A Direct Proof of the Riemann Hypothesis

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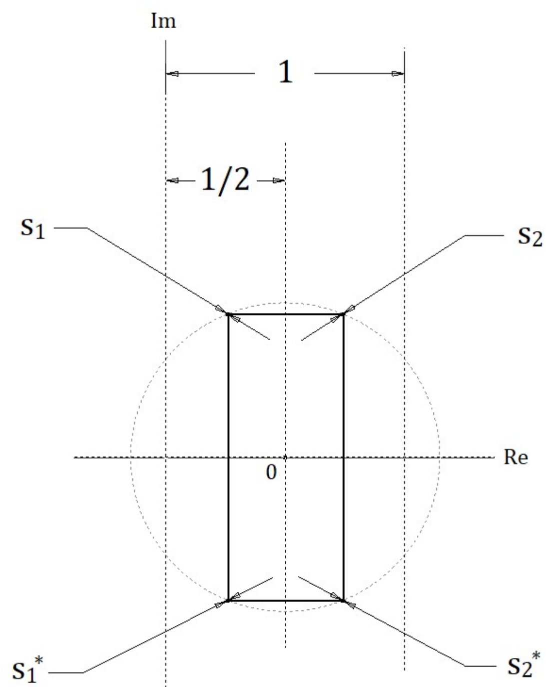
ABSTRACT. A zeta function $\zeta^{\rho}(s)$ is derived that shares all the non-trivial zeros of Riemann's zeta function $\zeta(s)$, and a novel representation of $\zeta(s)$ is presented that relates the two. From this it is shown how there can be only two types of zeros for $\zeta(s)$. A simple and direct proof of the Riemann hypothesis is obtained by setting both zeta functions to zero and solving for the general solution for all the non-trivial zeros directly.

1. Introduction

It is well known by B. Riemann's functional equation

$$\zeta(s) = 2^{s}\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \tag{1}$$

that any non-trivial zeros of the zeta function $\zeta(s)$ that do not have a real part one half must exist within the critical strip $0 < \Re(s) < 1$ at the vertices of rectangles, symmetric across the critical line $\Re(s) = 1/2$ and symmetric across the real axis. [1][2][3]



Graphic definition of Riemann's symmetric vertices property (not to scale).

This implies $|\Re(s_1) - 1/2| = |\Re(s_2) - 1/2|$ and $\Im(s_1) = \Im(s_2)$, which this paper refers to as "Riemann's symmetric vertices property".

In his paper [3] Riemann writes that the "symmetrical form" [2]

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}\zeta(1-s),$$

of his functional equation "induced" him to introduce the integral $\Gamma(s/2)$ in place of $\Gamma(s)$ in order to define the ξ function as

$$\xi(s) = \frac{s}{2}\,(s-1)\,\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\,\zeta(s).$$

This is an entire function that satisfies

$$\xi(s) = \xi(1-s),$$

which reveals the symmetry between $\xi(s)$ and $\xi(1-s)$. Because $\zeta(s)$ is a multiplicative factor of $\xi(s)$, and because s and s^* are reflections of each other across the real line, perpendicular to the reflection of s and $s-1$ across the critical line, by definition of $\xi(s)$ all non-trivial zeros of the Riemann zeta function must comply with Riemann's symmetric vertices property. [1][2][3].

This paper proposes that one can prove the non-existence of these hypothetical zeros off the critical line (that the Riemann hypothesis is true) by putting the Riemann zeta function in "oscillator form"

$$a + 2\,\omega\,\zeta^\rho\,b + \omega^2\,c = \zeta(s),$$

and solving for the general solution of $\zeta(s)$'s zeros directly.

Consider the following simplification in order to illustrate the motivation behind this representation. Because a whole is equal to the sum of its parts, there are only two types of solutions for

$$a + b + c = 0.$$

Type 1. Two terms negate each other and the third is zero, which has the geometric representation of a line ("Type 1 linear solution").

Type 2. Two terms negate the third, which has the geometric representation of a plane ("Type 2 planar solution").

This holds true for oscillator form as well. Given a, b, c and ω having no roots, when

$$a + 2\,\omega\,\zeta^\rho\,b + \omega^2\,c = 0$$

the Type 1 linear solution is

$$a = -\omega^2\,c, \qquad \zeta^\rho = 0,$$

and the Type 2 planar solution is

$$\zeta^\rho = \frac{-a - \omega^2 c}{2\,\omega\,\zeta^\rho\,b}, \qquad \zeta^\rho \neq 0.$$

Once defined for arguments s, by setting both ζ^ρ and $\zeta(s)$ to zero and solving for a general solution, proof of the Riemann hypothesis is obtained if the hypothetical zeros off the critical line are not mathematically possible.

