Addendum to "Dynamics of Renyi entanglement entropy in local quantum circuits with charge conservation"

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Abstract

In this addendum to [Huang, arXiv:1902.00977], I explicitly present results for qudit systems, i.e., spin systems with constant local dimension $d \geq 2$.

1 Introduction

This note is an addendum to my paper [2], which studies the dynamics of the Renyi entanglement entropy R_α in local quantum circuits with charge conservation. Initializing the system in random product states, Ref. [2] proves that R_α with Renyi index $\alpha > 1$ grows no faster than "diffusively" (up to a sublogarithmic correction) if charge transport is not faster than diffusive.

For simplicity, Ref. [2] only presents results for qubit or spin-1/2 systems. While the proof also works for qudit systems (i.e., spin systems with constant local dimension $d \geq 2$), this generalization is not explicitly presented in [2]. Due to recent interest [7], in this note I give an exposition so that readers who are only interested in the results do not have to spend their time verifying that every step of the proof in [2] remains valid for qudit systems. This note does not contain any essentially new ideas beyond those in [2].

For completeness and for the convenience of the reader, definitions and proofs are presented in full so that this note is technically self-contained, although this leads to a substantial amount of text overlap with the original paper [2]. It is not necessary to consult [2] before or during reading this note. However, in this note I do not discuss the conceptual aspects of the work. Such discussions are in [2].

I recommend related works [5, 6, 7], which study the same problem with a variety of analytical and numerical techniques. These works provide insights that are complementary to those in [2] and here.

The remainder of this note is organized as follows. In Section 2, I give the basic definitions. In Section 3, I present results with a complete proof for qudit systems.

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2 Preliminaries

Throughout this note, standard asymptotic notations are used extensively. Let $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ be two functions. One writes $f(x) = O(g(x))$ if and only if there exist constants $M, x_0 > 0$ such that $f(x) \leq Mg(x)$ for all $x > x_0$; $f(x) = \Omega(g(x))$ if and only if there exist constants $M, x_0 > 0$ such that $f(x) \geq Mg(x)$ for all $x > x_0$; $f(x) = \Theta(g(x))$ if and only if there exist constants $M_1, M_2, x_0 > 0$ such that $M_1 g(x) \leq f(x) \leq M_2 g(x)$ for all $x > x_0$.

For notational simplicity, we do not specify the base of the logarithm explicitly. All equations involving logarithms are valid as long as the base is an arbitrary but fixed constant.

Definition 1 (entanglement entropy)**.** The Renyi entanglement entropy R_α with index $\alpha \in (0,1) \cup (1, +\infty)$ of a bipartite pure state ρ_{AB} is defined as

$$R_\alpha(\rho_A) = \frac{1}{1-\alpha} \log \operatorname{tr}(\rho_A^\alpha) = \frac{1}{1-\alpha} \log \sum_{i \geq 1} \Lambda_i^\alpha, \tag{1}$$

where $\Lambda_1 \geq \Lambda_2 \geq \cdots \geq 0$ with $\sum_{i \geq 1} \Lambda_i = 1$ are the eigenvalues (in descending order) of the reduced density matrix $\rho_A = \operatorname{tr}_B \rho_{AB}$. The min-entropy is defined as

$$R_\infty(\rho_A) := \lim_{\alpha \to +\infty} R_\alpha(\rho_A) = -\log \Lambda_1. \tag{2}$$

Note that the von Neumann entanglement entropy is given by

$$\lim_{\alpha \to 1} R_\alpha(\rho_A) = -\operatorname{tr}(\rho_A \log \rho_A). \tag{3}$$

Lemma 1. *For $\alpha > 1$,*

$$R_\infty(\rho_A) \leq R_\alpha(\rho_A) \leq \frac{\alpha}{\alpha - 1} R_\infty(\rho_A). \tag{4}$$

Proof. For completeness, we give a proof of this well-known result. The first inequality is a consequence of the fact that R_α is monotonically non-increasing with respect to α (this is why R_∞ is called the min-entropy). The second inequality follows from

$$R_\alpha(\rho_A) = \frac{1}{1-\alpha} \log \sum_{i \geq 1} \Lambda_i^\alpha \leq \frac{1}{1-\alpha} \log(\Lambda_1^\alpha) = \frac{\alpha}{\alpha - 1} R_\infty(\rho_A). \tag{5}$$

