Sums of Powers of Fibonacci and Lucas Numbers and their Related Integer Sequences*

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Abstract

In this paper we will look at sums of odd powers of Fibonacci and Lucas numbers of even indices. Our motivation will be conjectures, now theorems, which go back to Melham. Using the simple approach of telescoping sums we will be able to give new proofs of those results. Along the way we will establish inverse relationships for such sums and discover new integer sequences.

Contents

1 Introduction

We define the Fibonacci numbers $(F_n)_{n\geq 1}$ by the initial values of

$$F_1 = F_2 = 1,$$

and, for $n \geq 2$, a general term of

$$F_{n-1} + F_n = F_{n+1}.$$

We define the Lucas numbers $(L_n)_{n\geq 1}$ identically as the Fibonacci numbers,

$$L_{n-1} + L_n = L_{n+1},$$

where $n \geq 2$, but with the different initial values of

$$L_1 = 1,\ L_2 = 3.$$

Ozeki [7] and Prodinger [8] might be the first places where we find explicit expressions for the sums of odd powers of Fibonacci and Lucas numbers of even indices. The result for Fibonacci numbers is

Theorem 1. *(Ozeki) for $m \geq 0$ and $n \geq 1$,*

$$\begin{aligned}\sum_{k=1}^{n} F_{2k}^{2m+1} \quad &= \quad \sum_{i=0}^{m} F_{2n+1}^{2i+1} \sum_{j=0}^{m-i} (-1)^{m+i} \frac{5^{i-m}}{L_{2m+1-2j}} \cdot \frac{2m-2j+1}{m-j+i+1} \binom{2m+1}{j} \binom{m-j+i+1}{2i+1} \\ &\quad + \sum_{j=0}^{m} (-1)^{j+1} 5^{-m} \cdot \frac{F_{2m+1-2j}}{L_{2m+1-2j}} \binom{2m+1}{j}.\end{aligned}$$

The result for Lucas numbers is

Theorem 2. *(Prodinger) for $m \geq 0$ and $n \geq 0$,*

$$\sum_{k=0}^{n} L_{2k}^{2m+1} = \sum_{l=0}^{m} L_{2n+1}^{2l+1} \sum_{j=0}^{m-l} \binom{2m+1}{j} \binom{m-j+l}{m-j-l} \frac{2m+1-2j}{2l+1} \cdot \frac{1}{L_{2m+1-2j}} - 4^m.$$

(We have stated the results in their original notation. We changed the "$+4$" to "-4." The original seems to contain a mistake. In this paper, for the sake of uniformity we will adopt the notation of Ozeki [7].)

Two conjectures in Melham [5] were the inspiration for these results. (The results are a bit removed from the original conjectures. That seems to be because, at the time, the authors were unaware of the exact statements of the conjectures. Melham [5] contains more information on this.) Since the conjectures have been proved, we state them as theorems. For the Fibonacci numbers we have

Theorem 3. *for $m \geq 0$ and $n \geq 1$,*

$$L_1 L_3 \cdots L_{2m-1} L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1} = (F_{2n+1} - 1)^2 \cdot P_{2k,2m+1},$$

where $P_{2k,2m+1}$ is a polynomial in F_{2n+1}, of degree $2m-1$, with integer coefficients.

For the Lucas numbers we have

Theorem 4. *for $m \geq 0$ and $n \geq 1$,*

$$L_1 L_3 \cdots L_{2m-1} L_{2m+1} \sum_{k=1}^{n} L_{2k}^{2m+1} = (L_{2n+1} - 1) \cdot Q_{2k,2m+1},$$

where $Q_{2k,2m+1}$ is a polynomial in L_{2n+1}, of degree $2m$, with integer coefficients.

Unlike Theorems 1 and 2, the result for Fibonacci numbers is considerably more difficult to obtain.

(Note: we have changed the notation slightly. There is little reason to keep track of the degrees of the polynomials. By Theorems 1 and 2 we know that $\sum_{k=1}^{n} F_{2k}^{2m+1}$ and $\sum_{k=1}^{n} L_{2k}^{2m+1}$ are polynomials in F_{2n+1} and L_{2n+1}, of degree $2m+1$. If we factor out a term of $(F_{2n+1}-1)^2$ or $L_{2n+1}-1$, the degree drops by 2 or 1.)

Sun, Xie, and Yang [9] seems to contain the first complete proof of Theorem 3. A byproduct of their work was another proof of Theorem 4....

...using the simple method of telescoping sums we will show that the original approach sketched in Melham [5] is sufficient to give a complete proof of Theorem 4. Expanding upon that approach to address the case of Fibonacci numbers, we will prove the inverse of Theorem 1:

Theorem 5. *for $m \geq 0$ and $n \geq 1$,*

$$F_{2n+1}^{2m+1} = 1 + \sum_{i=0}^{m} \left(\sum_{k=1}^{n} F_{2k}^{2i+1} \right) \sum_{j=i}^{m} \left[\binom{2m+1}{m-j} L_{2j+1} \cdot \frac{1}{5^{m-i}} \cdot \frac{2j+1}{j+i+1} \binom{j+i+1}{2i+1} \right].$$

Even though it is unnecessary for a proof of Theorem 4, we will do the same for Theorem 2:

Theorem 6. *for $m \geq 0$ and $n \geq 1$,*

$$L_{2n+1}^{2m+1} = 1 + \sum_{i=0}^{m} \left(\sum_{k=1}^{n} L_{2k}^{2i+1} \right) \sum_{j=i}^{m} \left[(-1)^{m+i} \binom{2m+1}{m-j} L_{2j+1} \frac{2j+1}{j+i+1} \binom{j+i+1}{2i+1} \right].$$

The result for Lucas numbers will give us a new integer sequence:

$$1, 1, 1, 1, -15, 4, 1, 125, -75, 11, \ldots. \qquad (1)$$

A full proof of Theorem 3, in line with our approach, requires establishing the following conjecture:

Conjecture 1. *in Theorem 5, the coefficients for* $\sum_{k=1}^{n} F_{2k}^{2i+1}$ *are integers only. In other words, for* $0 \leq i \leq m$, 5^{m-i} *divides*

$$\sum_{j=i}^{m} \left[\binom{2m+1}{m-j} L_{2j+1} \frac{2j+1}{j+i+1} \binom{j+i+1}{2i+1} \right].$$

At present we are unable to do that. We *are* able to establish a special case, which actually is a sharper result:

Lemma 1. *in Theorem 5, the coefficient for* $\sum_{k=1}^{n} F_{2k}$,

$$\frac{1}{5^m} \sum_{j=0}^{m} \left[\binom{2m+1}{m-j} L_{2j+1} \left(2j+1\right) \right],$$

is equal to $2m+1$.

Looking at the statement of Theorem 3, the difference between the special case and the general case of the conjecture is that the former gets us everything except for the coefficients being integers. We must settle for rational numbers. If we add the latter case, we get that last piece and discover a second integer sequence as well:

$$1, 1, 1, 1, 3, 4, 1, 5, 15, 11, \ldots. \tag{2}$$

Last, with regard to the overall presentation of the paper, we will place more emphasis on the inductive aspects of the material than is customary. This will provide motivation for the proofs and highlight how intermediate results arise from attempts to give new solutions to the original conjectures of Melham [5].

2 Lucas Numbers

We begin with the case for Lucas numbers. One reason is the simple approach sketched in Melham [5] will be sufficient to prove Theorem 4 completely. Another reason is the discussion will prepare us for the more difficult case of the Fibonacci numbers.

2.1 Telescoping Sums

Our starting point is the following observation:

$$\begin{aligned}
1 &= 1 \\
L_{2n+1} &= 1 + \binom{1}{0} L_1 \sum_{k=1}^{n} L_{2k} \\
L_{2n+1}^3 &= 1 - \binom{3}{1} L_1 \sum_{k=1}^{n} L_{2k} + \binom{3}{0} L_3 \sum_{k=1}^{n} L_{6k} \\
L_{2n+1}^5 &= 1 + \binom{5}{2} L_1 \sum_{k=1}^{n} L_{2k} - \binom{5}{1} L_3 \sum_{k=1}^{n} L_{6k} + \binom{5}{0} L_5 \sum_{k=1}^{n} L_{10k} \\
L_{2n+1}^7 &= 1 - \binom{7}{3} L_1 \sum_{k=1}^{n} L_{2k} + \binom{7}{2} L_3 \sum_{k=1}^{n} L_{6k} - \binom{7}{1} L_5 \sum_{k=1}^{n} L_{10k} + \binom{7}{0} L_7 \sum_{k=1}^{n} L_{14k}.
\end{aligned} \tag{3}$$

This is for $n \geq 1$. If we did not know it already,

$$L_1 \sum_{k=1}^{n} L_{2k} = L_{2n+1} - 1. \tag{4}$$

The significance of these relationships is they allow us to show $L_{2n+1} - 1$ divides

$$L_1 \sum_{k=1}^{n} L_{2k},\ L_3 \sum_{k=1}^{n} L_{6k},\ L_5 \sum_{k=1}^{n} L_{10k}, \ldots. \tag{5}$$

For example, we can write

$$\begin{aligned}
L_3 \sum_{k=1}^{n} L_{6k} &= L_{2n+1}^3 - 1 + \binom{3}{1} L_1 \sum_{k=1}^{n} L_{2k} \\
&= (L_{2n+1} - 1)\left(L_{2n+1}^2 + L_{2n+1} + 1\right) + (L_{2n+1} - 1)(3) \\
&= (L_{2n+1} - 1)\left(L_{2n+1}^2 + L_{2n+1} + 4\right).
\end{aligned}$$

Before we prove the result of (5), we must establish the relationships in (3). This we will do using telescoping sums.

