A Direct Proof of the Riemann Hypothesis

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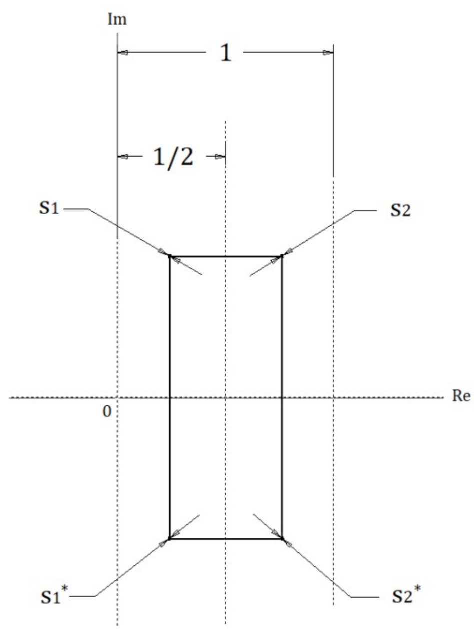
ABSTRACT. This paper presents a short and direct proof of the Riemann hypothesis, based on the previous longer [2]. A zeta function $\zeta^{\rho}(s)$ is defined that shares all the non-trivial zeros of the Riemann zeta function $\zeta(s)$, but none of the trivial. Proof is obtained by relating the two by functional equation, setting both to zero and solving for the general solution directly. The Riemann hypothesis is proven by a single claim.

1. Introduction

It is well known by B. Riemann's functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

that any non-trivial zeros of the zeta function $\zeta(s)$ that do not have a real part one half must exist within the critical strip $0 < \Re(s) < 1$, at the vertices of rectangles, symmetric across the critical line $\Re(s) = 1/2$ and symmetric across the real axis. [1][3]



Abstract representation of the hypothetical non-trivial zeros (not to scale).

The equation relates $\zeta(s)$ to $\zeta(1-s)$, which is a reflection through the point $s = 1/2$, and this paper proposes that one can prove the asymmetry of any hypothetical zeros across the critical line (that they do not exist and the Riemann hypothesis is true), using the novel representation

$$a_s + 2\zeta^{\rho}(s)b_s + \omega_0^2 c_s = \zeta(s).$$

As described in detail in [2], the above representation is one of two cases. The second case is only different in the second term

$$a_s + 2\omega_0 \zeta^{\rho 2}(s) b_s + \omega_0^2 c_s = \zeta(s).$$

Consider the following simplification in order to better illustrate the motivation behind this representation. Given

$$a + b + c = 0,$$

there are only two types of solutions.

Type 1. Two terms negate each other and the third is zero, which has the geometric representation of a line ("linear solution").

Type 2. Two terms negate the third, which has the geometric representation of a plane ("planar solution").

All of the above holds true as well for

$$a + b\,\zeta + c = 0.$$

Assuming a, b and c have no roots, the linear solution is

$$a = -c, \qquad \zeta = 0$$

and the planar solution is

$$\zeta = \frac{-a - c}{b}, \qquad \zeta \neq 0.$$

Similarly given a_s, b_s and c_s of the two cases above, having no roots, the linear solution is

$$a_s = -\omega_0^2 c_s$$

and the planar solution is

$$\zeta^\rho(s) = \frac{-a_s - \omega_0^2 c_s}{2b_s}, \qquad \zeta^\rho(s) \neq 0.$$

Setting the two cases equal to zero one is able to solve for the two types of Riemann zeta function zeros directly, which (once fully defined) would either prove or deny the hypothesis. Proof of the hypothesis follows by relating $\zeta^\rho(s)$ to $\zeta(s)$. By setting both $\zeta^\rho(s)$ and $\zeta(s)$ to zero and solving for a general solution proof is obtained if the hypothetical zeros off the critical line are not mathematically possible and $\Re(s) = 1/2,\ \zeta(s) = 0,\ \zeta^\rho(s) = 0$.