□

Definition 2 (local quantum circuit with charge conservation)**.** Consider a chain of N qudits or spins with constant local dimension $d \geq 2$. Assume without loss of generality that N is even. Let the time-evolution operator be

$$U(t, 0) = U(t, t-1)U(t-1, t-2) \cdots U(1, 0), \quad t \in \mathbb{Z}^+. \tag{6}$$

Each layer of the circuit consists of two sub-layers of local unitaries:

$$U(t, t-1) = \prod_{i=1}^{N/2-1} U_{2i,2i+1}^{(t)} \times \prod_{i=1}^{N/2} U_{2i-1,2i}^{(t)}. \tag{7}$$

Each unitary $U^{(t)}_{i,i+1}$ acts on two neighboring spins at sites $i, i{+}1$, and commutes with $S^z_i{+}S^z_{i+1}$, where S^z_i is the z component of the spin operator at site i. It should be clear that every $U^{(t)}_{i,i+1}$ and hence $U(t,0)$ preserve the total charge or the z component $\sum_{i=1}^N S^z_i$ of the total spin.

Let us consider some examples. For spin-1/2 ($d = 2$),

$$S^z_i = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

in the computational basis $\{|0\rangle, |1\rangle\}$, and $U^{(t)}_{i,i+1}$ is block diagonal in the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis:

$$U^{(t)}_{i,i+1} = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \tag{9}$$

where "$*$" denotes a possibly non-zero entry, i.e., $U^{(t)}_{i,i+1}$ is the direct sum of a phase factor, a unitary matrix of order 2, and another phase factor. For spin-1 ($d = 3$),

$$S^z_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{10}$$

in the basis $\{|0\rangle, |1\rangle, |2\rangle\}$, and $U^{(t)}_{i,i+1}$ is block diagonal

$$U^{(t)}_{i,i+1} = \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{pmatrix} \tag{11}$$

in the $\{|00\rangle, |01\rangle, |10\rangle, |02\rangle, |11\rangle, |20\rangle, |12\rangle, |21\rangle, |22\rangle\}$ basis.

Definition 3 (Haar-random local quantum circuit with charge conservation)**.** Recall that each local unitary $U^{(t)}_{i,i+1}$ in Eq. (7) commutes with $S^z_i + S^z_{i+1}$ and is therefore a block diagonal matrix in the computational basis. The ensemble of Haar-random local quantum circuits with charge conservation is defined by letting each block in each $U^{(t)}_{i,i+1}$ be an independent Haar-random unitary matrix.

3 Results for qudits

Recall that in our notation, the eigenstates of S^z_i are $\{|0\rangle_i, |1\rangle_i, \ldots, |d-1\rangle_i\}$ with $S^z_i|k\rangle_i = ((d-1)/2 - k)|k\rangle_i$. Let $Q_i := (d-1)/2 - S^z_i$ be the charge operator with $Q_i|k\rangle_i = k|k\rangle_i$ so

that $|k\rangle_i$ is interpreted as there being k charges on site i. Let X_i be the generalized Pauli X operator at site i defined as

$$X_i|k\rangle_i = |(k+1) \bmod d\rangle_i. \tag{12}$$

Let $\{|0\rangle_i, |1\rangle_i, \ldots, |d-1\rangle_i\}$ be eigenstates of X_i. It is not difficult to see that $|\langle k|k'\rangle| = 1/\sqrt{d}$ for any $k, k' = 0, 1, \ldots, d-1$, where we have omitted the subscript i for notational simplicity.

Diffusive transport means that the transport of conserved quantities satisfies the diffusion equation at large distance and time scales. It can be considered, e.g., in the linear response regime and in quantum quench, where the system is infinitely close to and far from equilibrium, respectively. However, it is unclear whether diffusive transport in one setting is equivalent to or implies that in another (it could be possible that transport is diffusive only in one setting but not in another). The proof of our results relies on the following necessary condition for no-faster-than-diffusive transport.