For the first step, we prove the following lemma:

Lemma 2. *for $m \geq 0$ and $n \geq 1$,*

$$L_{2n+1}^{2m+1} - 1 = \sum_{k=1}^{n} \left(L_{2k+1}^{2m+1} - L_{2k-1}^{2m+1} \right). \tag{6}$$

Proof. we proceed by mathematical induction. For the base case of $n = 1$ we have

$$\sum_{k=1}^{1} \left(L_{2k+1}^{2m+1} - L_{2k-1}^{2m+1} \right) = L_{2(1)+1}^{2m+1} - L_{1}^{2m+1} = L_{3}^{2m+1} - 1. \tag{7}$$

Assume that (7) is true for some $n \geq 1$. Then

$$\begin{aligned}\sum_{k=1}^{n+1} \left(L_{2k+1}^{2m+1} - L_{2k-1}^{2m+1} \right) &= \sum_{k=1}^{n} \left(L_{2k+1}^{2m+1} - L_{2k-1}^{2m+1} \right) + L_{2(n+1)+1}^{2m+1} - L_{2n+1}^{2m+1} \\ &= L_{2n+1}^{2m+1} - 1 + L_{2(n+1)+1}^{2m+1} - L_{2n+1}^{2m+1} \\ &= L_{2(n+1)+1}^{2m+1} - 1.\end{aligned}$$

□

For the next step, we rewrite the expression inside the parentheses of (6). In order to do that we will use the Binet forms:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \; L_n = \alpha^n + \beta^n, \tag{8}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Other relationships include

$$\alpha + \beta = 1, \; \alpha - \beta = \sqrt{5}, \; \alpha\beta = -1, \; 1 + \alpha = \alpha^2, \; 1 + \beta = \beta^2. \tag{9}$$

We will use the Binet forms to establish a number of results.

Lemma 3. *for $m \geq 0$ and $n \geq 1$,*

$$\sum_{k=1}^{n} \left(L_{2k+1}^{2m+1} - L_{2k-1}^{2m+1} \right) = \sum_{j=0}^{m} \left[(-1)^{m-j} \binom{2m+1}{m-j} L_{2j+1} \sum_{k=1}^{n} L_{2(2j+1)k} \right].$$

Proof. the calculation is cumbersome. We illustrate the case of $2m + 1 = 5$. The expression in parentheses we rewrite using the Binet forms:

$$\begin{aligned}L_{2k+1}^{5} - L_{2k-1}^{5} &= \left(\alpha^{2k+1} + \beta^{2k+1}\right)^5 - \left(\alpha^{2k-1} + \beta^{2k-1}\right)^5 \\ &= \binom{5}{0} \left(\alpha^{10k+5} - \alpha^{10k-5}\right) + \binom{5}{1} \left(\alpha^{8k+4}\beta^{2k+1} - \alpha^{8k-4}\beta^{2k-1}\right) \\ &+ \binom{5}{2} \left(\alpha^{6k+3}\beta^{4k+2} - \alpha^{6k-3}\beta^{4k-2}\right) + \binom{5}{3} \left(\alpha^{4k+2}\beta^{6k+3} - \alpha^{4k-2}\beta^{6k-3}\right) \\ &+ \binom{5}{4} \left(\alpha^{2k+1}\beta^{8k+4} - \alpha^{2k-1}\beta^{8k-4}\right) + \binom{5}{5} \left(\beta^{10k+5} - \beta^{10k-5}\right).\end{aligned}$$

We rewrite the first three terms as follows:

$$\binom{5}{0}\left(\alpha^{10k+5}-\alpha^{10k-5}\right) = \binom{5}{0}\alpha^{10k}\left(\alpha^{5}-\alpha^{-5}\right) = \binom{5}{0}\alpha^{10k}\left(\alpha^{5}-\frac{1}{\alpha^{5}}\right)$$
$$= \binom{5}{0}\alpha^{10k}\left(\alpha^{5}-\frac{1}{\left(\frac{-1}{\beta}\right)^{5}}\right) = \binom{5}{0}\alpha^{10k}\left(\alpha^{5}+\beta^{5}\right);$$

$$\binom{5}{1}\left(\alpha^{8k+4}\beta^{2k+1}-\alpha^{8k-4}\beta^{2k-1}\right) = \binom{5}{1}\alpha^{8k}\beta^{2k}\left(\alpha^{4}\beta^{1}-\alpha^{-4}\beta^{-1}\right)$$
$$= \binom{5}{1}\alpha^{6k}\left(\alpha\beta\right)^{2k}\left(\alpha^{3}\left(\alpha\beta\right)-\frac{1}{\alpha^{3}\left(\alpha\beta\right)}\right)$$
$$= \binom{5}{1}\alpha^{6k}\left(-\alpha^{3}+\frac{1}{\left(\frac{-1}{\beta}\right)^{3}}\right) = -\binom{5}{1}\alpha^{6k}\left(\alpha^{3}+\beta^{3}\right);$$

$$\binom{5}{2}\left(\alpha^{6k+3}\beta^{4k+2}-\alpha^{6k-3}\beta^{4k-2}\right) = \binom{5}{2}\alpha^{6k}\beta^{4k}\left(\alpha^{3}\beta^{2}-\alpha^{-3}\beta^{-2}\right)$$
$$= \binom{5}{2}\alpha^{2k}\left(\alpha\beta\right)^{4k}\left(\alpha\left(\alpha\beta\right)^{2}-\frac{1}{\alpha\left(\alpha\beta\right)^{2}}\right)$$
$$= \binom{5}{2}\alpha^{2k}\left(\alpha-\frac{1}{\frac{-1}{\beta}}\right) = \binom{5}{2}\alpha^{2k}\left(\alpha+\beta\right).$$

