Composition of Relativistic Gravitational Potential Energy

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Abstract

A relativistic composition of gravitational redshift can be implemented using the Volterra product integral. Using this composition as a model, expressions are developed for gravitational potential energy, escape velocity, and a metric. Each of these expressions alleviates a perceived defect in its conventional counterpart. Unlike current theory, relativistic gravitational potential energy would be limited to rest energy (Machian), escape velocity resulting from the composition would be limited to the speed of light, and the associated metric would be singularity-free. These ideal properties warrant investigation, at a foundational level, into relativistic compositions based on product integration.

Gravitational Redshift and the Volterra Product Integral

General relativity predicts Lorentzian form for gravitational time dilation and redshift, but these are found to have exponential form when composed primitively from the Einstein equivalence principle.

The total redshift, when it is influenced by more than one factor such as Doppler shift, gravitational shift and Hubble shift, can be found from the product [1],

$$1 + z_{\text{total}} \;=\; (1 + z_{\text{Doppler}})\,(1 + z_{\text{Gravity}})\,(1 + z_{\text{Hubble}}) \qquad (1)$$

When the redshifts are small, the total redshift can be approximated by the sum of the redshifts, but the multiplicative relativistic composition must be maintained for full accuracy.

Consider a sequence of N frames moving along a line, where the velocity of frame n with respect to frame $n-1$ is v_n, and where v_0 is a reference frame which may be presumed stationary. The relativistic redshift, z, of frame N with respect to the base frame can be expressed as a product of redshifts, z_n,

$$1 + z \;=\; (1 + z_1)\,(1 + z_2)\,\ldots\,(1 + z_N), \qquad 1 + z_n = \sqrt{\frac{1 + v_n/c}{1 - v_n/c}} \qquad (2)$$

The Volterra product integral is a continuous version of the above product. It

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takes a function, chops it into infinitesimal elements, adds one to each infinitesimal element to form a factor, then multiplies these factors to form a product. The continuous product of factors, $1 + f(x)$, over the interval $[a, b]$, given a function $f(x) = x$ for example, can be written as a product integral,

$$\prod_a^b \Big(1 + f(x)\,\mathrm{d}x\Big) \;\equiv\; \exp\Big(\int_a^b f(x)\,\mathrm{d}x\Big) \;=\; \exp\Big(\frac{b^2 - a^2}{2}\Big) \qquad (3)$$

where $\prod$ stands for product integration. Functionally, the product integral is simply an ordinary integral which is then exponentiated. Conceptually, integration is like calculating simple interest, while product integration is like compound interest.

Consider the gravitational redshift at a radial distance, R, from an ideal solid sphere of mass, M. Given a test particle of mass m, the classical element of potential energy due to an infinitesimal spherical shell of matter is $\mathrm{d}u = -F(r)\,\mathrm{d}r$ where $F(r)$ is the force of gravity between a shell of radius r and the test particle at R. By the Einstein equivalence principle, redshift due to the shell is given by $\mathrm{d}z = -\mathrm{d}u/mc^2$, where the rest energy is taken to be constant. The total redshift, $\widetilde{z}$, can be composed relativistically as the product integral of $\mathrm{d}z$,

$$1 + \widetilde{z} \;=\; \prod_r \Big(1 + \mathrm{d}z\Big) \;=\; \exp\Big(\int_r \mathrm{d}z\Big) \;=\; \exp\Big(\frac{GM}{Rc^2}\Big) \qquad (4)$$

where the integral is taken over all the shells, following the classical derivation of potential energy. The composite redshift due to a mass M, at a radial distance r, is then given by (5b), not the conventional relativistic (5a).

$$\text{(a) GR} : z \;=\; \Big(1 - \frac{2GM}{rc^2}\Big)^{-1/2} - 1 \qquad \text{(b) Mach} : \widetilde{z} \;=\; \exp\Big(\frac{GM}{rc^2}\Big) - 1 \quad (5)$$

Relativistic Gravitational Potential Energy

The corresponding relativistic gravitational potential energy can be obtained similarly, but must have opposite sign in the exponent to be consistent with the composition of relativistic gravitational redshift.

