A Direct Proof of the Riemann Hypothesis

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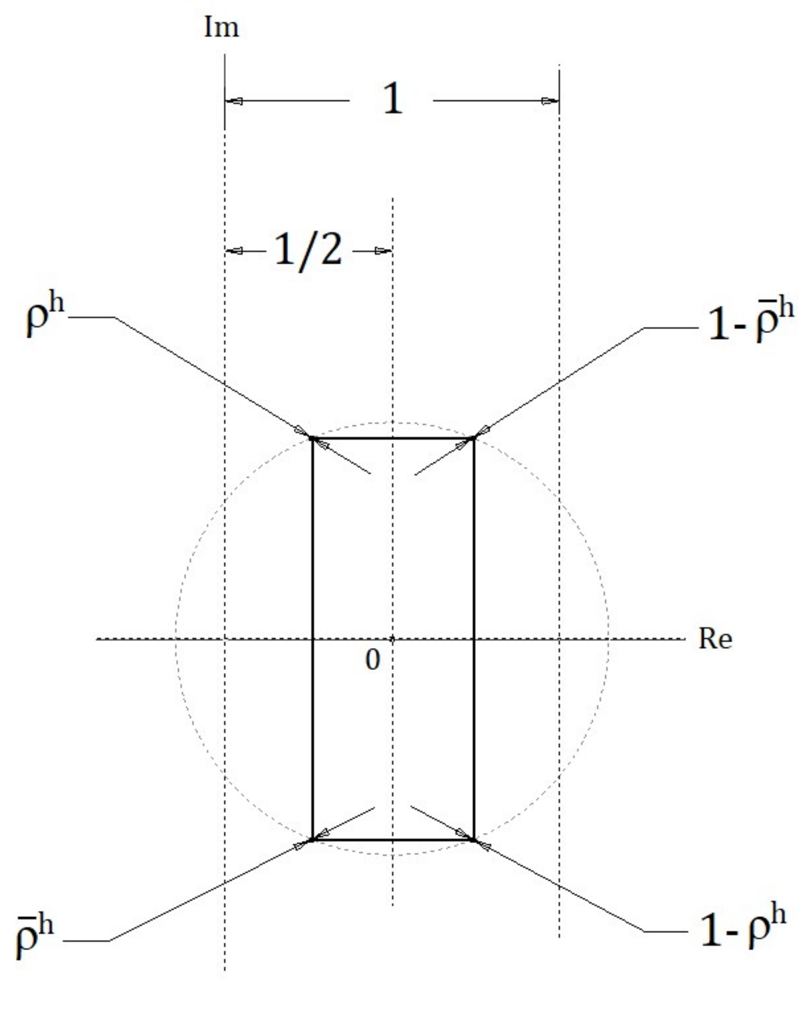
ABSTRACT. A function $\upsilon(s)$ is derived that shares all the non-trivial zeros of Riemann's zeta function $\zeta(s)$, and a novel representation of $\zeta(s)$ is presented that relates the two. From this the zeros of $\zeta(s)$ may be grouped according to two types: $\upsilon(s) = 0$ and $\upsilon(s) \neq 0$. A direct algebraic proof of the Riemann hypothesis is obtained by setting both zeta functions to zero and solving for two general solutions for all the non-trivial zeros.

Introduction. It is well known by B. Riemann's functional equation that any non-trivial zeros of the zeta function $\zeta(s)$ that do not have a real part one half must exist within the critical strip $0 < \Re(s) < 1$ at the vertices of rectangles, symmetric across the critical line $\Re(s) = 1/2$ and symmetric across the real axis. [1][2][3] This implies that for any two hypothetical non-trivial zeros ρ^h and $1 - \bar{\rho}^h$ symmetric across the critical line from one another, where $\bar{\rho}^h$ is the complex conjugate of ρ^h, $\left|\Re(\rho^h) - 1/2\right| = \left|\Re(1 - \bar{\rho}^h) - 1/2\right|$ and $\Im(\rho^h) = \Im(1 - \bar{\rho}^h)$, which this paper will refer to as "Riemann's symmetric vertices property".



