Fermat Point Violates Snell's Laws

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Abstract

Given three points one can find a fourth point such that the sum of its distances from the three points is minimal, using any of the many methods available in the literature. The solution point is called Fermat's Point (FP). The solution points to, two special cases. One is when the triangle formed from the three given points contains an angle equal to 120°. The other is when the triangle contains an angle greater than 120°. In both these cases the sum of the two distances i.e. the sum of the lengths of the sides containing that angle is minimum. This is well known. It is also well known that light travels between two given points by the minimum distance path viz. along the straight line connecting two points. If it suffers reflection at a point enroute, it travels by a two-segment broken line path. The path followed is such that the sum of the two segments is a minimum. The reflection phenomenon is governed by Snell's law of reflection. Reflection offers us an example of a natural phenomenon with three points and two distances connecting them. Therefore, we can compare the two minimal sums of distances given by FP and Snell's law. In this paper, we compare them and show that the two results are contradictory. Therefore, it follows that FP violates Snell's law. Snell's law of reflection and refraction are so connected that if one is violated the other is also violated.

Key words: Fermat Point, Reflection, Snell's law, Two line-segment path, Minimal distance path.

Introduction

Fermat posed a problem, at the end of an essay[1-4] on, "Method For The Study of Maxima And Minima", thus:

Given three points find a fourth in such a way that the sum of its distances from the three given points is a minimum[2-4].

It was solved by Torricelli[3,4]. Therefore, it is sometimes called Fermat-Torricelli point[3,4]. The solution is well known now and can be obtained by many methods[3-6]. This problem is concerned with three given points and three distances connecting them. The solution points to two special cases. One is when the triangle formed from the three given points contains an angle equal to 120°. The other is when the triangle contains an angle greater than 120°. In both these cases the sum of the two distances i.e. the sum of the lengths of the sides containing that angle is minimum. This result is well known[3-10]. It is also well known that light travels between two given points lying in the same medium, by the minimum distance path[4-8] Whether it travels along a straight line path or by a two segment broken line path as happens when it suffers reflection enroute at a point on a reflecting surface, light follows the minimum distance path. Reflection offers us an example of a natural phenomenon with three points and two distances connecting them. Therefore, we can compare the two minimal sums of distances given by FP and Snell's law.

In what follows we describe first the most popular method[3] of solution of FP briefly. For more details, literature at the end of the paper may be consulted. Then we describe Snell's law of reflection. Since FP deals with minimization of the sum of three distances in the general case, it is not suitable for comparison with Snell's laws that involve three points and two distances. Therefore, we formulate a simpler problem involving minimization of the sum of two distances connecting three points. We solve it. This solution enables us to compare the two claims of minimum sum of two distances connecting three points, one by FP and the other implied in Snell's law. In this paper, we compare the two claims and show that the two results are contradictory. It follows that FP violates Snell's law. Since Snell's law of reflection and refraction are so connected that if one is violated the other is also violated. Consequently, FP violates Snell's law of refraction and the associated Fermat's least time principle (FLTP). In a recent paper[6] we have shown the inconsistency between FP and FLTP.

Solution of Fermat Point

Let A, B, C be the three given points (see Fig. 1). We draw the triangle ABC. Let us assume that A, B, C are such that no angle of the triangle ABC is ≥ 120°. Our aim is to locate a fourth point P, such that the sum of its distances from P to A, B, C is a minimum. That is, P must satisfy the equation,

$$(PA + PB + PC) = a\ minimum \tag{1}$$

Construct an equilateral triangle on each of the sides of the triangle ABC so that they lie outside Δ ABC. Let them be ΔABF, ΔBCG, ΔCAH. Construct the three circumcircles of these triangles. The three circumcircles intersect at a point. Let it be denoted by P. It is the point we are seeking. P is the Fermat point.

Fermat's assertion is that P satisfies the equation (1).