Following the composition of gravitational redshift, relativistic gravitational potential energy is found to be

$$\widetilde{E}(r) \;=\; mc^2 \prod (1 + \mathrm{d}u/mc^2) \;=\; mc^2 \prod (1 - \mathrm{d}z) \;=\; mc^2 \exp\Big(\frac{-GM}{rc^2}\Big) \quad (6)$$

Simply attempting to form $\widetilde{E}(r)$ as $\prod(1 + \mathrm{d}u)$ would be improper. In (6), partitioning of $\mathrm{d}u = -mc^2\,\mathrm{d}z$ between mc^2 outside the product integral, and the dimensionless $-\mathrm{d}z$ inside, can be seen as a requirement that the product integral must be dimensionless in order to avoid exponentiated units.

Unlike Newtonian potential energy which is negative, relativistic gravitational potential energy is positive. Relativistic gravitational potential energy is an exponential map of the classical potential energy normalized by rest energy. In the absence of a gravitational field, relativistic gravitational potential energy is equal to rest energy. Gravitational potential energy is taken from that rest energy, and thus has a finite limit.

Classical potential energy is the first order term in a power series expansion:

$$\widetilde{E}(r) \;=\; mc^2\left[1-\frac{GM}{rc^2}+\frac{1}{2}\left(\frac{GM}{rc^2}\right)^2-\ldots\right] \tag{7}$$

Classical potential energy, U, is an approximation, $U \approx \widetilde{U}$.

$$\text{(a) GR}: \; U \;=\; \frac{-GMm}{r} \qquad \text{(b) Mach}: \; \widetilde{U} \;=\; mc^2\left[\exp\!\left(\frac{-GM}{rc^2}\right)\;-\;1\right] \tag{8}$$

Mach's principle [2] posits that rest energy of an object can be viewed as gravitational potential energy due to the elevation of that object from distant matter. Hence, the term Machian will be used to describe any quantities relating to gravity resulting from these compositions, but it is particularly appropriate in reference to relativistic gravitational potential energy.

Scale Factor

It is convenient to adopt a more compact notation. Based on gauge theoretic dimensional variability as outlined in Appendix A, a dimensionless scale factor for energy in general relativity, and in the Machian composition, can be given respectively by

$$\text{(a) GR}: \; \sigma(r)=\left(1-\frac{2GM}{rc^2}\right)^{1/2} \qquad \text{(b) Mach}: \; \widetilde{\sigma}(r)=\exp\!\left(\frac{-GM}{rc^2}\right) \tag{9}$$

Second order terms in their power series, *e.g.* (7), have opposite sign, presenting a possibility for decisive testing. Newtonian potential energy, U, can be expressed as $U = mc^2 \log\widetilde{\sigma}$, which is the functional inverse of the exponential map from classical to Machian gravitational potential energy, $\widetilde{E} = mc^2\,\widetilde{\sigma}$. Appendix B shows that Newtonian gravitational force is consistent with radial length contraction, naturally linking classical to modern gravitation.

Machian Escape Velocity from Potential Energy

Machian escape velocity can be derived from the condition that kinetic energy balances potential energy. Relativistic gravitational potential energy gives a radial escape velocity limited to the speed of light, as might be expected from a relativistic theory, whereas this condition is violated in both classical theory and general relativity. This can be demonstrated using a similar technique for both general relativity and the Machian composition.

Consider the relativistic energy, $(p^2c^2 + m^2c^4)^{1/2} = \gamma\, mc^2$, of a moving object of mass m, velocity v, relativistic momentum $p = \gamma\, mv$, and Lorentz factor

$\gamma = (1 - v^2/c^2)^{-1/2}$. In a gravitational field, this energy falls to $\widetilde{\sigma}\,\gamma\,mc^2$. The condition for escape is $\widetilde{\sigma}\,\gamma\,mc^2 = \widetilde{E}(\infty)$, or

$$\text{(a) GR}: \ \sigma\,\gamma \ = \ 1 \qquad \text{(b) Mach}: \ \widetilde{\sigma}\,\gamma \ = \ 1 \tag{10}$$

Solving (10b) for velocity, the Machian scale factor gives the escape velocity (11b) which is limited to c. Solving equation (10a) with the general relativity scale factor, assuming $E = mc^2\,\sigma$, would give the radial escape velocity as (11a) which is the same as Newtonian escape speed.

$$\text{(a) GR}: v_{\text{esc}}(r) = (2GM/r)^{1/2} \qquad \text{(b) Mach}: \widetilde{v}_{\text{esc}}(r) = c(1 - \widetilde{\sigma}^2)^{1/2} \tag{11}$$

See Appendix C for an alternative derivation as the composition of Machian escape velocity from gravitational acceleration.