Graphic definition of Riemann's symmetric vertices property (not to scale).

In his paper [3] Riemann writes that the "symmetrical form" [2]

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta(1-s),$$

of his functional equation

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \tag{1}$$

induced him to introduce the integral $\Gamma(s/2)$ in place of $\Gamma(s)$ in order to define the ξ function as

$$\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

This is an entire function that satisfies

$$\xi(s) = \xi(1-s),$$

which reveals the symmetry between $\xi(s)$ and $\xi(1-s)$.

Because $\zeta(s)$ is a multiplicative factor of $\xi(s)$, and because s and $1-s$ are reflections of each other through the real point one half, by definition of $\xi(s)$ all non-trivial zeros of the Riemann zeta function must comply with Riemann's symmetric vertices property. [1][2][3]. This definition may be stated as

$$D: \quad \begin{aligned} &\left|\Re(\rho^h)-\frac{1}{2}\right| = \left|\Re(1-\bar{\rho}^h)-\frac{1}{2}\right| \wedge \Im(\rho^h) = \Im(1-\bar{\rho}^h) \\ &\Leftarrow \exists\, \rho^h \\ &\wedge 1-\bar{\rho}^h\left(\Re(\rho^h) \neq \frac{1}{2} \wedge \Re(1-\bar{\rho}^h)\right. \\ &\left.\neq \frac{1}{2}, \rho^h \wedge 1-\bar{\rho}^h\right), (\rho^h \wedge 1-\bar{\rho}^h) \in \rho_n, \end{aligned}$$

which states that Riemann's symmetric vertices property is necessary for there to exist any non-trivial zeros ρ_n off the critical line. And this paper proposes that one can prove the non-existence of any hypothetical zeros off the critical line algebraically (that the Riemann hypothesis is true) by putting the Riemann zeta function in the form

$$a + 2\,b\,v\,\omega + c\,\omega^2 = \zeta(s),$$

and solving for the general solution of $\zeta(s)$'s zeros directly.

Motivation for this form. Given

$$a + b + c = 0,$$

there are only two types of solutions:

Type 1. Two terms negate each other and the third is zero, which has the geometric representation of a line ("Type 1 linear solution"), and

Type 2. Two terms negate the third, which has the geometric representation of a plane ("Type 2 planar solution").

This holds true for this paper's novel form as well. Given a, b, c having no roots,

$$a + 2\, b\, \upsilon\, \omega + c\, \omega^2 \; = 0,$$

implies

$$\upsilon = \frac{-a - c\, \omega^2}{2\, b\, \omega},$$

where the Type 1 linear solution is

$$a = -c\, \omega^2, \qquad \upsilon = 0,$$

and the Type 2 planar solution is

$$\upsilon = \frac{-a - c\, \omega^2}{2\, b\, \omega}, \qquad \upsilon \neq 0.$$

These two solutions may be considered grouped simply by $\upsilon = 0$ or $\upsilon \neq 0$ and stated as

$$\forall\, a + 2\, b\, \upsilon\, \omega + c\, \omega^2 = 0 \left(\exists\, \upsilon = 0 \;(\upsilon = 0, a + 2\, b\, \upsilon\, \omega + c\, \omega^2 = 0) \lor \exists\, \upsilon \neq 0 \;(\upsilon \neq 0, a + 2\, b\, \upsilon\, \omega + c\, \omega^2 = 0)\right).$$

Furthermore, υ need not be analytic (however it may be defined), as these solutions are purely algebraic. It is from this form then that the following claim may now be made.