The other three terms we can rewrite in an analogous fashion. In total we have

$$L_{2k+1}^{5}-L_{2k-1}^{5} = \binom{5}{0}\left(\alpha^{10k}+\beta^{10k}\right)\left(\alpha^{5}+\beta^{5}\right)-\binom{5}{1}\left(\alpha^{6k}+\beta^{6k}\right)\left(\alpha^{3}+\beta^{3}\right)$$
$$+\binom{5}{2}\left(\alpha^{2k}+\beta^{2k}\right)\left(\alpha+\beta\right),$$

which we rewrite as

$$L_{2k+1}^{5}-L_{2k-1}^{5}=\binom{5}{0}L_{5}L_{10k}-\binom{5}{1}L_{3}L_{6k}+\binom{5}{2}L_{1}L_{2k}.$$

If we change the order of the terms and sum from 1 to n, we get the desired result:

$$\sum_{k=1}^{n}\left(L_{2k+1}^{5}-L_{2k-1}^{5}\right) = \sum_{j=0}^{2}\left[(-1)^{2-j}\binom{5}{2-j}L_{2j+1}\sum_{k=1}^{n}L_{2(2j+1)k}\right].$$

□

Putting together Lemmas 2 and 3, we get the main result of (3):

Proposition 1. *for* $m \geq 0$ *and* $n \geq 1$*,*

$$L_{2n+1}^{2m+1} = 1 + \sum_{j=0}^{m} \left[(-1)^{m-j} \binom{2m+1}{m-j} L_{2j+1} \sum_{k=1}^{n} L_{2(2j+1)k} \right].$$

Now we are ready to prove Proposition 2, which we mentioned in (5). In order to do that, we clarify some terminology:

Definition 1. "$L_{2n+1} - 1$ divides $\sum_{k=1}^{n} L_{2(2m+1)k}$" means

$$\sum_{k=1}^{n} L_{2(2m+1)k} = (L_{2n+1} - 1) \cdot Q_{2(2m+1)k},$$

where $Q_{2(2m+1)k}$ is a polynomial in L_{2n+1}; that is,

$$Q_{2(2m+1)k} = r_s L_{2n+1}^{s} + r_{s-1} L_{2n+1}^{s-1} + \ldots + r_1 L_{2n+1} + r_0,$$

where the r_i are rational numbers, at least one of which is not equal to zero, and the s are non-negative integers. Notice that we allow for the possibility of

$$\sum_{k=1}^{n} L_{2k} = (L_{2n+1} - 1) \cdot 1.$$

Also, if we write something like

$$\begin{aligned} L_{2n+1}^{2m+1} - 1 &= (L_{2n+1} - 1) \left(L_{2n+1}^{2m} + L_{2n+1}^{2m-1} + \cdots + L_{2n+1} + 1 \right) \\ &= (L_{2n+1} - 1) \cdot Q_{2n+1,2m+1}, \end{aligned}$$

we mean "$L_{2n+1} - 1$ divides $L_{2n+1}^{2m+1} - 1$." Last, when no confusion will arise, we will abbreviate $Q_{2n+1,2m+1}$ by Q_{2n+1}.

Now we state and prove Proposition 2:

Proposition 2. *for* $m \geq 0$ *and* $n \geq 1$*,*

$$L_{2m+1} \sum_{k=1}^{n} L_{2(2m+1)k} = (L_{2n+1} - 1) \cdot Q_{2(2m+1)k},$$

where $Q_{2(2m+1)k}$ *is a polynomial in* L_{2n+1}*, with integer coefficients.*

Proof. we proceed by mathematical induction. Previously we showed that $L_{2n+1} - 1$ divides $L_1 \sum_{k=1}^{n} L_{2k}$ and $L_3 \sum_{k=1}^{n} L_{6k}$, and that the adjoining polynomials have integer coefficients. Assume that $L_{2n+1} - 1$ divides

$$L_1 \sum_{k=1}^{n} L_{2k},\ L_3 \sum_{k=1}^{n} L_{6k}, \ldots,\ L_{2m+1} \sum_{k=1}^{n} L_{2(2m+1)k}$$

for all $1, 3, \ldots, 2m+1$, and that the adjoining polynomials have integer coefficients. By Proposition 1 we have

$$L_{2n+1}^{2m+3} = 1 + \sum_{j=0}^{m+1} \left[(-1)^{m+1-j} \binom{2m+3}{m+1-j} L_{2j+1} \sum_{k=1}^{n} L_{2(2j+1)k} \right].$$

We rewrite it as follows:

$$\begin{aligned} L_{2m+3} \sum_{k=1}^{n} L_{2(2m+3)k} &= L_{2n+1}^{2m+3} - 1 - \sum_{j=0}^{m} \left[(-1)^{m+1-j} \binom{2m+3}{m+1-j} L_{2j+1} \sum_{k=1}^{n} L_{2(2j+1)k} \right] \\ &= (L_{2n+1} - 1) \left(L_{2n+1}^{2m+2} + L_{2n+1}^{2m+1} + \cdots + L_{2n+1} + 1 \right) \\ &\quad - \sum_{j=0}^{m} \left[(-1)^{m+1-j} \binom{2m+3}{m+1-j} L_{2j+1} \left(L_{2n+1} - 1 \right) Q_{2(2j+1)k} \right] \\ &= (L_{2n+1} - 1) \\ &\quad \times \left\{ Q_{2n+1} - \sum_{j=0}^{m} \left[(-1)^{m+1-j} \binom{2m+3}{m+1-j} L_{2j+1} \cdot Q_{2(2j+1)k} \right] \right\}. \end{aligned}$$