P lies inside the ΔABC if, as we assumed here, contains no angle ≥ 120°. If ΔABC contains an angle ≥ 120°. P lies at that vertex containing that angle. Therefore, we have two special cases here. i) when the ΔABC contains an angle equal to 120° and ii) when the ΔABC contains an angle greater than 120°. We use case i) during the course of our further discussions.

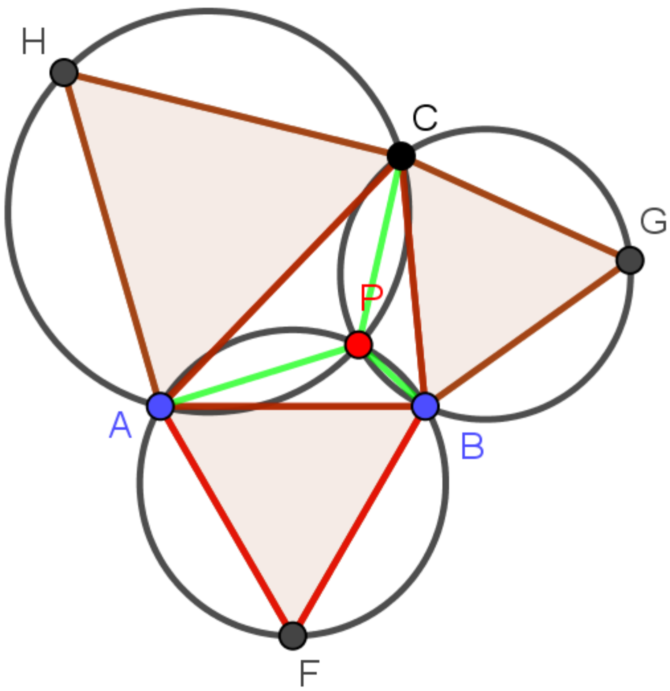

Fig.1. ABC is the triangle formed from the three given points A, B, C. Equilateral triangles ABF, BCG and CAH and their circumcircles are constructed. They intersect at point P. P is joined to A, B, C.

APBF is a cyclic quadrilateral. Since angle AFB is 60°, angle APB is 120°. Similarly, BPCG is a cyclic quadrilateral. Since angle BGC is 60°, angle BPC is 120° and CPAH is a cyclic quadrilateral. Since angle CHA is 60°, angle CPA is 120°. Thus, we see each of the triangles, APB, BPC and CPA contain an angle of 120°. In such cases where a triangle contains an angle of 120° the sum of the sides containing that angle is a minimum[3-10]. Therefore, P is the Fermat point of each of these triangles. Therefore, we get, according to FP:

$$(PA + PB)\ is\ a\ minimum \qquad (2)$$

$$(PB + PC)\ is\ a\ minimum \qquad (3)$$

$$(PC + PA)\ is\ a\ minimum \qquad (4)$$

Are these sums really minima?

We will compare theses sums with the sums given by Snell's law of reflection to confirm if they really are minimal.

Snell's Law of reflection[5-8]

A ray of light from a point A incident upon a reflecting surface MM'at point P is reflected at equal angles to the surface of reflection as shown in Fig. 2 to pass through point B. PP' is the normal to MM' at P. Angle APM = Angle BPM'. Angle APP' = Angle P'PB. The two rays lie in the plane normal to reflecting surface at the point of incidence.