Claim. Given an asymmetric Type 2 solution for the zeros of

$$a_s + 2\,\zeta^\rho(s) b_s + {\omega_0}^2 c_s = \zeta(s),$$

when $\zeta(s) = 0$ and $\zeta^\rho(s) = 0$ there exists the unique solution for all the non-trivial zeros of the Riemann zeta function.

$$2\,\zeta^{\rho}(s)b_s = -a_s - {\omega_0}^2 c_s + \zeta(s) = 0 \Leftrightarrow \Re(s) = \frac{1}{2}, \qquad \zeta(s) = 0,$$

where all the real parts of the non-trivial zeros of the Riemann zeta function have a real part equal to one half.

Proof outline.

1) Define $a_s + 2\omega_0\zeta^{\rho 2}(s)b_s + \omega_0^2 c_s$ so that it equals the Riemann zeta function.
2) All the zeros $\zeta(s) = 0$ exist of just two types: the Type 1 linear and the Type 2 planar solutions.
3) The Type 1 linear solution is $\Re(s) = 1/2$.
4) The Type 2 planar solution exists in pairs s_1 and s_2.
5) Given $\Im(s_1) = \Im(s_2)$ (hypothetical zeros symmetric across the real axis would necessarily share a common imaginary part), all the distances of the planar solution pairs are asymmetric across the critical line; $|\Re(s_1) - 1/2| \neq |\Re(s_2) - 1/2|$.
6) Therefore no non-trivial zeros are given by the planar solution.
7) Therefore all the non-trivial zeros have a $\Re(s) = 1/2$.

If all this is understood, then one can prove the Riemann hypothesis in this way.

Proof of the Claim. Let

$$a_s \equiv \frac{1}{s-1},$$

$$b_s \equiv -\frac{1}{(s-1)^2},$$

$$c_s \equiv \frac{1}{(s-1)^3},$$

$$\omega_0 \equiv i\,s^*,$$

where i is the imaginary number, s^* is the complex conjugate of s, and

$$\zeta^{\rho}(s) \equiv \frac{-2\,(s^*)^2 - (s-1)^3}{4\,(s-1)} - i\frac{1}{2}(s-1)^2 \int_0^{\infty} \frac{(1-i\,t)^s - (1+i\,t)^s}{(t^2+1)^s\,(e^{2\pi t}-1)}\,dt\,, \tag{1}$$

which converges for all s except $s = 1$, and is the Abel-Plana continuation of

$$-\frac{(2\,s-1)}{2\,(s-1)s} - \sum_{n=0}^{\infty} (-1)^s \frac{(s-1)^2}{2s}\frac{(n-1)^s}{(1-n^2)^s}.$$

Each of the above are derived in [2] and are not arbitrary. Given the definitions above, multiply $2b_s$ by (1), then add this to a_s plus $\omega_0^2 c_s$ and this gives the first case of the Riemann zeta function

$$a_s + 2\zeta^{\rho}(s)b_s + \omega_0^2 c_s = \zeta(s). \tag{2}$$

(2) is determined by $\zeta^{\rho}(s)$, so define $\zeta^{\rho 2}(s)$ as $\zeta^{\rho}(s)$ divided by ω_0,

$$\zeta^{\rho 2}(s) \equiv i\frac{2\,(s^*)^2 + (s-1)^3}{4\,(s-1)s^*} - \frac{(s-1)^2}{2s^*}\int_0^{\infty} \frac{(1-i\,t)^s - (1+i\,t)^s}{(t^2+1)^s\,(e^{2\pi t}-1)}dt\,, \tag{3}$$

such that

$$\omega_0\,\zeta^{\rho 2}(s) = \zeta^{\rho}(s), \qquad s \neq 0,$$

as (3) is undefined at $s = 0$. This gives the second case determined by $\zeta^{\rho 2}(s)$, which is

$$a_s + 2\omega_0\zeta^{\rho 2}(s)b_s + \omega_0^2 c_s = \zeta(s), \qquad s \neq 0. \tag{4}$$

In this way $\zeta^{\rho}(s)$ and $\zeta^{\rho 2}(s)$ share all the same zeros, and the first and third terms of the first and second cases are equal at the zeros of $\zeta^{\rho}(s)$ and $\zeta^{\rho 2}(s)$. Again, for a more detailed derivation of the above definitions, see [2].