Condition 1. Consider a chain of N qudits divided into two subsystems $C \otimes D$. Subsystem C is a consecutive region of m qudits, and subsystem D is the rest of the system. Suppose that C is initialized in the state $|0\rangle^{\otimes m}$ and that D is initialized in an arbitrary product state, i.e., each qudit in D is disentangled from all other qudits. Let i be the position of a qudit in the bulk of region C such that the distances from site i to the two endpoints of C are both $\Theta(m)$. If charge transport is not faster than diffusive, then

$$\langle\psi(t)|Q_i|\psi(t)\rangle \leq e^{-\Omega(m^2/t)}, \tag{13}$$

where $\psi(t)$ is the state (wave function) at time t.

Note that since

$$\|(1 - |0\rangle_i\langle 0|_i)|\psi(t)\rangle\|^2 = \langle\psi(t)|(1 - |0\rangle_i\langle 0|_i)|\psi(t)\rangle \leq \langle\psi(t)|Q_i|\psi(t)\rangle, \tag{14}$$

the inequality (13) implies that

$$\|(1 - |0\rangle_i\langle 0|_i)|\psi(t)\rangle\| \leq e^{-\Omega(m^2/t)}. \tag{15}$$

The inequality (13) can be intuitively understood as follows. At initialization $t = 0$, there is no charge in C, i.e., C is in the all-zero state. Any charge observed on site i at a later time t must be transported from D all the way to the bulk of C. The distance is $\Theta(m)$. The left-hand side of (13) is the amount of charge on site i at time t, and the right-hand side follows from the diffusion equation. In particular, a non-vanishing amount of charge requires that $t = \Omega(m^2)$. It should be clear that violating (13) unambiguously implies that charge transport is faster than diffusive.

As an instructive example, we show that

Lemma 2. *For any initial state $|\psi(0)\rangle$ with no charge in C,*

$$\Pr_{U(t,0)\in\mathcal{R}}\left(\langle\psi(0)|U^\dagger(t,0)Q_iU(t,0)|\psi(0)\rangle \leq e^{-\Omega(m^2/t)}\right) \geq 1 - e^{-\Omega(m^2/t)}, \tag{16}$$

where $\mathcal{R}$ is the ensemble of Haar-random local quantum circuits with charge conservation (Definition 3).

Proof. It is proved [3, 4] that the distribution of charge

$$\left\{ \mathop{\mathbb{E}}_{U(t,0)\in\mathcal{R}} \langle\psi(0)|U^\dagger(t,0)Q_iU(t,0)|\psi(0)\rangle \right\}_{i=1}^{N} \tag{17}$$

after averaging over the ensemble $\mathcal{R}$ evolves as an unbiased discrete random walk. Hence,

$$\mathop{\mathbb{E}}_{U(t,0)\in\mathcal{R}} \langle\psi(0)|U^\dagger(t,0)Q_iU(t,0)|\psi(0)\rangle \le e^{-\Omega(m^2/t)} \tag{18}$$

if site i is in the bulk of region C. Then, (16) follows from Markov's inequality. □

We are ready to state and prove the main result.

Theorem 1. *Consider a chain of N qudits as a bipartite quantum system $A \otimes B$. Assume without loss of generality that N is even. Subsystem A consists of qudits at sites $1, 2, \ldots, N/2$, and subsystem B is the rest of the system (we study the entanglement across the middle cut). Initialize the system in a random product state $|\psi_{\rm ini}\rangle$ in the generalized Pauli X basis, i.e., each spin is independently in $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ with equal probability. Let $\alpha > 1$ and $\rho_A(t) := \mathrm{tr}_B(U(t,0)|\psi_{\rm ini}\rangle\langle\psi_{\rm ini}|U^\dagger(t,0))$ be the reduced density matrix of subsystem A at time t. If charge transport under the dynamics $U(t,0)$ is not faster than diffusive in the sense of Condition 1, then*

$$R_\alpha(\rho_A) \le \frac{\alpha}{\alpha-1} O(\sqrt{t \log t}) \tag{19}$$

holds with probability $\ge 1 - 1/p(t)$, where p is a polynomial of arbitrarily high degree.