Claim. Given a Type 2 planar solution for the zeros of

$$a_s + 2\,\omega_s\,\zeta^\rho(s)\,b_s + \omega_s^2\,c_s = \zeta(s)$$

that does not comply with Riemann's symmetric vertices property, when $\zeta(s) = 0$ and $\zeta^\rho(s) = 0$ there exists a unique solution for all the non-trivial zeros of the Riemann zeta function.

$$2\,\omega_s\,\zeta^\rho(s)\,b_s = -a_s - {\omega_s}^2 c_s + \zeta(s) = 0 \iff \Re(s) = \frac{1}{2}, \qquad \zeta(s) = 0.$$

See the appendix for an outline of this proof.

Proof of the Claim. Begin by defining the functions of the oscillator form of the Riemann zeta function. Multiply both sides of (1) by $(s-1)^3$, subtract one, then multiply by i. This gives

$$i(\zeta(s)(s-1)^3 - 1) = i\left(2^s \pi^{s-1}(s-1)^3 \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) - 1\right).$$

Then add $-2\,\mathrm{Im}(s)$ to both sides and multiply by $\sin(\arg(s))$.

$$\frac{\Im(s)\big(i((s-1)^3\zeta(s) - 1) - 2\,\Im(s)\big)}{|s|}$$
$$= \sin(\arg(s))\Bigg(-2\,\mathrm{Im}(s)$$
$$+ i\left(2^s \pi^{s-1}(s-1)^3 \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) - 1\right)\Bigg).$$

Add $2\,\mathrm{Im}(s)\,(2\,\mathrm{Im}(s) + i)\cos(\arg(s))$ to both sides, and divide both sides by $s^* \sin(\arg(s))$

$$\frac{i\left((s^*)^2 + (s-1)^2((s-1)\zeta(s) - 1)\right)}{s^*}$$
$$= \frac{1}{s^* \sin(\arg(s))}\Bigg(\sin(\arg(s))\Bigg(-2\,\mathrm{Im}(s)$$
$$+ i\left(2^s \pi^{s-1}(s-1)^3 \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) - 1\right)\Bigg)$$
$$+ 2\,\mathrm{Im}(s)(2\,\mathrm{Im}(s) + i)\cos(\arg(s))\Bigg)$$

Divide both sides by $2(s-1)$ and let the right hand side be defined as $\zeta^\rho(s)$.

$$\frac{i\left((s^*)^2 + (s-1)^2((s-1)\zeta(s) - 1)\right)}{2\,(s-1)\,s^*} = \zeta^\rho(s), \qquad s \neq 0. \tag{2}$$

Multiply both sides of (2) by $-2\,i\,(s\ -\ 1)\,s^*$, then add $(s\ -\ 1)^2-(s^*)^2$ and divide by $(s\ -\ 1)^3$ in order to solve back for the Riemann zeta function.

$$\zeta(s)=\frac{(s\ -\ 1)^2-2\,i\,(s\ -\ 1)\zeta^\rho(s)\,s^*-\ (s^*)^2}{(s\ -\ 1)^3},\qquad s\neq 0.$$

Now define the other functions of the oscillator form of $\zeta(s)$. Subtract $(s^*\,(-s^*\ -\ 2\,i\,(s\ -\ 1)\,\zeta^\rho(s)))/(s\ -\ 1)^3$ from both sides and let the right hand side be defined as a_s.

$$\begin{aligned}\zeta(s)-\frac{s^*\left(-s^*\ -\ 2\,i\,(s\ -\ 1)\zeta^\rho(s)\right)}{(s\ -\ 1)^3} &= \frac{1}{s-1}.\\ &= a_s\end{aligned}$$

Divide both sides by $(1-s)$ and let the right hand side be defined as b_s.

$$\begin{aligned}-\frac{2\,i\,(s\ -\ 1)\zeta^\rho(s)\,s^*+\ (s^*)^2+\ (s\ -\ 1)^3\zeta(s)}{(s\ -\ 1)^4} &= -\frac{1}{(s-1)^2}.\\ &= b_s\end{aligned}$$

Divide both sides once more by $(1-s)$ and let the right hand side be defined as c_s.

$$\begin{aligned}\frac{2\,i\,(s\ -\ 1)\,\zeta^\rho(s)\,s^*+\ (s^*)^2+\ (s\ -\ 1)^3\,\zeta(s)}{(s\ -\ 1)^5} &= \frac{1}{(s-1)^3}.\\ &= c_s\end{aligned}$$