Upon examination of $\zeta^{\rho}(s)$ from (1), one finds it does not share any of the trivial zeros of the Riemann zeta function

$$\zeta^{\rho}(-2n) = -\frac{5}{6}, -\frac{9}{10}, -\frac{13}{14}, -\frac{17}{18}, -\frac{21}{22}, \ldots = \frac{1}{4n+2} - 1 \neq 0, \qquad n \in \mathbb{N},$$

and therefore neither does $\zeta^{\rho 2}(s)$ from (3), as they share the same zeros. Considering first the second case in (4) that $\zeta^{\rho 2}(s)$ determines, solve for $\zeta^{\rho 2}(s)$. One gets

$$-\frac{i\left((s-1)^2\big(1-(s-1)\zeta(s)\big) - (s^*)^2\right)}{2\,(s-1)s^*} = \zeta^{\rho 2}(s). \tag{5}$$

Because $\big(1-(s-1)\zeta(s)\big) = 1, \zeta(s) = 0$ in the numerator, (5) reduces to

$$\zeta^{\rho 2}(s) = -\frac{i\,((s-1)^2 - (s^*)^2)}{2\,(s-1)s^*} \tag{6}$$

for all the zeros (trivial and non-trivial) of the Riemann zeta function. As proposed, (6) consists of two types. The Type 1 linear solution is

$$a_s = -\omega_0^2 c_s, \qquad \zeta^{\rho 2}(s) = 0.$$

Because

$$-\frac{i\,((s-1)^2 - (s^*)^2)}{2\,(s-1)s^*} = 0 \iff \Re(s) = \frac{1}{2},$$

one gets

$$\Re(s) = \frac{1}{2}, \qquad \zeta^{\rho 2}(s) = 0. \tag{7}$$

The Type 2 planar solution is

$$\zeta^{\rho 2}(s) = \frac{-a_s - \omega_0^2 c_s}{2\,\omega_0 b_s}, \qquad \zeta^{\rho 2}(s) \neq 0,$$

whereby solving for the real part of s gives

$$\Re(s) = \frac{\pm\sqrt{-(2\,\Im(s)\ +\ i)^2(\zeta^{\rho 2}(s)^2 -\ 1)} +\ 2\,\Im(s)\ +\ \zeta^{\rho 2}(s)\ +\ i}{2\,\zeta^{\rho 2}(s)}\ , \qquad (8)$$

$$\zeta^{\rho 2}(s) \neq 0,$$

such that (8) provides a pair of $\Re(s)'s$ across the critical line from each other.

Upon examination of the square root, because

$$-(2\,\Im(s) +\ i)^2(\zeta^{\rho 2}(s)^2 -\ 1) = (\zeta^{\rho 2}(s)^2 -\ 1)(s^* -\ s\ +\ 1)^2,$$

one can also express (8) as

$$\Re(s) = \frac{\pm\sqrt{(\zeta^{\rho 2}(s)^2 -\ 1)(s^* -\ s\ +\ 1)^2} +\ 2\,\Im(s)\ +\ \zeta^{\rho 2}(s)\ +\ i}{2\,\zeta^{\rho 2}(s)},$$

which also provides a pair of $\zeta^{\rho 2}(s)$'s

$$\zeta^{\rho 2}(s) = \frac{\pm\sqrt{(2\,\mathrm{Im}(s) +\ i)^2}\ |1\ -\ 2\,\mathrm{Re}(s)| +\ (s^* +\ s\ -\ 1)(2\,\mathrm{Im}(s) +\ i)}{4\,(|(s)^2| -\ s^*)} \qquad (9)$$

that correspond to a total of four possible hypothetical zeros (two sets of pairs) across the real and critical lines from each other, as were graphically defined at the beginning of this paper.