Proof. We divide the system into two subsystems $C \otimes D$. Subsystem C consists of m qudits at sites $N/2 - m/2 + 1, N/2 - m/2 + 2, \ldots, N/2 + m/2$ near the cut, where m is an even positive integer to be determined later. Subsystem D is the rest of the system. The initial state can be factored into

$$|\psi_{\rm ini}\rangle = |\psi_{\rm ini}\rangle_C \otimes |\psi_{\rm ini}\rangle_D, \tag{20}$$

where $|\psi_{\rm ini}\rangle_C$ and $|\psi_{\rm ini}\rangle_D$ are random product states in subsystems C and D, respectively. Define

$$|\psi_0\rangle = |0\rangle_C^{\otimes m} \otimes |\psi_{\rm ini}\rangle_D \tag{21}$$

so that $|\langle\psi_0|\psi_{\rm ini}\rangle| = d^{-m/2}$. Since $U(t,0)$ is unitary,

$$|\langle U(t,0)\psi_0, U(t,0)\psi_{\rm ini}\rangle| = d^{-m/2}. \tag{22}$$

The left-hand side of this equation is the absolute value of the inner product between $U(t,0)|\psi_0\rangle$ and $U(t,0)|\psi_{\rm ini}\rangle$. Occasionally we do not use the standard Dirac notation because it is cumbersome. Let $P := |0\rangle_{N/2}\langle 0|_{N/2} \otimes |0\rangle_{N/2+1}\langle 0|_{N/2+1}$. Using (15) twice,

$$\begin{aligned} \|(1-P)U(t,0)|\psi_0\rangle\| &\le \|(1 - |0\rangle_{N/2}\langle 0|_{N/2})U(t,0)|\psi_0\rangle\| \\ &\quad + \||0\rangle_{N/2}\langle 0|_{N/2}(1 - |0\rangle_{N/2+1}\langle 0|_{N/2+1})U(t,0)|\psi_0\rangle\| \le 2e^{-\Omega(m^2/t)}. \end{aligned} \tag{23}$$

Assume without loss of generality that $N/2$ is odd. The only local unitary in $U(t,t-1)$ acting on both subsystems A and B is in the second product on the right-hand of Eq. (7). Define a modified local quantum circuit as

$$V(t,0) = V(t,t-1)V(t-1,t-2)\cdots V(1,0), \tag{24}$$

$$V(t,t-1) = \prod_{i=1}^{N/2-1} U^{(t)}_{2i,2i+1} \prod_{i=1}^{(N/2-1)/2} U^{(t)}_{2i-1,2i} u^{(t)}_{N/2,N/2+1} \prod_{i=(N/2+3)/2}^{N/2} U^{(t)}_{2i-1,2i}, \tag{25}$$

where $u^{(t)}_{N/2,N/2+1} := \langle 00|U^{(t)}_{N/2,N/2+1}|00\rangle$ is a complex number. It is easy to see that

$$U(t,t-1)P = V(t,t-1)P. \tag{26}$$

Therefore,

$$\begin{aligned} U(t,0)|\psi_0\rangle &= U(t,t-1)U(t-1,0)|\psi_0\rangle \approx U(t,t-1)PU(t-1,0)|\psi_0\rangle \\ &= V(t,t-1)PU(t-1,0)|\psi_0\rangle \approx V(t,t-1)U(t-1,0)|\psi_0\rangle, \end{aligned} \tag{27}$$

where each approximation step incurs an additive error upper bounded by (23). Iterating this process,

$$\||\Delta_t\rangle\| \leq 4te^{-\Omega(m^2/t)}, \quad |\Delta_t\rangle := U(t,0)|\psi_0\rangle - V(t,0)|\psi_0\rangle. \tag{28}$$

Recall that both $|\psi_{\text{ini}}\rangle_C$ and $|\psi_{\text{ini}}\rangle_D$ are random product states in the generalized Pauli X basis. We now fix the latter but not the former. Then, $|\psi_0\rangle$ and $|\Delta_t\rangle$ are fixed but $|\psi_{\text{ini}}\rangle$ is not. Let

$$S = \{|0\rangle, |1\rangle, \ldots, |d\rangle\}_C^{\otimes m} \otimes |\psi_{\text{ini}}\rangle_D \tag{29}$$

be the set of all possible initial states consistent with $|\psi_{\text{ini}}\rangle_D$ so that $|S| = d^m$. Since the states in S are pairwise orthogonal,

$$\frac{1}{|S|} \sum_{|\psi_{\text{ini}}\rangle \in S} |\langle \Delta_t|U(t,0)|\psi_{\text{ini}}\rangle|^2 \leq d^{-m}\||\Delta_t\rangle\|^2. \tag{30}$$