On the right-hand side we have $L_{2n+1} - 1$ times a sum of polynomials in L_{2n+1}. That means $L_{2n+1} - 1$ divides $L_{2m+3} \sum_{k=1}^{n} L_{2(2m+3)k}$. Also, the final polynomial has integer coefficients. □

2.2 Proof of Theorem 4

Now we are ready to prove Theorem 4. We follow the approach sketched in Melham [5]. In order to set up our previous work we need the following result:

Proposition 3. *for $m \geq 0$ and $n \geq 1$,*

$$L_n^{2m+1} = \sum_{j=0}^{m} (-1)^{jn} \binom{2m+1}{j} L_{(2m+1-2j)n}.$$

Proof. we use the Binet forms:

$$\begin{aligned} L_n^{2m+1} &= (\alpha^n + \beta^n)^{2m+1} = \sum_{j=0}^{2m+1} \binom{2m+1}{j} \alpha^{(2m+1-j)n} \beta^{jn} \\ &= \sum_{j=0}^{m} \binom{2m+1}{j} \left(\alpha^{(2m+1-j)n} \beta^{jn} + \alpha^{jn} \beta^{(2m+1-j)n} \right) \\ &= \sum_{j=0}^{m} \binom{2m+1}{j} (\alpha\beta)^{jn} \left(\alpha^{(2m+1-2j)n} + \beta^{(2m+1-2j)n} \right) \\ &= \sum_{j=0}^{m} (-1)^{jn} \binom{2m+1}{j} L_{(2m+1-2j)n}. \end{aligned}$$

□

We begin the proof of Theorem 4:

Proof. in Proposition 3 we replace the n by $2k$ and sum from 1 to n:

$$\begin{aligned}\sum_{k=1}^{n} L_{2k}^{2m+1} &= \binom{2m+1}{0}\sum_{k=1}^{n} L_{2(2m+1)k} + \binom{2m+1}{1}\sum_{k=1}^{n} L_{2(2m-1)k} \quad (10)\\ &+\cdots+\binom{2m+1}{m-1}\sum_{k=1}^{n} L_{6k} + \binom{2m+1}{m}\sum_{k=1}^{n} L_{2k}\\ &= \sum_{j=0}^{m}\left[\binom{2m+1}{j}\sum_{k=1}^{n} L_{2(2m+1-2j)k}\right].\end{aligned}$$

Previously, in Proposition 2 we established that, for $m \geq 0$, $L_{2n+1}-1$ divides $L_{2m+1}\sum_{k=1}^{n} L_{2(2m+1)k}$ and the adjoining polynomial has integer coefficients. That allows us to rewrite (10) as

$$\begin{aligned}L_1L_3\cdots L_{2m-1}L_{2m+1}\sum_{k=1}^{n} L_{2k}^{2m+1} &= L_1L_3\cdots L_{2m-1}L_{2m+1}\sum_{j=0}^{m}\left[\binom{2m+1}{j}\sum_{k=1}^{n} L_{2(2m+1-2j)k}\right]\\ &= L_1L_3\cdots L_{2m-1}\binom{2m+1}{0}L_{2m+1}\sum_{k=1}^{n} L_{2(2m+1)k}\\ &+\cdots+L_3\cdots L_{2m-1}L_{2m+1}\binom{2m+1}{m}L_1\sum_{k=1}^{n} L_{2k}\\ &= L_1L_3\cdots L_{2m-1}\binom{2m+1}{0}(L_{2n+1}-1)\cdot Q_{2(2m+1)k}\\ &+\cdots+L_3\cdots L_{2m-1}L_{2m+1}\binom{2m+1}{m}(L_{2n+1}-1)\cdot Q_{2k}\\ &= (L_{2n+1}-1)\cdot Q_{2k,2m+1},\end{aligned}$$

where $Q_{2(2m+1)k},\ldots,Q_{2k}$ and $Q_{2k,2m+1}$ are polynomials in L_{2n+1}, with integer coefficients. □

We have proved Theorem 4, the original conjecture of Melham [5] for Lucas numbers. For convenience, we state it again:

Theorem. *for $m \geq 0$ and $n \geq 1$,*

$$L_1L_3\cdots L_{2m-1}L_{2m+1}\sum_{k=1}^{n} L_{2k}^{2m+1} = (L_{2n+1}-1)\cdot Q_{2k,2m+1},$$

where $Q_{2k,2m+1}$ is a polynomial in L_{2n+1}, of degree $2m$, with integer coefficients.

3 Fibonacci Numbers - First Attempt

Now we turn to Fibonacci numbers. We mention from the start that the approach of Melham [5], which we just applied to Lucas numbers, will be insufficient to give a complete proof of Theorem 3. There are two reasons for this:

1. the approach tells us $F_{2n+1}-1$ divides such sums, not $(F_{2n+1}-1)^2$;
2. the appearance of a divisor of 5^m makes it difficult to determine whether or not the adjoining polynomials have integer coefficients.

Nevertheless, we will state the analogous results for Fibonacci numbers because they will serve as the foundation for a new approach in the next section. (Note: we will state the results without proofs. The proofs are analogous to those given for Lucas numbers.)