Machian Metric

For Machian gravitational potential and its associated escape velocity, the coordinate system of Gullstrand and Painlevé [3] gives the metric in geometrized units ($c = G = 1$) as

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + (\mathrm{d}r + \widetilde{\beta}\mathrm{d}\tau)^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{12}$$

where τ is proper time in the frame of an object in free fall, initially at rest at infinity. These coordinates are determined by escape velocity, $\widetilde{\beta} = (1 - \widetilde{\sigma}^2)^{1/2}$.

Since $\widetilde{\beta}$ is less than the speed of light for any non-zero radius, there is no absolute event horizon. In static Schwarzschild coordinates, instead of $g_{tt} = -\sigma^2$ that term becomes $g_{tt} = -\widetilde{\sigma}^2$, and the resulting metric equation is then given by

$$\mathrm{d}s^2 = -\widetilde{\sigma}^2\mathrm{d}t^2 + \widetilde{\sigma}^{-2}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{13}$$

where t is proper time in the frame of a motionless object. The Machian metric is the same as a limiting case of the Brans-Dicke metric.

Conclusion

Brans-Dicke theory [4] was formulated with the Machian idea that G should not be treated as a constant, but rather as a field that varies with the density of surrounding matter. However, inertial and gravitational mass in Brans-Dicke theory would differ slightly, by a presently undetectable amount.

Brans-Dicke includes an explicit term to enforce Machian relations, while the relativistic composition of gravitational potential energy is implicitly Machian. Therefore, instead of pursuing Brans-Dicke, it seems reasonable to attempt to reformulate the general theory of relativity so that there is a correspondence to Machian potential energy instead of classical gravitational potential energy.

Appendix A

Dimensional Variability in General Relativity

Bowler [5] has shown that general relativity is a gauge theory in which the fundamental dimensions of length, time and mass vary radially with the dimensionless scale factor respectively as $L = L_o\sigma$, $T = T_o\sigma^{-1}$ and $M = M_o\sigma^{-3}$. As shown in Table 1, which demonstrates the variability of some physical quantities, energy becomes

$$E(r) \;=\; E_o\,\sigma(r) \tag{14}$$

in the presence of a gravitational field at radius, r, compared to the original energy, E_o, sufficiently far from the field where $\sigma(\infty) = 1$. The radial variability corresponds to the escape example under consideration.

Table 1: Dimensional Variability in General Relativity

	Radial	Transverse
Length: L	$L_o\,\sigma$	L_o
Time: T	$T_o\,\sigma^{-1}$	$T_o\,\sigma^{-1}$
Energy: E	$E_o\,\sigma$	$E_o\,\sigma$
Momentum: p	$p_o\,\sigma^{-1}$	p_o
Mass: M	$M_o\,\sigma^{-3}$	$M_o\,\sigma^{-1}$
Velocity: v	$v_o\,\sigma^2$	$v_o\,\sigma$
Acceleration: a	$a_o\,\sigma^3$	$a_o\,\sigma^2$
Force: f	f_o	$f_o\,\sigma$
Power: P	$P_o\,\sigma^2$	$P_o\,\sigma^2$
Newton: G	$G_o\,\sigma^8$	$G_o\,\sigma^3$
Planck: h	h_o	h_o
$-GM/rc^2$: Φ	Φ_o	Φ_o

The table is a straightforward dimensional analysis having no context. For example, energy could refer to potential energy, or the energy of a photon in a gravitational field.

Energy contraction or time dilation could explain Pound-Rebka results. The Shapiro time delay experiment is consistent with the table's radial variability of velocity (of light) in a gravitational field.

Planck's constant and products of complementary pairs like Energy×Time or Momentum×Length have no scale factor, nor does the measure of gravitational field strength, Φ.

