-1)+(2n-3)+\cdots+1\Big)$. Also by the same argument, it follows that $\mathcal{A} = \frac{a_1^2 n^2}{2}$. Thus by comparing the two estimates, the result follows immediately. □

Corollary 1. *For all natural number n, $\sum_{j=1}^{2n-1} 1 = n$, where $(2,j)=1$.*

Proof. It follows from Theorem 1 $\sum_{j=1}^{2n-1}(2n-j) = n^2$. By arranging the sum, we have that $2n\sum_{j=1}^{2n-1} 1 = n^2 + \sum_{j=1}^{2n-1} j$. Applying Theorem 1 once again, with $(2,j)=1$ the result follows immediately. □

Corollary 2. *Every positive integer n has a representation in the form $n = p_i + \sum_{j=2p_i+1}^{2n-1} 1$ with $(2,j)=1$, where p_i is any prime number in the set of all primes no more than n.*

Proof. By Corollary 1, the sum $n = \sum_{j=1}^{2n-1} 1 = \sum_{j=1}^{2p_i-1} 1 + \sum_{j=2p_i}^{2n-1} 1 = p_i + \sum_{j=2p_i+1}^{2n-1} 1$, where $(2,j)=1$, and n is in the desired form. □

3. Relation with the Goldbach conjecture

Conjecture 1. *Let $\wp_n$ be the set of all odd primes no more than n. Then there exists a prime $p_i \in \wp_n$ such that $\sum_{j=2p_i+1}^{2n-1} 1 = p_s$, where $p_s \in \wp_n$ for $(2,j)=1$.*

Idea of proof. Let n be any even number and let $\wp_n := \{p_1, \ldots, p_m\}$, the set of all odd primes no bigger than n. Now, we can write $\wp_n = \{p_1, p_2, \ldots, p_m\}$, where $3 = p_1 < p_2 \cdots < p_m < n$. Let us consider the set $\overline{\wp_n} := \{2p_1, 2p_2, \ldots, 2p_m\}$ and construct the sets of right-translates of $\overline{\wp_n}$, where each element in the set of right-translates are no more than $2n-1$. Consider the set of right-translates $\overline{\wp_n}^{+1} := \{2p_1+1, 2p_2+1, \ldots, 2p_m+1\}$, $\overline{\wp_n}^{+3} := \{2p_1+3, 2p_2+3, \ldots, 2p_m+3\}, \ldots, \overline{\wp_n}^{+l_m} := \{2p_m + l_m\}$, where l_m is the odd number such that $2p_m + l_m = 2n-1$ and $2p_m + (l_m+2) > 2n-1$ for some prime, and where the lower right-translate terminates as we keep on adding odd numbers to the chain. The only last surviving right-translate is the set $\overline{\wp_n}^{+l_m} := \{2p_m + l_m\}$, with only a single element. Let us now consider each of the following quantities $\mathcal{D}_1 = \#\{2p_1+1, 2p_1+3, \ldots, 2p_1+l_1\}$, $\mathcal{D}_2 = \#\{2p_2+1, 2p_2+3, \ldots, 2p_2+l_2\}, \ldots, \mathcal{D}_m = \#\{2p_m+1, \ldots, 2p_m+l_m\}$, where $l_m > \cdots > l_2 > l_1$ and $2p_1 + l_1 = 2p_2 + l_2 = 2p_3 + l_3 = \cdots = 2p_m + l_m = 2n-1$, by virtue of the way the right-translates were constructed. In combination with Corollary 2, we see that these quantities are $\pi(n)-1$ distinct odd numbers no bigger than n. We claim that at least one of these quantities must be prime. Suppose none is prime, then by letting $\mathcal{P}_n := \{p_i + k_i | p_i + k_i = n, \quad p_i \in \wp_n, \quad k_i \notin \wp_n\}$, it follows from Corollary 2 that $\#\mathcal{P}_n = \pi(n) - 1$. This assumption leads to yet again an equivalent formulation of the Goldbach problem. □

Conjecture 2. *Let $n \geq 4$ be any even number and let $\wp_n$ denotes the set of all odd primes less than n. Then the inequality*

$$1 + \sum_{\substack{p_i + k_i = n \\ k_i \notin \wp_n \\ p_i \in \wp_n}} 1 < \pi(n),$$

is valid.

Remark 2. It is very important to notice that, this inequality relates the Goldbach problem to the prime counting function $\pi(n)$.

Now let us see why this inequality is an equivalent formulation of the Goldbach problem. The inequality on the left gives us the number of representations of an even number as a sum of a prime and a composite. Thus the inequality claims that the number of representations of an even number n as a sum of a prime and composite is always $< \pi(n) - 1$. Without loss of generality, the even prime 2 is excluded since it does not form a summand of n. So in principle we only considered the set of all odd primes. Yet again, the number of such representations is $< \pi(n) - 1$, so we are made to suspect that there is a representation of n as a sum of two primes. Suppose such a representation does not exists. Then for the missing odd primes p_j, we have the representations of the forms $n = p_j + (n - p_j)$. Clearly, $n - p_j$ cannot be composite since p_j is missing in the above representations; otherwise, by some degree of freedom we would have by adjoinment $\sum_{\substack{p_i+k_i=n \\ p_i \in \wp_n \\ k_i \notin \wp_n}} 1 = \pi(n) - 1$, thereby arriving at a contradiction.

References

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