Second order differential equations with variable coefficients and related physical problems

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Abstract

A theory of second order linear differential equations with variable coefficients is proposed in this work. The theory is shown to be useful to solve boundary value problems and Schrödinger eigenvalue problems in terms of elementary functions.

1. Theory

Consider the second order differential equation

$$y''(\tau)+c\,y(\tau)=0 \tag{1}$$

where prime means differentiation with respect to τ, and c a free parameter. Under the nonlocal transformation [1]

$$y(\tau)=u(x)g^{\ell}(x), \qquad d\tau=g^{\gamma}(x)dx \tag{2}$$

the equation (1) turns into the second order differential equation with variable coefficients

$$u''(x)+(2\ell-\gamma)\frac{g'(x)}{g(x)}u'(x)+\left[c\,g^{2\gamma}(x)+\ell\frac{g''(x)}{g(x)}+(\ell^2-\ell\gamma-\ell)\left(\frac{g'(x)}{g(x)}\right)^2\right]u(x)=0 \tag{3}$$

One may notice that the equation (3) may be used to solve several problems of boundary value problems and Schrödinger eigenvalue problems in terms of elementary functions

2. Applications

2.1 Schrödinger eigenvalue problems

2.1.1 Analysis of the inverse square root potential

This paragraph is devoted to the study of the inverse square root potential under the position-dependent mass formalism.

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To this end, let $g(x)=x^{\beta}$. Then (3) reduces to

$$u''(x)+\frac{(2\ell-\gamma)\beta}{x}u'(x)+x^{\beta}\left[cx^{\beta(2\gamma-1)}+\frac{\ell\beta(\beta-1)+\beta^2(\ell^2-\ell\gamma-\ell)}{x^{\beta+2}}\right]u(x)=0 \tag{4}$$

Applying $\gamma=\frac{1}{2}$, (4) transforms into

$$u''(x)+\frac{\beta}{2}(4\ell-1)\frac{u'(x)}{x}+x^{\beta}\left[c+\frac{\ell\beta\left(\ell\beta-\frac{\beta}{2}-1\right)}{x^{\beta+2}}\right]u(x)=0 \tag{5}$$

Now substituting $\ell=\frac{1}{2}$, and $\beta=-\frac{3}{2}$, into (5), leads to

$$u''(x)-\frac{3}{4x}u'(x)+x^{-\frac{3}{2}}\left[c+\frac{3}{4\sqrt{x}}\right]u(x)=0 \tag{6}$$

The equation (6) is the Schrödinger equation for the potential

$$V(x)=-\frac{3}{4}\frac{1}{\sqrt{x}} \tag{7}$$

under the position-dependent mass

$$m(x)=x^{-3/2} \tag{8}$$

The equation (6) may be easily solved using the above nonlocal transformation in terms of elementary functions.

References

[1] J. Akande, D. K. K. Adjaï and M. D. Monsia, On Schrödinger equations equivalent to constant coefficient equations, (2018), Math.Phys.,viXra.org.1804.0222v2.pdf