Claim. Given only these two types of zeros of the Riemann zeta function

$$\forall\, \zeta(s) = 0 \left(\exists\, \upsilon(s) = \frac{-a_s - c_s\, \omega_s^2}{2\, b_s\, \omega_s} \left(\upsilon(s) = \frac{-a_s - c_s\, \omega_s^2}{2\, b_s\, \omega_s}, \zeta(s) = 0\right)\right),$$

grouped according to whether or not $\upsilon(s)$ equals zero

$$\forall\, \zeta(s) = 0 \left(\exists\, \upsilon(s) = 0 \;(\upsilon(s) = 0, \zeta(s) = 0) \lor \exists\, \upsilon(s) \neq 0 \;(\upsilon(s) \neq 0, \zeta(s) = 0)\right),$$

if the first type $\upsilon(s) = 0$ contains all the critical zeros

$$\Re(s) = \frac{1}{2} \land \zeta(s) = 0 \Leftrightarrow \upsilon(s) = 0 \land \zeta(s) = 0,$$

and the second type $\upsilon(s) \neq 0$ does not comply with Riemann's symmetric vertices property

$$\left|\Re(\rho^h) - \frac{1}{2}\right| = \left|\Re(1 - \bar{\rho}^h) - \frac{1}{2}\right| \wedge \Im(\rho^h) = \Im(1 - \bar{\rho}^h) \;\forall\; \upsilon(s) \neq 0 \wedge \zeta(s) = 0,$$

then the Riemann hypothesis is necessarily true

$$\forall\, \rho_n \left(\exists\, \Re(s) = \frac{1}{2} \left(\Re(s) = \frac{1}{2}, \rho_n\right)\right).$$

See the appendix for an outline of this proof.

Proof of the Claim. Begin by bringing all of the irrational properties of Riemann's functional equation (1), including $\zeta(1-s\,)$, into a single function $\upsilon(s)$ so that $\upsilon(s)$ relates to $\zeta(s)$ only by rational functions. First multiply both sides of (1) by $(s-1)^3$, subtract one, then multiply by i. This gives

$$i(\zeta(s)(s-1)^3 - 1) \;= i\;\left(2^s \pi^{s\,-\,1}(s-1)^3 \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) - \,1\right).$$

Then add $-2\ \mathrm{Im}(s)$ to both sides and multiply by $\sin\big(\arg(s)\big)$.

$$\begin{aligned}&\frac{\Im(s)\big(i((s\,-\,1)^3\zeta(s) -\,1) -\,2\,\Im(s)\big)}{|s|}\\ &\qquad\qquad = \sin\big(\arg(s)\big)\Bigg(-2\ \mathrm{Im}(s)\\ &\qquad\qquad +\ i\ \left(2^s\pi^{s\,-\,1}(s-1)^3\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) -\,1\right)\Bigg).\end{aligned}$$

The $|s|$ in the denominator suggests that neither side of the equation may meet the conditions necessary for complex differentiation going forward. Again, because this is an algebraic proof, there will be no need to apply the Cauchy-Riemann equations for the above or anything that follows. Therefore, continue by adding $2\ \mathrm{Im}(s)\ (2\ \mathrm{Im}(s) +\ i)\ \cos\big(\arg(s)\big)$ to both sides, and dividing both sides by $\sin\big(\arg(s)\big)\ \bar{s}$

$$\begin{aligned}&\frac{i\ \Big((s^*)^2 +\ (s\ -\ 1)^2\big((s\ -\ 1)\zeta(s) -\ 1\big)\Big)}{\bar{s}}\\ &\qquad\qquad = \frac{1}{\sin\big(\arg(s)\big)\,\bar{s}}\Bigg(\sin\big(\arg(s)\big)\Bigg(-2\ \mathrm{Im}(s)\\ &\qquad\qquad +\ i\ \left(2^s\pi^{s\,-\,1}(s-1)^3\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) -\ 1\right)\Bigg)\\ &\qquad\qquad +\ 2\ \mathrm{Im}(s)(2\ \mathrm{Im}(s) +\ i)\cos\big(\arg(s)\big)\Bigg)\end{aligned}$$