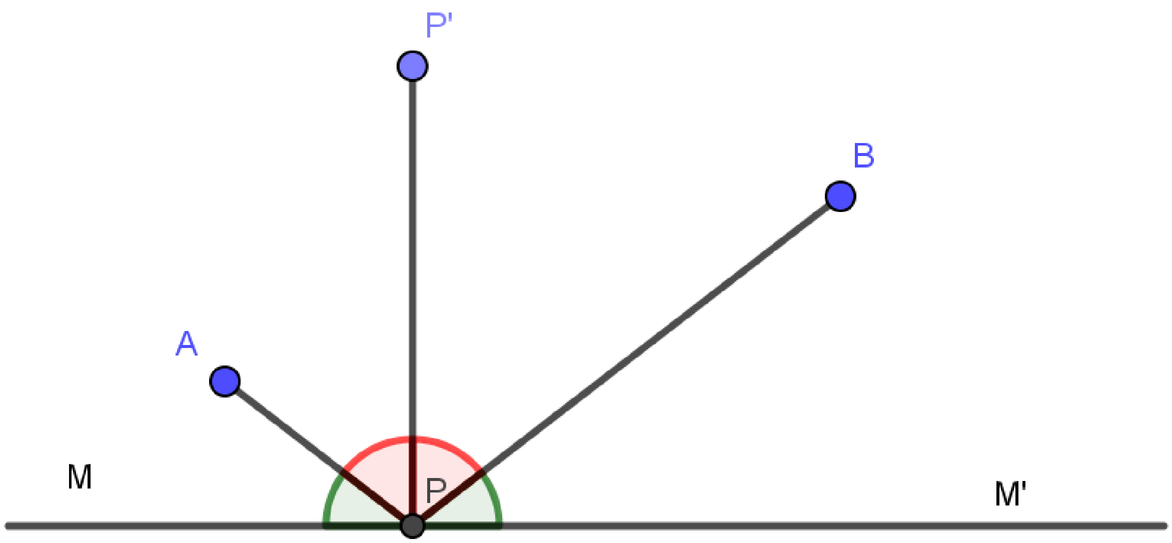

Fig.2 . Points A, B and the line MM' not passing through them, all in one plane are given. Point P is the point of incidence. PP' is the normal to MM' at P. Angle APM = Angle BPM'. Angle APP' = Angle P'PB.

Aside

Hero of Alexandria introduced the concept of motion of light during the 1st century[5-7]. Using that concept he arrived at the result that light takes the least distance path in going from a point A to another point B after reflection at point P on its way; both points being in the same medium.

In 17th century Fermat developed a method for the study of maxima and minima[1]. He applied it to refraction of light rays and showed that light takes the least time path to satisfy Snell's law of refraction. This result enabled him to assert that, light takes the least time path in going from a point A to another point B whether directly along the line joining A and B or along a broken line path of two segments AP and PB after reflection at point P when A and B lie in the same medium or by refraction when A and B lie in different media[7-8]. This result came to be known as Fermat's least time principle (FLTP). Thinking that he also solved the problem of minimization of the sum of two distances in his *Method*[1], Fermat posed a challenge problem that involved the minimization of the sum of three distances, in a letter to Torricelli[4]. Torricelli solved the problem. This is the reason the solution point is sometimes called Torricelli point. The solution point is called Fermat point (FP). *End of aside*

Before proceeding further, we pose a three-point two-distance (3-point 2-distant) problem that is simpler than the Fermat's three-point three-distance (3-point 3-distant) problem. Fermat neither posed this simpler problem nor solved it. But he applied his 3-point 3-distant problem in solving 3-point 2-distance problem associated with reflection and refraction of light[1].

A new simpler problem involving three points and two distances

Given two points find a third non collinear point in such a way that the sum of the distances from it to the two given points is a minimum.

Solution

Let A, B be the two given points (see Fig. 3). Our aim is to find a third point P such that (PA + PB) is a minimum.

Construct an equilateral triangle ABC with side equal to AB. Draw its circumcircle. Draw the diameter through C. Let the other end of the diameter be P. P is the point we are seeking. P is the point that minimizes the sum of the distances AP, PB.

Proof

Draw the line through one of the two given points A or B, say A, and P (see Fig. 3). Draw the tangent PT to the circumcircle at P and the line through A, P. Construct an equilateral triangle on PB. We note that CAPB is a cyclic quadrilateral. Angle ACB = 60°. Angle APB = 120°. Angle APB' = 180°. Therefore, B' falls on AP. Triangles ACP and BCP are congruent. Therefore, angle CPB = 60°. Angle PBB' is also = 60°. Therefore, CP is parallel to BB'. PT is perpendicular to BB'. Therefore, triangles PBT and PB'T are congruent (AAS). Therefore, PB = PB'.