Our starting point is

Proposition 4. *for $m \geq 0$ and $n \geq 1$,*

$$F_{2n+1}^{2m+1} = 1 + \frac{1}{5^m}\sum_{j=0}^{m}\left[\binom{2m+1}{m-j}L_{2j+1}\sum_{k=1}^{n}F_{2(2j+1)k}\right].$$

Some special cases include

$$\begin{aligned}
1 &= 1 \\
F_{2n+1} &= 1 + \binom{1}{0}L_1\sum_{k=1}^{n}F_{2k} \\
F_{2n+1}^3 &= 1 + \frac{1}{5}\binom{3}{1}L_1\sum_{k=1}^{n}F_{2k} + \frac{1}{5}\binom{3}{0}L_3\sum_{k=1}^{n}F_{6k} \qquad (11)\\
F_{2n+1}^5 &= 1 + \frac{1}{5^2}\binom{5}{2}L_1\sum_{k=1}^{n}F_{2k} + \frac{1}{5^2}\binom{5}{1}L_3\sum_{k=1}^{n}F_{6k} + \frac{1}{5^2}\binom{5}{0}L_5\sum_{k=1}^{n}F_{10k} \\
F_{2n+1}^7 &= 1 + \frac{1}{5^3}\binom{7}{3}L_1\sum_{k=1}^{n}F_{2k} + \frac{1}{5^3}\binom{7}{2}L_3\sum_{k=1}^{n}F_{6k} + \frac{1}{5^3}\binom{7}{1}L_5\sum_{k=1}^{n}F_{10k} + \frac{1}{5^3}\binom{7}{0}L_7\sum_{k=1}^{n}F_{14k}.
\end{aligned}$$

(Strictly speaking, $1 = 1$ is not a special case. Like before, we include it merely for the purpose of symmetry.) Once again,

$$L_1\sum_{k=1}^{n}F_{2k} = F_{2n+1} - 1. \qquad (12)$$

We establish Proposition 4 with two lemmas. The first is

Lemma 4. *for $m \geq 0$ and $n \geq 1$,*

$$F_{2n+1}^{2m+1} - 1 = \sum_{k=1}^{n}\left(F_{2k+1}^{2m+1} - F_{2k-1}^{2m+1}\right).$$

The second is

Lemma 5. *for $m \geq 0$ and $n \geq 1$,*

$$\sum_{k=1}^{n}\left(F_{2k+1}^{2m+1}-F_{2k-1}^{2m+1}\right)=\frac{1}{5^m}\sum_{j=0}^{m}\left[\binom{2m+1}{m-j}L_{2j+1}\sum_{k=1}^{n}F_{2(2j+1)k}\right].$$

With Proposition 4 we are able to prove the following result:

Proposition 5. *for $m \geq 0$ and $n \geq 1$,*

$$L_{2m+1}\sum_{k=1}^{n}F_{2(2m+1)k}=(F_{2n+1}-1)\cdot P_{2(2m+1)k},$$

where $P_{2(2m+1)k}$ is a polynomial in F_{2n+1}, with integer coefficients.

Unfortunately, this is as strong as we are able to make the result. For example, by (11) we have

$$F_{2n+1}^3=1+\frac{1}{5}\cdot\binom{3}{1}L_1\sum_{k=1}^{n}F_{2k}+\frac{1}{5}\cdot\binom{3}{0}L_3\sum_{k=1}^{n}F_{6k},$$

which we can rewrite as

$$L_3\sum_{k=1}^{n}F_{6k}=(F_{2n+1}-1)\left(5F_{2n+1}^2+5F_{2n+1}+2\right).$$

For the special case of $n=4$ we have

$$\begin{aligned}L_3\sum_{k=1}^{4}F_{6k} &= 4\,(8+144+2584+46368)=196416\\ &=(34-1)\left(5\cdot 34^2+5\cdot 34+2\right)\\ &=(33)\,(5952)\,,\end{aligned}$$

where $F_9=34$. $(34-1)$ does not divide 5942 another time.

Now we turn to the approach of Melham [5]. Once again we need a starting point:

Proposition 6. *for $m \geq 0$ and $n \geq 1$,*

$$F_n^{2m+1}=\frac{1}{5^m}\sum_{j=0}^{m}(-1)^{j(n+1)}\binom{2m+1}{j}F_{(2m+1-2j)n}.$$

Like last time, we replace n by $2k$ and sum from 1 to k:

$$\begin{aligned}\sum_{k=1}^{n} F_{2k}^{2m+1} &= \frac{1}{5^m}\binom{2m+1}{0}\sum_{k=1}^{n} F_{2(2m+1)k} - \frac{1}{5^m}\binom{2m+1}{1}\sum_{k=1}^{n} F_{2(2m-1)k} \\ &\pm \cdots \pm \frac{1}{5^m}\binom{2m+1}{m-1}\sum_{k=1}^{n} F_{6k} \pm \frac{1}{5^m}\binom{2m+1}{m}\sum_{k=1}^{n} F_{2k} \\ &= \frac{1}{5^m}\sum_{j=0}^{m}\left[(-1)^j\binom{2m+1}{j}\sum_{k=1}^{n} F_{2(2m+1-2j)k}\right].\end{aligned}$$