Divide both sides by $2(s-1)$ and let the right hand side be $\upsilon(s)$, such that

$$\frac{i\left((\bar{s})^2 + (s-1)^2\left((s-1)\zeta(s)-1\right)\right)}{2\,(s-1)\,\bar{s}} = \upsilon(s), \qquad s \neq 0. \tag{2}$$

Solve back for the Riemann zeta function from (2).

$$\zeta(s) = \frac{(s-1)^2 - 2\,i\,(s-1)\,\upsilon(s)\,\bar{s} - (\bar{s})^2}{(s-1)^3}, \qquad s \neq 0.$$

Now expand the right hand side to

$$\zeta(s) = \frac{1}{s-1} - \frac{2\,i\,\upsilon(s)\,\bar{s}}{(s-1)^2} - \frac{(\bar{s})^2}{(s-1)^3}, \qquad s \neq 0$$

in order to define the rational functions in the terms above. Add the second two terms on the right hand side to both sides and let the right hand side be a_s, such that

$$\begin{aligned} \zeta(s) + \frac{2\,i\,\upsilon(s)\,\bar{s}}{(s-1)^2} + \frac{(\bar{s})^2}{(s-1)^3} &= \frac{1}{s-1}. \\ &= a_s \end{aligned}$$

Divide both sides by $1-s$ and let the right hand side be b_s, such that

$$\begin{aligned} \frac{\zeta(s)}{1-s} + \frac{2\,i\,\upsilon(s)\,\bar{s}}{(1-s)(s-1)^2} + \frac{(\bar{s})^2}{(1-s)(s-1)^3} &= \frac{a_s}{1-s} \\ -a_s\,\zeta(s) - a_s\frac{2\,i\,\upsilon(s)\,\bar{s}}{(s-1)^2} - a_s\frac{(\bar{s})^2}{(s-1)^3} &= -\frac{1}{(s-1)^2}. \\ &= b_s \end{aligned}$$

Divide both sides once more by $1-s$ and let the right hand side be c_s, such that

$$\begin{aligned} -a_s\frac{\zeta(s)}{1-s} - a_s\frac{2\,i\,\upsilon(s)\,\bar{s}}{(1-s)(s-1)^2} - a_s\frac{(\bar{s})^2}{(1-s)(s-1)^3} &= \frac{b_s}{1-s} \\ -b_s\,\zeta(s) - b_s\frac{2\,i\,\upsilon(s)\,\bar{s}}{(s-1)^2} - b_s\frac{(\bar{s})^2}{(s-1)^3} &= \frac{1}{(s-1)^3}. \\ &= c_s \end{aligned}$$

Multiply both sides by $i\,(s-1)^3\bar{s}$ and let the right hand side be ω_s, such that

$$\begin{aligned} -i\,(s-1)^3\,\bar{s}\,\zeta(s)\,b_s + 2\,b_s\,(s-1)\,\upsilon(s)\,(\bar{s})^2 - i\,b_s\,(\bar{s})^3 &= i\,(s-1)^3\bar{s}\,c_s \\ i\frac{\zeta(s)\,\bar{s}}{a_s} - 2\,b_s\,(1-s)\,\upsilon(s)\,(\bar{s})^2 - i\,c_s\,(1-s)(\bar{s})^3 &= i\,\bar{s}. \\ &= \omega_s \end{aligned}$$

Because

$$i\frac{\zeta(s)\,\bar{s}}{a_s} - 2\,b_s\,(1\,-\,s)\,v(s)\,(\bar{s})^2 - i\,c_s\,(1\,-\,s)(\bar{s})^3 = \frac{\omega_s}{a_s}\ \ \zeta(s)\ + 2\,a_s\,v(s)\,\omega_s^2 + b_s{\omega_s}^3,$$

solving for $\zeta(s)$ from

$$\frac{\omega_s}{a_s}\ \ \zeta(s)\ + 2\,a_s\,v(s)\,\omega_s^2 + b_s{\omega_s}^3 = \omega_s$$

gives the desired form

$$P_1:\qquad \begin{aligned}\zeta(s) &= -a_s\,(2\,a_s\,\omega_s\,v(s)\ +\ b_s\,\omega_s^2 -\ 1) \\ &= a_s + 2\,b_s\,\omega_s\,v(s)\ + c_s\,{\omega_s}^2\end{aligned}.$$