The difference between radial and transverse scale factors for mass is a bit puzzling, but not relevant to energy balance in the radial escape example.

Perhaps the most curious thing in the table is the radial scale factor to the *eighth* power for Newton's G.

Appendix B

Radial length contraction from a dual formulation

The calculation of Machian gravitational potential energy using the product integral for the free fall example can be rewritten as

$$\widetilde{E}(R) \;=\; mc^2\,\widetilde{\sigma}(R) \;=\; mc^2 \exp\!\left(\frac{-1}{mc^2}\int_{\infty}^{R} F(r)\,\mathrm{d}r\right) \tag{15}$$

But, from the identity $\int \mathrm{e}^u(\mathrm{d}u/\mathrm{d}x)\mathrm{d}x = \mathrm{e}^u$, there is also the identity

$$\int \exp\!\left(\frac{-GM}{rc^2}\right)\frac{GM}{r^2c^2}\,\mathrm{d}r \;=\; \widetilde{\sigma}(r) \tag{16}$$

Eq. (15) can then be expressed in the form of a definite integral of the gravitational force, $F(r) = -GMm/r^2$, as

$$\widetilde{E}(R) \;=\; -\int_{\infty}^{R} F(r)\,\widetilde{\sigma}(r)\,\mathrm{d}r \tag{17}$$

Eq. (16) also leads to another definite integral,

$$-\int_{0}^{R} F(r)\,\widetilde{\sigma}(r)\,\mathrm{d}r \;=\; \widetilde{E}(R) - mc^2 \tag{18}$$

which is approximately equal to the Newtonian potential energy.

Since a radial force, because of its dimensions, is independent of the scale factor, $\widetilde{\sigma}(r)$ can be associated with the element of radius, $\mathrm{d}r$, in (17) and (18).

Thus, the potential energy function calculated by the redshift rule can be interpreted as a spatial integration of the Newtonian gravitational force from a point mass in which the element of radial distance is contracted by the exponential scale factor.

In other words, Newtonian gravitational force is consistent with radial length contraction by virtue of the mathematical identity leading to (16), a relation that is unique to Machian gravitation. This dual formulation for energy indicates a natural link to modern gravitation in which the concept of Newtonian force would have renewed relevance.

Appendix C

Machian Escape Velocity as Relativistic Composition

In the above presentation, Machian escape velocity was determined by a balance between potential and kinetic energies. Alternatively, Machian escape velocity can be composed relativistically from the point of view of an observer in a stationary frame viewing events in a gravitationally accelerated frame [6]. There are related assumptions.

Gravitational force will be applied in the rest frame where measurements are made. Also, time dilation and length contraction due to free fall velocity of an object in a gravitational field are assumed not to be involved, in accord with Gullstrand-Painlevé coordinates and the river model of gravity.

A relativistic velocity can be composed by considering time as intervals of Δt. Let $g\Delta t$ be the change in velocity that would be brought about in one time interval by an acceleration, $g = -GM/r^2$, expected from the force of gravity under classical Galilean assumptions. Taking g as constant over the interval, a relativistic velocity for interval n can be composed approximately as

$$v_n \;=\; \frac{v_{n-1} + g\Delta t}{1 + v_{n-1}\, g\Delta t/c^2} \tag{19}$$

and the distance traveled can be approximated similarly as

$$r_n \;=\; r_{n-1} + v_n \Delta t \tag{20}$$

Given appropriate initial values, the resulting velocity corresponds to Machian escape velocity associated with relativistic potential energy calculated using the product integral. To verify that this composition of velocity produces Machian escape velocity, the recursion (19) can be rearranged to give an acceleration

$$a \;=\; \lim_{\Delta t \to 0} \frac{v_n - v_{n-1}}{\Delta t} \;=\; g\,\gamma^{-2} \tag{21}$$

which is the same as the time derivative of Machian escape velocity (11b).