(We changed $(-1)^{j(2k+1)}$ to $(-1)^j$.) Then we rewrite the expression:

$$\begin{aligned}L_1L_3\cdots L_{2m-1}L_{2m+1}\sum_{k=1}^{n} F_{2k}^{2m+1} &= L_1L_3\cdots L_{2m-1}L_{2m+1} \\ &\times\frac{1}{5^m}\sum_{j=0}^{m}\left[(-1)^j\binom{2m+1}{j}\sum_{k=1}^{n} F_{2(2m+1-2j)k}\right] \\ &= L_1L_3\cdots L_{2m-1}\cdot\frac{1}{5^m}\binom{2m+1}{0}L_{2m+1}\sum_{k=1}^{n} F_{2(2m+1)k} \\ &\pm\cdots\pm L_3\cdots L_{2m-1}L_{2m+1}\cdot\frac{1}{5^m}\binom{2m+1}{m}L_1\sum_{k=1}^{n} F_{2k} \\ &= L_1L_3\cdots L_{2m-1}\cdot\frac{1}{5^m}\binom{2m+1}{0}(F_{2n+1}-1)\cdot P_{2(2m+1)k} \\ &\pm\cdots\pm L_3\cdots L_{2m-1}L_{2m+1}\cdot\frac{1}{5^m}\binom{2m+1}{m}(F_{2n+1}-1)\cdot P_{2k} \\ &= \frac{1}{5^m}(F_{2n+1}-1)\,P_{2k,2m+1},\end{aligned}$$

where $P_{2(2m+1)k},\ldots,P_{2k}$ and $P_{2k,2m+1}$ are polynomials in F_{2n+1}, with integer coefficients. Unfortunately, we do not know whether or not 5^m divides those coefficients evenly.

In conclusion, due to the reasons mentioned previously, we can state only a weak result:

Theorem 7. *(Fibonacci, weak) for $m \geq 0$ and $n \geq 1$,*

$$L_1L_3\cdots L_{2m-1}L_{2m+1}\sum_{k=1}^{n} F_{2k}^{2m+1} = (F_{2n+1}-1)\cdot P_{2k,2m+1},$$

where $P_{2k,2m+1}$ is a polynomial in F_{2n+1}, of degree $2m$, with rational coefficients.

The aim of the next section is to improve upon this.

4 Fibonacci Numbers - Second Attempt

4.1 Discovery

If we have experience solving problems of this kind (Edwards [2], Zielinski [11]), upon being given explicit expressions for $\sum_{k=1}^{n} F_{2k}^{2m+1}$, our first impulse might be to look for recursive relationships among all such sums. For example, Melham [5] tells us

$$\begin{aligned} 4\sum_{k=1}^{n} F_{2k}^{3} &= \left(F_{2n+1}-1\right)^2\left(F_{2n+1}-2\right) \qquad (13) \\ &= F_{2n+1}^3 - 3F_{2n+1} + 2. \end{aligned}$$

If we substitute $F_{2n+1} = 1 + \sum_{k=1}^{n} F_{2k}$ then we get

$$F_{2n+1}^3 = 1 + 3\sum_{k=1}^{n} F_{2k} + 4\sum_{k=1}^{n} F_{2k}^3.$$

For the next case we have

$$\begin{aligned} 44\sum_{k=1}^{n} F_{2k}^{5} &= \left(F_{2n+1}-1\right)^2\left(4F_{2n+1}^3 + 8F_{2n+1}^2 - 3F_{2n+1} - 14\right) \qquad (14) \\ &= 4F_{2n+1}^5 - 15F_{2n+1}^3 + 25F_{2n+1} - 14. \end{aligned}$$

If we substitute the previous expressions for F_{2n+1} and F_{2n+1}^3 then we get

$$F_{2n+1}^5 = 1 + 5\sum_{k=1}^{n} F_{2k} + 15\sum_{k=1}^{n} F_{2k}^3 + 11\sum_{k=1}^{n} F_{2k}^5.$$

For a fuller picture, we introduce matrix notation:

$$\begin{bmatrix} 1 \\ \sum F_{2k} \\ \sum F_{2k}^3 \\ \sum F_{2k}^5 \\ \sum F_{2k}^7 \\ \sum F_{2k}^9 \end{bmatrix} = \begin{bmatrix} 1 & & & & & 0 \\ -1 & 1 & & & & \\ \frac{1}{2} & -\frac{3}{4} & \frac{1}{4} & & & \\ -\frac{7}{22} & \frac{25}{44} & -\frac{15}{44} & \frac{1}{11} & & \\ \frac{139}{638} & -\frac{553}{1276} & \frac{455}{1276} & -\frac{56}{319} & \frac{1}{29} & \\ -\frac{1877}{12122} & \frac{8055}{24244} & -\frac{4083}{12122} & \frac{5625}{24244} & -\frac{189}{2204} & \frac{1}{76} \end{bmatrix} \times \begin{bmatrix} 1 \\ F_{2n+1} \\ F_{2n+1}^3 \\ F_{2n+1}^5 \\ F_{2n+1}^7 \\ F_{2n+1}^9 \end{bmatrix}. \qquad (15)$$