Because a_s, b_s and c_s are multiplicative inverses of $(s-1)$ to some power, and zero has no reciprocal, a_s, b_s and c_s have no roots. And because $(s\,-\,1)\zeta(s) -\ 1 = -1, \zeta(s) = 0$ in the numerator of (2), (2) reduces to

$$\begin{aligned}v(s) &= \frac{-a_s - c_s\omega_s^2}{2\,b_s\,\omega_s} \\ &= \frac{i\,((s^*)^2 -\ (s\,-\,1)^2)}{2\,(s\,-\,1)\,\bar{s}}\end{aligned} \tag{3}$$

for all the zeros (trivial and non-trivial) of $\zeta(s)$, which can be stated as

$$P_2:\qquad \forall\,\zeta(s) = 0\left(\exists\,v(s) = \frac{-a_s - c_s\,{\omega_s}^2}{2\,b_s\,\omega_s}\left(v(s) = \frac{-a_s - c_s\,{\omega_s}^2}{2\,b_s\,\omega_s}, \zeta(s) = 0\right)\right).$$

$P1$ is not only an alternative form of (1), but also contains just two types of solutions for $\zeta(s) = 0$ that may be grouped according to the zeros of $v(s)$

$$P_3:\qquad \begin{aligned}&\forall\,\zeta(s) = 0\,\big(\exists\,v(s) = 0\,(v(s) = 0, \zeta(s) = 0) \vee\ \exists\,v(s) \\ &\neq 0\,(v(s) \neq 0, \zeta(s) = 0)\big),\end{aligned}$$

which are 1) the linear solution and 2) the planar solution. Solve first for the first type $v(s) = 0$.

$$\begin{aligned}0 &= \frac{-a_s - c_s\,\omega_s^2}{2\,b_s\,\omega_s} \\ &= \frac{i\,((\bar{s})^2 -\ (s\,-\,1)^2)}{2\,(s\,-\,1)\,\bar{s}} \\ &= \frac{i\,\big(1\,-\,2\,i\,\Im(s)\big)(2\,\Re(s)\,-\,1)}{2\,(s\,-\,1)\,\bar{s}} \Longrightarrow \Re(s) = \frac{1}{2} : a_s = -c_s\,\omega_s^2\end{aligned}.$$

Because $2\,\Re(s) - 1 = 0$ implies $\Re(s) = 1/2$, and no other part of (3) could equal zero by the definitions of complex arithmetic, the Type 1 solution $\upsilon(s) = 0$ is linear on the critical line.

$$\Re(s) = \frac{1}{2}, \qquad \upsilon(s) = 0, \tag{4}$$

which can be stated as

$$P_4\colon \qquad \forall\, \upsilon(s) = 0 \wedge \zeta(s) = 0 \ \left(\exists\, \Re(s) = \frac{1}{2}\left(\Re(s) = \frac{1}{2}, \upsilon(s) = 0 \wedge \zeta(s) = 0\ \right)\right).$$