Multiply both sides by $i\,(s\ -\ 1)^3s^*$ and let the right hand side be defined as ω_s.

$$\begin{aligned}\frac{i\,s^*\left(2\,i\,(s\ -\ 1)\,\zeta^\rho(s)\,s^*+\ (s^*)^2+\ (s\ -\ 1)^3\,\zeta(s)\right)}{(s\ -\ 1)^2} &= i\,s^*.\\ &= \omega_s\end{aligned}$$

Finally, square both sides, multiply by c_s and add $a_s+2\omega_s\zeta^\rho(s)b_s$. Because of the trigonometric and complex arithmetic definitions used in the above algebraic manipulations, one gets the oscillator form of $\zeta(s)$.

$$\zeta(s)=a_s+2\,\omega_s\,\zeta^\rho(s)\,b_s+\omega_s^2\,c_s,\qquad s\neq 0. \tag{3}$$

And because a_s, b_s and c_s are multiplicative inverses of $(s-1)$ to some power, and zero has no reciprocal, a_s, b_s and c_s have no roots.

Now one can solve for the Riemann zeta function's zeros directly. Because $(s\ -\ 1)\zeta(s)-1=-1, \zeta(s)=0$ in the numerator of (2), (2) reduces to

$$\begin{aligned}\zeta^{\rho}(s) &= \frac{-a_s - \omega_s^2 c_s}{2\,\omega_0 b_s} \\ &= \frac{i\,((s^*)^2 - (s - 1)^2)}{2\,(s - 1)s^*}\end{aligned} \tag{4}$$

for all the zeros (trivial and non-trivial) of $\zeta(s)$. Because a whole is equal to the sum of its parts, there are only two types of solutions for $\zeta(s) = 0$. The first solution is linear on the critical line.

$$\begin{aligned}0 &= \frac{-a_s - \omega_s^2 c_s}{2\,\omega_s\, b_s} \\ &= \frac{i\,((s^*)^2 - (s - 1)^2)}{2\,(s - 1)s^*} \\ &= \frac{i\left(1 - 2\,i\,\Im(s)\right)(2\,\Re(s) - 1)}{2\,(s - 1)s^*} \Longrightarrow \Re(s) = \frac{1}{2} : a_s = -\omega_s^2 c_s\end{aligned}.$$

Because $2\,\Re(s) - 1 = 0$ implies $\Re(s) = 1/2$, and no other part of (4) could equal zero by the definitions of complex arithmetic, the Type 1 linear solution is

$$\Re(s) = \frac{1}{2}, \qquad \zeta^{\rho}(s) = 0. \tag{5}$$

Solving for the real part of s from the Type 2 planar solution

$$\zeta^{\rho}(s) = \frac{-a_s - \omega_0^2 c_s}{2\,\omega_0 b_s}, \qquad \zeta^{\rho}(s) \neq 0, \tag{6}$$

given the quadratic formula applied to (6), one gets

$$\Re(s) = \frac{\pm\sqrt{-(2\,\Im(s) + i)^2(\zeta^{\rho}(s)^2 - 1)} + 2\,\Im(s) + \zeta^{\rho}(s) + i}{2\,\zeta^{\rho}(s)}, \quad \zeta^{\rho}(s) \neq 0, \tag{7}$$

where all the zeros other than $\Re(s) = 1/2, \zeta^{\rho}(s) = 0$ consist of pairs across the critical line from each other.

Upon examination of the square root, because

$$-(2\,\Im(s) + i)^2(\zeta^{\rho}(s)^2 - 1) = (\zeta^{\rho}(s)^2 - 1)(s^* - s + 1)^2,$$

one can also express (7) as

$$\Re(s) = \frac{\pm\sqrt{(\zeta^{\rho}(s)^2 - 1)(s^* - s + 1)^2} + 2\,\Im(s) + \zeta^{\rho}(s) + i}{2\,\zeta^{\rho}(s)},$$

which also provides a pair of $\zeta^{\rho}(s)$'s, given by

$$\zeta^{\rho}(s) = \frac{\pm\sqrt{(2\,\mathrm{Im}(s) + i)^2}\,|1 - 2\,\mathrm{Re}(s)| + (s^* + s - 1)(2\,\mathrm{Im}(s) + i)}{4\,(|(s)^2| - s^*)}.$$

This gives a total of four possible hypothetical zeros (two sets of pairs) across the real and critical lines from each other, as were graphically defined at the beginning of this paper. Now one can ask, given any hypothetical non-trivial zero s_1 off the critical line, *is it possible for any* s_2 *to be symmetric to* s_1 *across the critical line?* That is; for any $\Re(s_1) \neq 1/2$, *is it possible given the two solutions in (7) to have a* $\Re(s_2) \neq 1/2$ *equidistant to* s_1 *from the critical line?*

