On the Exact Periodic Solution of a Truly Nonlinear oscillator Equation

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Abstract

We calculate for the first time the exact and general solution of a well-known equation which is assumed to be a truly nonlinear oscillator and to have only periodic solutions. We find complex-valued functions as solutions. As a result the supposed qualities of this equation are not consistent.

Key words: Exact periodic solutions, truly nonlinear oscillator equation, first integral, complex-valued function.

Introduction

The truly nonlinear oscillator equations are subject of an intensive study in the literature since they could not be solved exactly in general and by application of standard perturbation methods in particular [1].In his book [1] Mickens said that the nonlinear differential equation

$$\ddot{x} + ax + bx^{1/3} = 0 \tag{1}$$

where a and b are non-negative, with the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$, is a truly nonlinear oscillator as well as several other nonlinear equations. To prove the existence of periodic solutions, Mickens investigated for simplicity reason, the form

$$\ddot{x} + x + x^{1/3} = 0 \tag{2}$$

using phase plane method. Although the equation (2) has no known exact solutions, Mickens and Wilkerson [2] and Mickens [1] showed exact and approximate values of the period. Such an investigation leads Mickens to claim that all the solutions of (1) or (2) are periodic. However, a nonlinear equation widely investigated as oscillator equation in the literature [3-5] has been recently shown unable to exhibit smooth periodic solution [6,7]. In this respect it becomes reasonable to ask whether the equation (1) or (2) has effectively only periodic solutions as claimed by Mickens [1]. Therefore the objective in this paper is to show that the Mickens proposition fails from mathematical point of view by exhibiting for the first time, the exact and general solution to (2) as complex-valued function and secondly that even a modification of sign in (2) could not also lead to physically acceptable real-valued periodic solution. To perform this task, we first state the required theory of differential equations (section 2) and secondly formulate a theorem for the exact integrability of (2) (section 3). Finally a modified Mickens equation is shown to have the ability to exhibit real-valued periodic solutions but not physically acceptable (section 4) and a conclusion of the work is drawn.

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2. Proposed theory of differential equations

Let us consider the time-independent first integral introduced recently by Monsia et al. [8]

$$I = I(x, \dot{x}) = \dot{x}^{\ell} x^{q} - \beta x^{\alpha} \tag{3}$$

where β, ℓ, q and α are arbitrary constants. From (3) one may secure by differentiation with respect to time the nonlinear equation

$$\ddot{x} + \frac{q}{\ell x}\left(\beta x^{\alpha-q} + I x^{-q}\right)^{\frac{2}{\ell}} - \frac{\beta\alpha}{\ell} x^{\alpha-q-1}\left(\beta x^{\alpha-q} + I x^{-q}\right)^{\frac{2-\ell}{\ell}} = 0 \tag{4}$$

where $\ell \neq 0$. This equation may also be established from the Lagrangian developed by Monsia et al. [8]. Using (4), exact integrability theorem of (2) may be formulated.

3. Integrability theorem of the equation (2)

To formulate the theorem which assures the exact and general solution to (2), let us consider $q = -2,\ \ell = 2,$ and $\alpha = -\frac{2}{3}$. Thus (4) reduces to the generalized nonlinear equation

$$\ddot{x} - Ix - \frac{2}{3}\beta x^{1/3} = 0 \tag{5}$$

For $I = -a$, and $b = -\frac{2}{3}\beta$, one may recover the generalized Mickens equation (1). From (5) one may state the following theorem for the exact integrability of (2).

Theorem 1

If $\beta = -\frac{3}{2}$, *and* $I = -1$, *then the equation (5) turns into (2), and is exactly integrable and the exact and general solutions are*

$$x = \pm i\frac{3\sqrt{6}}{4}\sin^{3}\left[\frac{1}{3}(t + K_1)\right] \tag{6}$$

and

$$x = \pm i\frac{3\sqrt{3}}{8}\left[1 \pm \sin[\frac{2}{3}(t + K_2)]\right]^{\frac{3}{2}} \tag{7}$$

where K_1, *and* K_2 *are arbitrary constants.*

Proof

In the context of (2) the first integral (3) becomes

$$I = \dot{x}^{2} x^{-2} - \beta x^{-\frac{2}{3}} \tag{8}$$

which leads to

$$\frac{dx}{x\sqrt{I+\beta x^{-\frac{2}{3}}}}=\pm dt \tag{9}$$

The change of variable

$$X=x^{\frac{1}{3}} \tag{10}$$

reduces (9) to

$$3\int\frac{dX}{\sqrt{I\,X^2+\beta}}=\mp(t+K_1) \tag{11}$$

Setting $I=-\gamma$ and $\beta=-\frac{3}{2}$ (11) becomes [9]

$$-\frac{3i}{\sqrt{\gamma}}\arcsin h\left[\frac{2\gamma\,X}{\sqrt{6\gamma}}\right]=\pm(t+K_1) \tag{12}$$

so that, one may secure the exact and general solution to (5) as

$$x=-i\frac{3\sqrt{6\gamma}}{4\gamma^2}\sin^3\left[\pm\frac{\sqrt{\gamma}}{3}(t+K_1)\right] \tag{13}$$

where K_1 and γ are arbitrary parameters.