I.e., given $\widetilde{v}_{\text{esc}} \;=\; c\,(1 - \widetilde{\sigma}^2)^{1/2}$, and recalling that $\gamma^{-1} \;=\; \widetilde{\sigma}$ for escape, the acceleration (21) can be found from

$$a \;=\; \frac{\mathrm{d}\widetilde{v}_{\text{esc}}}{\mathrm{d}t} \;=\; \frac{\mathrm{d}\widetilde{v}_{\text{esc}}}{\mathrm{d}\widetilde{\sigma}}\,\frac{\mathrm{d}\widetilde{\sigma}}{\mathrm{d}r}\,\frac{\mathrm{d}r}{\mathrm{d}t} \tag{22}$$

where these derivatives are given by

$$\frac{\mathrm{d}\widetilde{v}_{\text{esc}}}{\mathrm{d}\widetilde{\sigma}} \;=\; \frac{-c\,\widetilde{\sigma}}{(1 - \widetilde{\sigma}^2)^{1/2}} \qquad \frac{\mathrm{d}\widetilde{\sigma}}{\mathrm{d}r} \;=\; \frac{GM\,\widetilde{\sigma}}{r^2 c^2} \qquad \frac{\mathrm{d}r}{\mathrm{d}t} \;=\; \widetilde{v}_{\text{esc}} \tag{23}$$

References

[1] Clark S., Redshift, p. 178. University of Hertfordshire Press, Hatfield (1997)

[2] Bondi H., Cosmology. Cambridge University Press, Cambridge (1952)

[3] Hamilton A. and Lisle J., The river model of black holes, Am. J. Phys. 76, 519 (2008) arXiv:gr-qc/0411060

[4] Brans Carl H., The roots of scalar-tensor theory: an approximate history. (2005) arXiv:gr-qc/0505063

[5] Bowler M.G., Gravitation and Relativity. Pergamon Press, Oxford (1976)

[6] Walker C., A Tale of Two Relativities (2018) http://fqxi.org/community/forum/topic/3071

Related Material

▷ One consequence of Machian gravitation is that cosmological inflation would be untenable. In Walther Nernst's alternative to inflationary cosmology, the redshift of light can be viewed as evidence of a quantum mechanical harmonic oscillator by Planck's hypothesis, in which light energy decays exponentially by losing a quantum of energy, hH, every cycle. It is proposed that the primary obstacle to tired light posed by supernova data can be overcome by complementarity between distant time dilation and received light energy.

Uncertainty and complementarity in the cosmological redshift (2015)
https://fqxi.org/community/forum/topic/2292

▷ Machian gravitational relativity, discussed in Appendix C, implies a rest frame compatible with quantum mechanics in which to compose Machian escape velocity. This universal rest frame could correspond to a plenum of energy populated by fundamental quanta at the inferred zero point of electromagnetic radiation, $hH/2$. The properties of space, if not space itself, could be due to these quanta. This is [6] in the References above.

A Tale of Two Relativities (2018) fqxi.org/community/forum/topic/3071

▷ By combining the complex analytic Cauchy-Riemann derivative with the Cayley-Dickson construction of a quaternion, possible formulations of a quaternion derivative are explored with the goal of finding an analytic quaternion derivative having conjugate symmetry. Two such analytic derivatives can be found. This finding may have significance in areas of quantum mechanics where quaternions are fundamental, especially regarding the enigmatic phenomenon of complementarity, where a quantum process presents two essential aspects which would be mutually exclusive classically.

Seeking the Analytic Quaternion (2017)
fqxi.org/community/forum/topic/2822 or vixra.org/abs/2111.0167

▷ A simple noisy vector model is shown to be in accord with Robert McEachern's hypothesis that Bell correlations are associated with processes which can provide only one bit of information per sample. Unlike Richard Gill's treatment of Pearle's Hidden-Variable Model (arXiv:1505.04431), this classical model does not quite approach the expectation of quantum mechanics as the number of trials is increased. However, the noisy vector model has the advantage of an obvious separation of signal from noise used to measure information. It has yet to be shown if the Gill-Pearle model satisfies the one-bit criterion.

How Well Do Classically Produced Correlations Match Quantum Theory? (2017) `http://vixra.org/abs/1701.0621`

▷ Simulating Bell correlations by Monte Carlo methods can be time-consuming due to the large number of trials required to produce reliable statistics. For a noisy vector model, formulating the vector threshold crossing in terms of geometric probability can eliminate the need for trials, with inferred probabilities replacing statistical frequencies.

Simulated Bell-like Correlations from Geometric Probability (2017) `http://vixra.org/abs/1705.0377`