Because

$$r = \sqrt{(v - x)^2 + (w - y)^2}$$

gives the distance r between any two points $v + i\,w$ and $x + i\,y$ on the complex plane, the distance r_{cr} from any point s to the nearest point $1/2 + i\,\Im(s)$ on the critical line is given by

$$r_{cr} = \left|\Re(s) - \frac{1}{2}\right|.$$

One can then verify if symmetry between the positive and negative solutions of (7) is mathematically possible. Setting the two distances equal to each other

$$\left|\Re(s_1) - \frac{1}{2}\right| = \left|\Re(s_2) - \frac{1}{2}\right|,$$

where s_1 is either the positive or negative solution to (7) and s_2 is either the positive or negative as well, one gets

$$\begin{gathered} \left|\frac{\pm\sqrt{-(2\,\Im(s_1) + i)^2(\zeta^\rho(s_1)^2 - 1)} + 2\,\Im(s_1) + \zeta^\rho(s_1) + i}{2\,\zeta^\rho(s_1)} - \frac{1}{2}\right| \\ = \left|\frac{\pm\sqrt{-(2\,\Im(s_1) + i)^2(\zeta^\rho(s_2)^2 - 1)} + 2\,\Im(s_1) + \zeta^\rho(s_2) + i}{2\,\zeta^\rho(s_2)} - \frac{1}{2}\right|, \\ s_1 = s_2 \quad \vee \quad \zeta^\rho(s_1) = \pm 1 \quad \vee \quad \zeta^\rho(s_2) = \pm 1 \end{gathered} \tag{8}$$

The imaginary part of s_1 is on both sides of the equation because Riemann's symmetric vertices property not only implies $|\Re(s_1) - 1/2| = |\Re(s_2) - 1/2|$, but also $\Im(s_1) = \Im(s_2)$. Now one can evaluate the solutions of (8). The value inside the square root reduces to zero for $\zeta^\rho(s) = \pm 1$, but because

$$\frac{i\,((s^*)^2 - (s - 1)^2)}{2\,(s - 1)s^*} = \pm 1 \Rightarrow s \pm i\,s^* = 1$$

from (6) is false by the definitions of complex arithmetic, this solution is extraneous. The only other possible solution $s_1 = s_2$ in (8) is meaningless, as the only arguments that could apply would be the critical zeros, which would give $\zeta^\rho(s) = 0$ for $\Re(s) = 1/2$, leaving (8) undefined. Therefore

$$\left|\Re(s_1) - \frac{1}{2}\right| \neq \left|\Re(s_2) - \frac{1}{2}\right|, \qquad \zeta^\rho(s) \neq 0. \tag{9}$$

The Type 2 planar solution (6) does not comply with Riemann's symmetric vertices property, and the negative solution to (7) also becomes extraneous with the positive only applying to the trivial zeros. Because all non-trivial zeros must comply with Riemann's symmetric vertices property, and there are only two types of solutions for $\zeta(s) = 0$, all the non-trivial zeros of the Riemann zeta function must be restricted to the Type 1 linear solution (5). Therefore, when $\zeta(s) = 0$ and $\zeta^{\rho}(s) = 0$ there exists a unique solution for all the non-trivial zeros of the Riemann zeta function.

$$2\,\omega_s\,\zeta^{\rho}(s)\,b_s = -a_s - {\omega_s}^2 c_s + \zeta(s) = 0 \iff \Re(s) = \frac{1}{2}, \qquad \zeta(s) = 0,$$

And the Riemann hypothesis is correct. ■

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Appendix. Proof outline.

This proof follows the direct form $P_1 \wedge \ldots \wedge P_n \Rightarrow Q$.

P_1: All non-trivial zeros of the Riemann zeta function must comply with Riemann's symmetric vertices property.

P_2: $a_s + 2\,\omega_s\,\zeta^{\rho}(s)\,b_s + \omega_s^2\,c_s = \zeta(s), s \neq 0.$

P_3: a_s, b_s and c_s have no roots.

P_4: There are only two types of solutions for P_2 equal to zero: 1) a Type 1 linear solution and 2) a Type 2 planar solution.

P_5: The Type 1 linear solution is $\Re(s) = 1/2$.

P_6: The Type 2 planar solution consists of pairs s_1 and s_2.

P_7: The Type 2 planar solution pairs do not comply with Riemann's symmetric vertices property.

P_8: All the non-trivial zeros of P_1 must be restricted to the Type 1 linear solution.

Q: The Riemann hypothesis is correct.