Substituting $I=-1$, that is $\gamma=1$, into (13), one may recover the exact and general complex-valued solution (6) to the equation (2). To establish the second kind of solutions to (2), let us consider the change of variable

$$X=x^{\frac{2}{3}} \tag{14}$$

which leads to

$$\frac{3}{2}\int\frac{dX}{\sqrt{IX^2+\beta X}}=\pm(t+K_2) \tag{15}$$

Setting $I=-\gamma<0$, and $\beta=-\frac{3}{2}$, (15) becomes

$$-\frac{3}{2\sqrt{\gamma}}\arcsin\left[-\frac{4\gamma X}{3}-1\right]=\pm(t+K_2) \tag{16}$$

from which one may get the following exact and general solution to (5)

$$x = \mp i\frac{3\sqrt{3}}{8\gamma\sqrt{\gamma}}\left[1 \pm \sin\left[\frac{2\sqrt{\gamma}}{3}(t+K_2)\right]\right]^{\frac{3}{2}} \tag{17}$$

where K_2 and γ are arbitrary constants.

Applying $\gamma = 1$, that is $I = -1$, into (17), the complex-valued solutions (7) to (2) may be obtained.

Thus the above theorem is proved. Now consider the problem of the modification of Mickens equation (2). The Mickens equation (2) is nothing but the harmonic oscillator with a nonlinear additional term $x^{1/3}$. This suggests that one may consider also the harmonic oscillator equation with the additional term $-x^{1/3}$ in order to investigate the existence of real-valued periodic solutions to (1) in the context of negative b. This investigation is carried out in the following section.

4. Proposed modified Mickens equation

Consider in this part the modified Mickens equation

$$\ddot{x} + x - x^{1/3} = 0 \tag{18}$$

In this respect, the following theorem holds.

Theorem 2

If $\beta = \frac{3}{2}$, *and* $I = -1$, *then (5) transforms into (18) and admits two kinds of exact real-valued solutions*

$$x = \pm\frac{3\sqrt{6}}{4}\sin^3\left[\frac{1}{3}(t+K_3)\right] \tag{19}$$

and

$$x = \frac{3\sqrt{3}}{8}\left[1 \mp \sin[\frac{2}{3}(t+K_4)]\right]^{\frac{3}{2}} \tag{20}$$

where K_3, *and* K_4 *are arbitrary constants.*

Proof

From the first integral (3), one may write

$$J = \frac{dx}{x\sqrt{I + \beta x^{-\frac{2}{3}}}} = \pm dt \tag{21}$$

for $q=-2,\ \ell=2,\ \ \alpha=-\dfrac{2}{3}.$

Putting $\beta=\dfrac{3}{2}$, into (21), J leads to

$$J=\int\frac{dx}{x\sqrt{I+\frac{3}{2}x^{-\frac{2}{3}}}}=\pm(t+K) \tag{22}$$

where K is an arbitrary constant. The integral J may be computed using two methods. First let $X=x^{\frac{1}{3}}$, then (22) yields

$$J_1=3\int\frac{dX}{\sqrt{I\,X^2+\frac{3}{2}}} \tag{23}$$

Applying $I=-\gamma<0$, one may write

$$J_1=3\int\frac{dX}{\sqrt{\frac{3}{2}-\gamma\,X^2}} \tag{24}$$

to obtain [9]

$$J_1=\frac{3}{\sqrt{\gamma}}\arcsin\left[X\sqrt{\frac{2\gamma}{3}}\right] \tag{25}$$

From that one may read, using (22)

$$X=\sqrt{\frac{3}{2\gamma}}\sin\left[\pm\frac{\sqrt{\gamma}}{3}(t+K_3)\right] \tag{26}$$

which leads to the exact and general solutions

$$x=\pm\frac{3\sqrt{6\gamma}}{4\gamma^2}\sin^3\left[\frac{\sqrt{\gamma}}{3}(t+K_3)\right] \tag{27}$$

where $K_3=K$, and γ are arbitrary constants.

For $\gamma=1$, that is $I=-1$, the solution (19) is recovered. Consider secondly the change of variable $X=x^{\frac{2}{3}}$. Then (22) leads to

$$J_2=\frac{3}{2}\int\frac{dX}{\sqrt{I\,X^2+\frac{3}{2}X}} \tag{28}$$

Setting $I = -\gamma < 0,\ J_2$ turns into

$$J_2 = \frac{3}{2}\int \frac{dX}{\sqrt{\frac{3}{2}X - \gamma X^2}} \tag{29}$$

to yield [9]

$$J_2 = -\frac{3}{2\sqrt{\gamma}}\arcsin\left[-\frac{4\gamma X}{3} + 1\right] \tag{30}$$

Using the equation (22), one may write

$$1 - \frac{4\gamma X}{3} = \sin\left[\mp\frac{2\sqrt{\gamma}}{3}(t + K_4)\right] \tag{31}$$

so that

$$X = \left(\frac{3}{4\gamma}\right)\left[1 \pm \sin\left[\frac{2\sqrt{\gamma}}{3}(t + K_4)\right]\right] \tag{32}$$

and $x(t)$ becomes

$$x = \frac{3\sqrt{3\gamma}}{8\gamma^2}\left[1 \pm \sin\left[\frac{2\sqrt{\gamma}}{3}(t + K_4)\right]\right]^{\frac{3}{2}} \tag{33}$$

where $K_4 = K,$ and γ are arbitrary constants.

For $\gamma = 1,$ that is $I = -1,$ the solution (20) is obtained. Thus the theorem 2 is proved. Since these equations (19) and (20) are established with a prescribed negative value of the first integral, they cannot be considered as solutions of the oscillator equation for which the time-independent first integral represents the Hamiltonian, that is the energy having positive values.

Conclusion

This work is designed to show that the proposition following which some truly nonlinear oscillator equations have only real-valued periodic solutions is not consistent from mathematical point of view. In this regard it has been shown that one of these equations may exhibit complex-valued solutions. It has been also shown that a modification of sign in this equation could not lead to obtain physically acceptable periodic solutions.

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