Solve next for the real part of s from the Type 2 planar solution

$$\upsilon(s) = \frac{-a_s - c_s\,\omega_s^2}{2\,b_s\,\omega_s}, \qquad \upsilon(s) \neq 0, \tag{5}$$

given the quadratic formula applied to (5). One gets

$$\Re(s) = \frac{\pm\sqrt{-(2\,\Im(s)\,+\,i)^2(\upsilon(s)^2 - 1)} + 2\,\Im(s)\,+\,\upsilon(s)\,+\,i}{2\,\upsilon(s)}, \qquad \upsilon(s) \neq 0, \tag{6}$$

where all the zeros other than $\Re(s) = 1/2,\ \upsilon(s) = 0$ consist of pairs across the critical line from each other, which in terms of the hypothetical non-trivial zeros may be stated as

$$P_5\colon \qquad \forall\, \upsilon(\rho^h) \neq 0 \left(\exists\, \rho^h \wedge 1 - \bar{\rho}^h \left(\rho^h \wedge 1 - \bar{\rho}^h, \upsilon(\rho^h) \neq 0\right)\right).$$

The trivial zeros are applicable to (6) as positive solutions, but because any counterpart to these could not exist symmetrically across the critical line inside the critical strip, much less on the real line, their negative solution counterparts are extraneous.

Upon examination of the square root in (6), because

$$-(2\,\Im(s) + \,i)^2(\zeta^\rho(s)^2 - 1) = (\upsilon(s)^2 - 1)(\bar{s} - \,s\,+\,1)^2,$$

one can also express (6) as

$$\Re(s) = \frac{\pm\sqrt{(\upsilon(s)^2 - 1)(\bar{s} - \,s\,+\,1)^2} + 2\,\Im(s)\,+\,\upsilon(s)\,+\,i}{2\,\upsilon(s)},$$

which also provides a pair of $\upsilon(s)$'s, given by

$$\upsilon(s) = \frac{\pm\sqrt{(2\,\mathrm{Im}(s) + \,i)^2}\,|1\,-\,2\,\mathrm{Re}(s)| + \,(\bar{s} + \,s\,-\,1)(2\,\mathrm{Im}(s) + \,i)}{4\,(|(s)^2| - \,\bar{s})}.$$

This gives a total of four possible hypothetical zeros (two sets of pairs) across the real and critical lines from each other, as were graphically defined at the beginning of this paper, and as implied by (1). Now one can ask the question, given any hypothetical non-trivial zero ρ^h off the critical line, *is it possible for any* $1 - \bar{\rho}^h$ *to be symmetric to* ρ^h *across the critical line?* That is; for any $\Re(\rho^h) \neq 1/2$, *is* *it possible given the two solutions in (6) to have a $\Re(1-\bar{\rho}^h) \neq 1/2$ equidistant to ρ^h from the critical line, considering what has been presented so far?*

This is elementary to verify. Because

$$r = \sqrt{(v-x)^2 + (w-y)^2}$$

gives the distance r between any two points $v + i\,w$ and $x + i\,y$ on the complex plane, the distance r_{cr} from any point s to the nearest point $1/2 + i\,\Im(s)$ on the critical line is given by

$$r_{cr} = \left|\Re(s) - \frac{1}{2}\right|.$$

One can then check if symmetry between the positive and negative solutions of (6) is mathematically possible. Setting the two distances equal to each other

$$\left|\Re(\rho^h) - \frac{1}{2}\right| = \left|\Re(1-\bar{\rho}^h) - \frac{1}{2}\right|,$$

where ρ^h is either the positive or negative solution to (6) and $1-\bar{\rho}^h$ is either the positive or negative as well, one gets

$$\begin{aligned}&\left|\frac{\pm\sqrt{-(2\,\Im(\rho^h)\,+\,i)^2(v(\rho^h)^2-\,1)} +\,2\,\Im(\rho^h)\,+\,v(\rho^h)\,+\,i}{2\,v(\rho^h)} - \frac{1}{2}\right| \\ &= \left|\frac{\pm\sqrt{-(2\,\Im(\rho^h)\,+\,i)^2(v(1-\bar{\rho}^h)^2-\,1)} +\,2\,\Im(\rho^h)\,+\,v(1-\bar{\rho}^h)\,+\,i}{2\,v(1-\bar{\rho}^h)} - \frac{1}{2}\right|, \\ &\rho^h = 1-\bar{\rho}^h \quad \vee \quad v(\rho^h) = \pm 1 \quad \vee \quad v(1-\bar{\rho}^h) = \pm 1\end{aligned} \tag{7}$$

