AN INTEGRAL EQUATION FOR THE GRAMM SERIES AND INTEGRAL TRANSFORMS

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ABSTRACT : In this paper we give an integral equation satisfied by the gramm series based on the use of the Borel transform

In Mathematics the Gramm series is define as the infinite series

$$G(x) = 1 + \sum_{n=1}^{\infty} \frac{(\log x)^n}{nn!\zeta(n+1)} \qquad (1)$$

Let be the following integral equation

$$-\log\left(1 - \frac{1}{s}\right) = s\int_1^{\infty} dx \frac{g(x)}{x^s - 1} \cdot \frac{1}{x} \qquad (2)$$

Then with a simple change of variable $x = e^t$, we have the following integral equation

$$-\log\left(1 - \frac{1}{s}\right) = \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{s^n} = s\int_0^{\infty} dt \frac{g(e^t)}{e^{st} - 1} \qquad (3)$$

Using our method described in Paper [1] , which uses the Borel generalized transform to solve integral equations of this kind

$$g(s) = s\int_0^{\infty} dt K(st) f(t) \qquad (4)$$

With a solution given by the power series $\quad f(t) = \sum_{n=1}^{\infty} \frac{c_n t^n}{M(n+1)} \quad (5)$

With $\quad c(n) = \frac{1}{2\pi i}\int_C g(z)z^{n-1} \qquad g(s) = \sum_{n=0}^{\infty} \frac{c(n)}{s^n} \qquad M(n+1) = \int_0^{\infty} dt K(t)t^n \qquad (6)$

The Kernel, K(t) inside (4) does not need to be a smooth function, this can include step function of

the form $\quad H(ax - b) = \begin{cases} 1 & \text{if } x > \frac{b}{a} \\ 0 & \text{if } x < \frac{b}{a} \end{cases}$

Then,we can find a series solution for this integral equation as follows

$$G(x) - 1 = g(x) = \sum_{n=1}^{\infty} \frac{(\log x)^n}{n n! \zeta(n+1)} \quad (7)$$

Which is precisely the Gramm series (minus a constant 1), so the Integral given in formula (3) is just the integral equation satisfied by the Gramm function

Given a function $g(s) = s \int_{0}^{\infty} dt K(st) f(t)$, then if the function $f(t)$ is analytic near 0 , then by integration by parts we can give an asymptotic expansion for the function g as

$$\frac{g(s)}{s} = \sum_{n=0}^{\infty} \frac{K_{n+1}(0)}{s^{n+1}} (-1)^{n+1} \frac{d^n f(0)}{dx^n} \qquad (8)$$

The functions $K_n(t)$ are given by the recurrence equation $\frac{dK_{n+1}(0)}{dt} = K_n(t)$, and $K_0(t) = K(t)$ is the Kernel of our Integral equation.

Our method [1] can be also useful to find integral equations satisfied for some functions of series, for example for the Riesz Function

$$Riesz(x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{(k-1)! \zeta(2n)} \qquad 1 - e^{-x} = \int_{0}^{\infty} \frac{dt}{t} \left[\sqrt{\frac{x}{t}} \right] Riesz(t) \qquad (9)$$

It is not hard to prove the integral equation inside (9) just use the expansion of the function $1 - e^{-\frac{1}{x}}$ in powers of $\frac{1}{x}$, and then apply the Generalized Borel transform with the Kernel $K(t) = \left[\frac{1}{\sqrt{t}} \right]$, since the Mellin transform representation of the Riemann zeta function

$$\zeta(2n) = n \int_{0}^{\infty} dt \left[\frac{1}{\sqrt{t}} \right] t^{n-1} = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \quad \text{is valid for n=1,2,3,4,5,6} \qquad (10)$$

References:

[1]   Garcia J.J "Borel resummation and the solution of integral equatio· e-print vixra.org/abs/1304.0013

[2]   Ingham, A. E. Ch. 5 in *"The Distribution of Prime Numbers"*. New York: Cambridge University Press, 1990.

[3]   Weisstein, Eric W. "Gram Series." From *MathWorld*--A Wolfram Web Resource. https://mathworld.wolfram.com/GramSeries.html