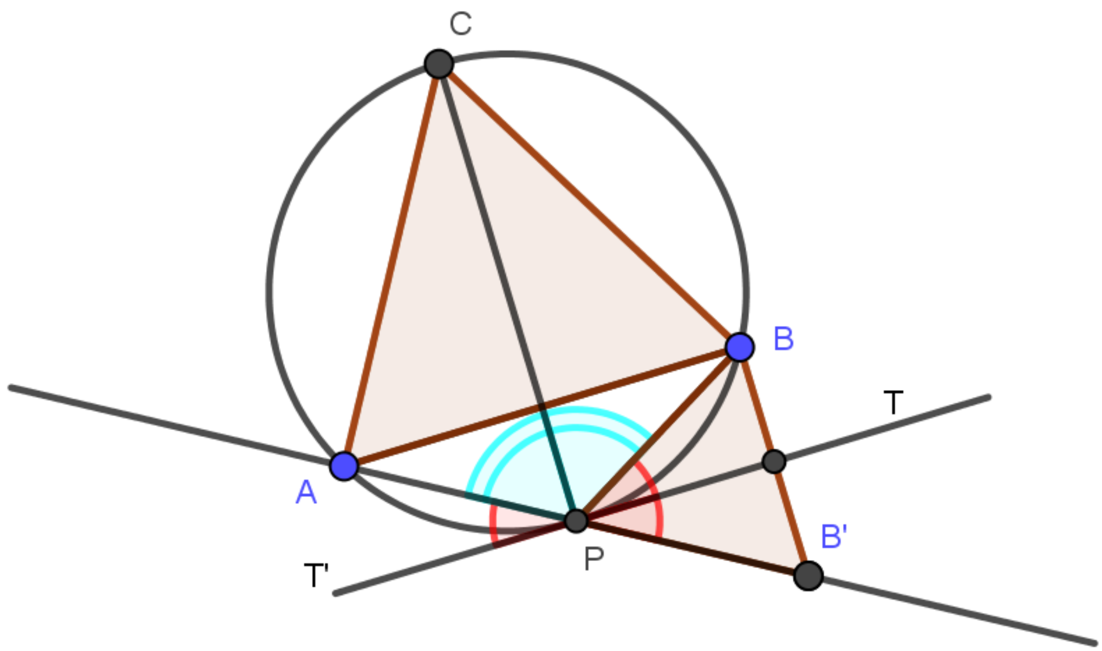

Fig. 3. A, B are two given points. ABC is an equilateral triangle. Its circumcircle is drawn. The diameter through C whose other end is P is drawn.The tangent PT to the circle at P and the line through A, P is drawn. an equilateral triangle is constructed on PB. The third vertex B', falls on line AP.

$$(AP + PB) = (AP + PB') = AB' = a\ minimum \qquad (5)$$

This result is same as the one we get using given by Snell's law of reflection. TPT' being the tangent and PC the diameter, PT and PC are orthogonal. Angle APT' = Angle BPT. Therefore, a light ray from A incident at P on the reflecting surface T'T is reflected to pass through B. AP-PB forms a reflection ray couple. Thus, the solution of our 3-point 2-distant problem is identical with the solution of Snell's law of reflection. Having thus identified that Snell's law solution is the same as the solution of our 3-point 2-distant problem, we proceed with the task of testing the claims of FP and Snell's law.

Since triangle APB contains angle APB = 120°, FP asserts that P is the point that minimizes the sum of the sides enclosing this angle i.e., (AP + PB) is a minimum. Snell's law of reflection also gives the same result. The results of FP and Snell's law of reflection agree and the two are in harmony for this point. Let us proceed further.