The imaginary part of ρ^h is on both sides of the equation because Riemann's symmetric vertices property not only requires $|\Re(\rho^h) - 1/2| = |\Re(1-\bar{\rho}^h) - 1/2|$, but also $\Im(\rho^h) = \Im(1-\bar{\rho}^h)$. Now one can evaluate the solutions of (7). The value inside the square root reduces to zero for $v(s) = \pm 1$, but because

$$\frac{i\,((\bar{s})^2 -\,(s\,-\,1)^2)}{2\,(s\,-\,1)\,\bar{s}} = \pm 1 \Longrightarrow s \pm i\,\bar{s} = 1$$

from (5) is false, this solution is extraneous. The only other possible solution $\rho^h = 1-\bar{\rho}^h$ in (7) is meaningless, as the only arguments that could apply would be the critical zeros, which would give $v(\rho^h) = 0$ for $\Re(\rho^h) = 1/2$, leaving (7) undefined, which can be written as

P_6: $$\forall\, v(\rho^h) \neq 0\ \left(\nexists\, \Re(\rho^h) = \frac{1}{2}\left(\Re(\rho^h) = \frac{1}{2}, v(\rho^h) \neq 0\right)\right),$$

and as such

P_7: $$\Re(s) = \frac{1}{2} \wedge \zeta(s) = 0 \Leftrightarrow v(s) = 0 \wedge \zeta(s) = 0,$$

and

P_8: $$\left|\Re(\rho^h) - \frac{1}{2}\right| \neq \left|\Re(1-\bar{\rho}) - \frac{1}{2}\right| \ \forall\, \upsilon(\rho^h) \neq 0.$$

The Type 2 planar solution (6) does not comply with Riemann's symmetric vertices property, and the negative solution also becomes extraneous with the positive only applying to the trivial zeros. Because all non-trivial zeros must comply with Riemann's symmetric vertices property in D, no non-trivial zeros exist as Type 2 planar solutions

P_9: $$\forall\, \rho \left(\nexists\, \upsilon(s) \neq 0\ (\upsilon(s) \neq 0, \rho)\right)\ .$$

And because there are only two types of solutions for $\zeta(s) = 0$ (which follows from P_3), all the non-trivial zeros of the Riemann zeta function must be restricted to the Type 1 linear solution

P_{10}: $$\forall\, \rho \left(\exists\, \upsilon(\rho) = 0\ (\upsilon(\rho) = 0, \rho)\right).$$

Since the Type 1 solution is the real part of s equal to one half, all the non-trivial zeros of the Riemann zeta function have a real part one half

$$\forall\, \rho \left(\exists\, \Re(\rho) = \frac{1}{2}\left(\Re(\rho) = \frac{1}{2}, \rho\right)\right),$$

and the Riemann hypothesis is correct. ■

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Appendix. This proof follows the direct form $P_1 \wedge \ldots \wedge P_n \Rightarrow Q$ and depends on the following definition of Riemann's symmetric vertices property: $\left|\Re(\rho^h) - 1/2\right| = \left|\Re\left(1-\bar{\rho}^h\right) - 1/2\right| \wedge \Im(\rho^h) = \Im\left(1-\bar{\rho}^h\right) \Leftarrow \exists\, \rho^h \wedge 1-\bar{\rho}^h \left(\Re(\rho^h) \neq \frac{1}{2} \wedge \Re\left(1-\bar{\rho}^h\right) \neq \frac{1}{2}, \rho^h \wedge 1-\bar{\rho}^h\right), \left(\rho^h \wedge 1-\bar{\rho}^h\right) \in \rho_n$.