Now one can ask, *given any hypothetical non-trivial zero s_1 off the critical line, is it possible for any s_2 to be symmetric to s_1 across the critical line? That is; for any $\Re(s_1) \neq 1/2$, is it possible given the two solutions in (8) to have a $\Re(s_2) \neq 1/2$ equidistant from the critical line?* If it is mathematically impossible, then the Riemann hypothesis is necessarily true, as the Type 1 zeros are the only other possibility, given by (7). Now one can prove the Riemann hypothesis. Because

$$r = \sqrt{(v-x)^2 + (w-y)^2}$$

gives the distance between any two points $v + i\,w$ and $x\ +\ i\,y$ on the complex plane, the distance r_{cr} from any point s to the nearest point $1/2 + i\,\Im(\mathrm{s})$ on the critical line is given by

$$r_{cr} = \left|\Re(s) - \frac{1}{2}\right|.$$

One can verify if symmetry between the positive and negative solutions of (8) is possible or impossible. Setting the two distances equal to each other

$$\left|\Re(s_1) - \frac{1}{2}\right| = \left|\Re(s_2) - \frac{1}{2}\right|,$$

where s_1 is either the positive or negative solution to (8) and s_2 is either the positive or negative as well, one gets

$$\left|\frac{\pm\sqrt{-(2\,\Im(s_1)\,+\,i)^2(\zeta^{\rho 2}(s_1)^2-\,1)}+\,2\,\Im(s_1)\,+\,\zeta^{\rho 2}(s_1)\,+\,i}{2\,\zeta^{\rho 2}(s_1)}-\frac{1}{2}\right|$$
$$=\left|\frac{\pm\sqrt{-(2\,\Im(s_1)\,+\,i)^2(\zeta^{\rho 2}(s_2)^2-\,1)}+\,2\,\Im(s_1)\,+\,\zeta^{\rho 2}(s_2)\,+\,i}{2\,\zeta^{\rho 2}(s_2)}-\frac{1}{2}\right|, \tag{10}$$
$$s_1 = s_2 \quad \vee \quad \zeta^{\rho 2}(s_1) = \pm 1 \quad \vee \quad \zeta^{\rho 2}(s_2) = \pm 1$$

The value inside the square root reduces to zero for $\zeta^{\rho 2}(s) = \pm 1$, but for $\zeta^{\rho 2}(s) = \pm 1$ no solutions exist for (6). These solutions are also outside the critical strip. The only other possible solution $s_1 = s_2$ in (10) is meaningless, as the only argument that would make sense would be the critical zeros. However, $\zeta^{\rho 2}(s) = 0$ for $\Re(s) = 1/2$ leaves (10) undefined. Therefore the only solutions to (10) are extraneous. Thus,

$$\left|\Re(s_1) - \frac{1}{2}\right| \neq \left|\Re(s_2) - \frac{1}{2}\right|, \qquad \zeta^{\rho 2}(s) \neq 0.$$

No non-trivial zeros exist off the critical line at the vertices of rectangles symmetric across both the real and critical line. Their existence is mathematically impossible. The negative solution to (8) is therefore also extraneous and the positive solution only applies to the trivial zeros. This should be obvious upon examination of the above.

Because $\omega_0\,\zeta^{\rho 2}(s) = \zeta^{\rho}(s), s \neq 0$, where the two zeta functions share all the same zeros, all of the above holds true for $\zeta^{\rho}(s)$ as well, as the two oscillator representations of the Riemann zeta function are identical when $\zeta^{\rho 2}(s) = 0, \zeta^{\rho}(s) = 0$. Therefore, letting also $\zeta^{\rho}(s) = 0, \zeta(s) = 0$ one gets the general solution for the real part of all the non-trivial zeros of the Riemann zeta function, which proves the claim of this paper. Given an asymmetric Type 2 solution for the zeros of

$$a_s + 2\,\zeta^{\rho}(s) b_s + {\omega_0}^2 c_s = \zeta(s),$$

when $\zeta(s) = 0$ and $\zeta^{\rho}(s) = 0$ there exists the unique solution for all the non-trivial zeros of the Riemann zeta function.

$$2\,\zeta^{\rho}(s) b_s = -a_s\,-\,{\omega_0}^2 c_s +\,\zeta(s)\;=0 \quad \Leftrightarrow \quad \Re(s) = \frac{1}{2}, \qquad \zeta(s) = 0,$$

where all the real parts of the non-trivial zeros of the Riemann zeta function have a real part equal to one half.

No non-trivial zeros could exist in the critical strip except those on the critical line, those provided by the Type 1 line solution, as the Type 2 solutions are asymmetric across the critical line from each other. There are no other types of solutions for the Riemann zeta function. The Riemann hypothesis is therefore correct. ■

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