We now move P along the minor arc PA or PB. For any and every position P_i of P, angle AP_iB = 120°. Therefore, FP asserts that $(AP_i + P_iB)$ is a minimum. But $(AP_i + P_iB) \neq (AP_j + P_jB)$. Therefore, it is impossible for FP to satisfy the condition of minimality of sum of two distances connecting 3 noncollinear points. We also note that for no point $P_i \neq P$ do we get a reflection ray couple AP_i - P_iB, showing that Snell's law of reflection does not give the result ($AP_i + P_iB$) is a minimum

Thus it is clear that given two points A and B Snell's law of reflection uniquely fixes a third point P in such a way that the sum of its distances to the two given points is a minimum. By giving multiple (infinity) points P_i that give ($AP_i + P_iB$), ($AP_j + P_jB$), the sums of the distances from A and B are all minima, FP violates Snell's law of reflection.

We now proceed to prove the result for the general case of three points three distances problem posed by Fermat.

Let us recall Fig. 1 and make constructions as shown in Fig. 4.

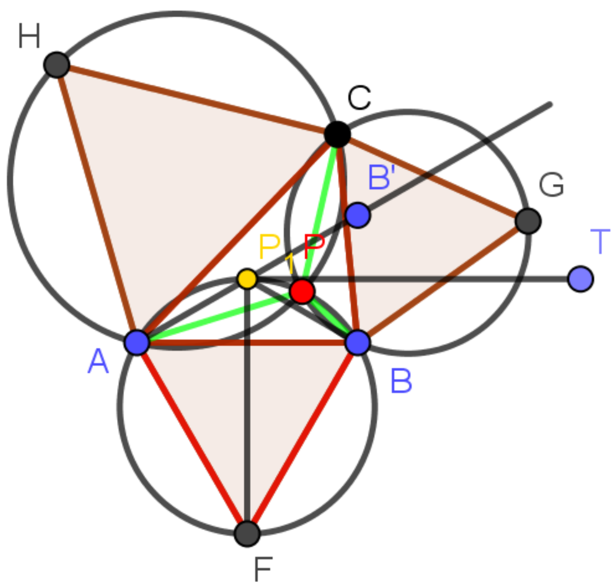

Fig. 4. ABC is the triangle formed from the three given points A, B, C. Equilateral triangles ABF, BCG and CAH and their circumcircles are constructed. They intersect at point P. P is joined to A, B, C.

Draw the diameter of the circumcircle of triangle ABF through F. Let the other end of the diameter be P_1. Draw the line through A, P_1. Draw the tangent P_1T to the circumcircle at P_1. Angle AP_1F equals angle FP_1B. Angles subtended by equal arcs AF and BF. Since the tangent and the diameter at P are perpendicular, the ray AP is reflected to pass along PB. This indicates AP_1 and P_1B is a reflection ray couple. PB = PB' as shown earlier. Thus, we get, ($P_1A + P_1B$) = ($P_1A + P_1B'$) = AB' = a minimum.

Since angle APB = 120°, FP asserts (AP + PB) is a minimum. If it were to be true, then AP, PB must be a reflection ray couple. Also ray CP must get reflected at P to retrace its path along PC. But this is impossible since CP is not the normal to the surface at P.

Because P_1 and P are different points on the circle, it follows that,

$$(PA + PB) \neq (P_1A + P_1B) = a\ minimum \qquad (6)$$

Therefore, FP violates Snell's Law

We now take B, C as the two given points and see if P is the point from which the sum of the distances to B and C is a minimum.