Define a subset of S as

$$S' := \left\{ |\psi_{\text{ini}}\rangle \in S : |\langle \Delta_t|U(t,0)|\psi_{\text{ini}}\rangle| \leq d^{-m/2}\||\Delta_t\rangle\|\sqrt{p(t)} \right\}. \tag{31}$$

Markov's inequality implies that

$$|S'|/|S| \geq 1 - 1/p(t). \tag{32}$$

It suffices to prove (19) for all (initial) states in S'. To this end, we make use of

Lemma 3 (Eckart-Young theorem [1])**.** *Let*

$$|\psi\rangle = \sum_{i\geq 1} \lambda_i |a_i\rangle_A \otimes |b_i\rangle_B \tag{33}$$

be the Schmidt decomposition of the state $|\psi\rangle$*, where* $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ *with* $\sum_{i\geq 1} \lambda_i^2 = 1$ *are the Schmidt coefficients in descending order. Any state* $|\phi\rangle$ *of Schmidt rank* D *satisfies that*

$$|\langle\phi|\psi\rangle| \leq |\langle\psi'|\psi\rangle| = \sqrt{\sum_{i=1}^{D} \lambda_i^2} \tag{34}$$

where

$$|\psi'\rangle := \frac{1}{\sqrt{\sum_{i=1}^{D} \lambda_i^2}} \sum_{i=1}^{D} \lambda_i |a_i\rangle_A \otimes |b_i\rangle_B. \tag{35}$$

For any state $|\psi_{\text{ini}}\rangle \in S'$,

$$\begin{aligned} &|\langle V(t,0)\psi_0, U(t,0)\psi_{\text{ini}}\rangle| = |\langle U(t,0)\psi_0, U(t,0)\psi_{\text{ini}}\rangle - \langle\Delta_t|U(t,0)|\psi_{\text{ini}}\rangle| \\ &\geq d^{-m/2} - |\langle\Delta_t|U(t,0)|\psi_{\text{ini}}\rangle| \geq d^{-m/2}\left(1 - \||\Delta_t\rangle\|\sqrt{p(t)}\right) \\ &\geq d^{-m/2}\left(1 - 4te^{-\Omega(m^2/t)}\sqrt{p(t)}\right). \end{aligned} \tag{36}$$

Let λ_1 be the largest Schmidt coefficient of $U(t,0)|\psi_{\text{ini}}\rangle$, and $\Lambda_1 = \lambda_1^2$ be the largest eigenvalue of the reduced density matrix $\rho_A(t) = \text{tr}_B(U(t,0)|\psi_{\text{ini}}\rangle\langle\psi_{\text{ini}}|U^\dagger(t,0))$. Since none of the local unitaries in $V(t,t-1)$ or $V(t,0)$ act on both subsystems A and B, $V(t,0)$ does not generate entanglement so that $V(t,0)|\psi_0\rangle$ is a product state between A and B (i.e., a state of Schmidt rank 1). Combining this observation with (36) and Lemma 3,

$$\lambda_1 \geq d^{-m/2}\left(1 - 4te^{-\Omega(m^2/t)}\sqrt{p(t)}\right). \tag{37}$$

Lemma 1 implies that

$$R_\alpha(\rho_A) \leq \frac{\alpha}{\alpha-1} R_\infty(\rho_A) = -\frac{\alpha}{\alpha-1}\ln\Lambda_1 = -\frac{2\alpha}{\alpha-1}\ln\lambda_1. \tag{38}$$

We complete the proof by choosing $m = O(\sqrt{t\log t})$ with a sufficiently large pre-factor hidden in the Big-O notation such that the factor in parentheses on the right-hand side of (37) is lower bounded by a positive constant. □

Combined with Lemma 2, the conclusion of Theorem 1 applies in particular to a Haar-random local quantum circuit with charge conservation.

As stated in [2], it is straightforward to extend Theorem 1 to two and higher spatial dimensions.

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