$$\begin{aligned}
\zeta(s) \quad &= \quad a_s + 2\, b_s\, \upsilon(s)\, \omega_s + c_s {\omega_s}^2 \colon a_s = \tfrac{1}{s-1}, b_s = -\tfrac{1}{(s-1)^2}, c_s = \tfrac{1}{(s-1)^3}, \omega_s = i\, \bar{s} && P_1 \\
&\Rightarrow \quad \forall\, \zeta(s) = 0 \left(\exists\, \upsilon(s) = \frac{-a_s - c_s\, {\omega_s}^2}{2\, b_s\, \omega_s}\left(\upsilon(s) = \frac{-a_s - c_s\, {\omega_s}^2}{2\, b_s\, \omega_s}, \zeta(s) = 0\right)\right) && P_2 \\
&\Rightarrow \quad \forall\, \zeta(s) = 0 \left(\exists\, \upsilon(s) = 0\ (\upsilon(s) = 0, \zeta(s) = 0) \vee \exists\, \upsilon(s) \neq 0\ (\upsilon(s) \neq 0, \zeta(s) = 0)\right) && P_3 \\
&\Rightarrow \quad \forall\, \upsilon(s) = 0 \wedge \zeta(s) = 0\ \left(\exists\, \Re(s) = \frac{1}{2}\left(\Re(s) = \frac{1}{2}, \zeta^\rho(s) = 0 \wedge \zeta(s) = 0\ \right)\right) && P_4
\end{aligned}$$

$$\Rightarrow \quad \forall\, \upsilon(\rho^h) \neq 0 \left(\exists\, \rho^h \wedge 1-\bar{\rho}^h(\rho^h \wedge 1-\bar{\rho}^h, \zeta^\rho(\rho^h) \neq 0)\right) \qquad P_5$$

$$\Rightarrow \quad \forall\, \upsilon(\rho^h) \neq 0 \left(\nexists\, \Re(\rho^h) = \frac{1}{2}\left(\Re(\rho^h) = \frac{1}{2}, \upsilon(\rho^h) \neq 0\right)\right) \qquad P_6$$

$$\Rightarrow \quad \Re(s) = \frac{1}{2} \wedge \zeta(s) = 0 \Leftrightarrow \upsilon(s) = 0 \wedge \zeta(s) = 0 \qquad P_7$$

$$\left|\Re(\rho) - \frac{1}{2}\right| \quad \neq \quad \left|\Re(1-\bar{\rho}) - \frac{1}{2}\right| \;\forall\, \upsilon(1-\bar{\rho}) \neq 0 \qquad P_8$$

$$\Rightarrow \quad \forall\, \rho \left(\nexists\, \upsilon(s) \neq 0\; (\upsilon(s) \neq 0, \rho)\right) \qquad P_9$$

$$\Rightarrow \quad \forall\, \rho \left(\exists\, \upsilon(\rho) = 0\; (\upsilon(\rho) = 0, \rho)\right) \qquad P_{10}$$

$$\Rightarrow \quad \forall\, \rho \left(\exists\, \Re(\rho) = \tfrac{1}{2} \left(\Re(\rho) = \tfrac{1}{2}, \rho\right)\right) \qquad \blacksquare$$

P_1	Because of the definitions of algebra, trigonometry and complex arithmetic
P_2	Because of the implication of P_1 and because anything multiplied by zero is zero
P_3	Because of the implication of P_2 and because zero has no reciprocal
P_4	Because of the implication of P_3 and the definitions of complex arithmetic
P_5	Because of the implication of P_3 and the definition of the quadratic formula
P_6	Because of the implication of P_5 and the definitions of complex arithmetic
P_7	Because of the reverse implication of P_4 and P_6
P_8	Because of the implication of P_5 and the definitions of complex arithmetic
P_9	Because of the implication of P_8 and the definition of Riemann's symmetric vertices property
P_{10}	Because of the implication of P_3 and P_9
Q	Because of the implication of P_{10}