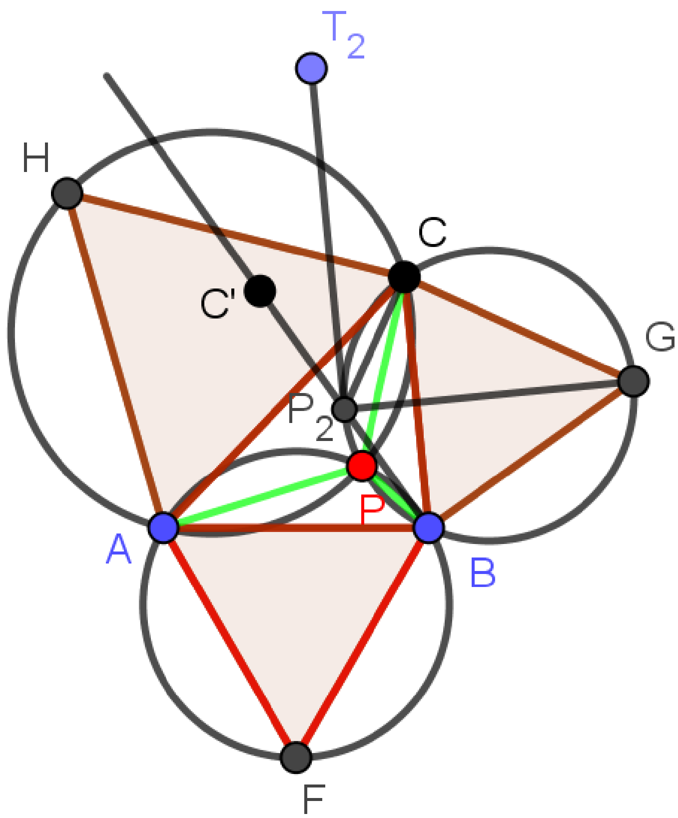

Fig.5. BCG is an equilateral triangle and its circumcircle is constructed. GP_2 is the diameter through G. The line through B, P_2 and, the tangent to the circle at P_2 are drawn.

We take the equilateral triangle BCG and its circumcircle. Draw the diameter of the circumcircle. through G (see Fig. 5). Let the other end of the diameter be P_2. Draw the line through B, P_2. Draw the tangent P_2T_2 to the circumcircle at P_2. Angle BP_2P equals angle CP_2T_2. Therefore, reflection of C falls on line BP_2. Therefore, BP_2, P_2C form a reflection ray couple. Therefore. $(P_2B + P_2C)$ is a minimum.

If BP, PC were to be a reflection ray couple, then GP, PG must also be one. Therefore, GP must get reflected to retrace the path along PG. But this is impossible since PG is not the normal to the surface at P.

Since P_2 and P are different points on the circle,

$$(PB + PC) \neq (P_2B + P_2C) = a\ minimum \qquad (7)$$

Therefore, FP again violates Snell's Law.

Similarly, we can prove that P is not a point whose sum of the distances from C, A is a minimum. That is we can show that,

$$(PC + PA) \neq (P_3C + P_3A) = a\ minimum \qquad (8)$$

Therefore, FP yet again violates Snell's Law.

Thus, failing to be a point of minimal sum distance from any of the three pairs of points A, B; B, C; C, A; P fails to be a point of minimal sum from the three given points.

$$(AP + PB) \neq a\ minimum \qquad (9)$$

$$(PB + PC) \neq a\ minimum \qquad (10)$$

$$(PC + PA) \neq a\ minimum \qquad (11)$$

$$2(AP + PB + PC) \neq a\ minimum \qquad (12)$$

$$(AP + PB + PC) \neq a\ minimum \qquad (13)$$

FP asserts that each pair is a minimum and the sum of the distances from three given points is a minimum. Therefore, FP violates Snell's law which gives the point of minimal sum from each pair of the three given points.

Failing to be the point that minimizes the sum of the distances from three given points or from any pair of points, FP fails to provide a minimal time path from any of the three given points to any other of the three given points through a fourth point.

It follows that the (theory underlying) Fermat's method of maxima and minima in proving the least time principle for the path of reflection of a light ray is invalid. Snell's law of reflection and the law of refraction are so connected that if one is violated then the other is violated. Therefore, if Fermat's method violates Snell's law of reflection it also violates Snell's law of refraction. Therefore, FLTP violates Snell's laws.

Thus, both FP and FLTP violate Snell's laws.

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