(We abbreviate $\sum_{k=1}^{n} F_{2k}^{2m+1}$ by $\sum F_{2k}^{2m+1}$.) This is a special case of Theorem 1. Since the matrix is lower triangular, it has an inverse:

$$\begin{bmatrix} 1 \\ F_{2n+1} \\ F_{2n+1}^3 \\ F_{2n+1}^5 \\ F_{2n+1}^7 \\ F_{2n+1}^9 \end{bmatrix} = \begin{bmatrix} 1 & & & & & 0 \\ 1 & 1 & & & & \\ 1 & 3 & 4 & & & \\ 1 & 5 & 15 & 11 & & \\ 1 & 7 & 35 & 56 & 29 & \\ 1 & 9 & 66 & 171 & 189 & 76 \end{bmatrix} \times \begin{bmatrix} 1 \\ \sum F_{2k} \\ \sum F_{2k}^3 \\ \sum F_{2k}^5 \\ \sum F_{2k}^7 \\ \sum F_{2k}^9 \end{bmatrix}. \qquad (16)$$

This is a reason to get excited! Starting with complicated fractions, we have arrived at simple, whole numbers. Also, it is an improvement upon the old approach. Immediately we have

$$\begin{aligned}4\sum_{k=1}^{n} F_{2k}^3 &= F_{2n+1}^3 - 1 - 3\sum_{k=1}^{n} F_{2k} \\ &= \left(F_{2n+1}-1\right)\left(F_{2n+1}^2 + F_{2n+1} - 2\right).\end{aligned}$$

Now, there is something important to notice. Unlike the previous relationships for Fibonacci numbers in (11), we can factor out another term of $F_{2n+1} - 1$:

$$4\sum_{k=1}^{n} F_{2k}^3 = \left(F_{2n+1}-1\right)^2\left(F_{2n+1} - 2\right),$$

which was what we had for (13). In fact, we can do this for all higher powers. For example,

$$\begin{aligned}11\sum_{k=1}^{n} F_{2k}^5 &= F_{2n+1}^5 - 1 - 5\sum_{k=1}^{n} F_{2k} - 15\sum_{k=1}^{n} F_{2k}^3 \\ &= \left(F_{2n+1}-1\right)\left(F_{2n+1}^4 + F_{2n+1}^3 + F_{2n+1}^2 + F_{2n+1} + 1\right) - 5\left(F_{2n+1}-1\right) - 15\sum_{k=1}^{n} F_{2k}^3 \\ &= \left(F_{2n+1}-1\right)\left(F_{2n+1}^4 + F_{2n+1}^3 + F_{2n+1}^2 + F_{2n+1} - 4\right) - 15\sum_{k=1}^{n} F_{2k}^3 \\ &= \left(F_{2n+1}-1\right)^2\left(F_{2n+1}^3 + 2F_{2n+1}^2 + 3F_{2n+1} + 4\right) - 15\left(F_{2n+1}-1\right)^2\left(\frac{F_{2n+1}}{4} - \frac{2}{4}\right).\end{aligned}$$

If we multiply both sides by 4 and simplify the right side, we get

$$44\sum_{k=1}^{n} F_{2k}^5 = \left(F_{2n+1}-1\right)^2\left(4F_{2n+1}^3 + 8F_{2n+1}^2 - 3F_{2n+1} - 14\right),$$

which was (14). What allowed us to factor out another term of $F_{2n+1} - 1$ was that the coefficient for $\sum_{k=1}^{n} F_{2k}$ was the next odd integer.

How do we prove this rigorously in the general case? That is the purpose of the next section. Unfortunately, rewriting

$$\sum_{k=1}^{n}\left(F_{2k+1}^{2m+1} - F_{2k-1}^{2m+1}\right)$$

of Lemma 5 directly into the expressions of (16) might be asking too much. Instead, we will try something else.

4.2 Proof

We borrow an idea from Ozeki [7]. Theorem 1 of Jennings [4] is the following:[1]

Theorem 8. *for $j \geq 0$ and $n \geq 0$,*

$$F_{(2j+1)n} = \sum_{i=0}^{j} (-1)^{(j+i)n} \frac{2j+1}{j+i+1} 5^i \binom{j+i+1}{2i+1} F_n^{2i+1}.$$

If we replace n by $2k$ and sum from 1 to n, we get

$$\sum_{k=1}^{n} F_{2(2j+1)k} = \sum_{i=0}^{j} \left[5^i \frac{2j+1}{j+i+1} \binom{j+i+1}{2i+1} \sum_{k=1}^{n} F_{2k}^{2i+1} \right].$$

Next, we insert this expression into the previous result of Proposition 4:

$$\begin{aligned} F_{2n+1}^{2m+1} &= 1 + \frac{1}{5^m} \sum_{j=0}^{m} \left[\binom{2m+1}{m-j} L_{2j+1} \sum_{k=1}^{n} F_{2(2j+1)k} \right] \\ &= 1 + \frac{1}{5^m} \sum_{j=0}^{m} \left[\binom{2m+1}{m-j} L_{2j+1} \left(\sum_{i=0}^{j} 5^i \frac{2j+1}{j+i+1} \binom{j+i+1}{2i+1} \sum_{k=1}^{n} F_{2k}^{2i+1} \right) \right]. \end{aligned}$$

If we change the order of summation and take notice of the divisor of 5, we get Theorem 5:

Theorem. *for $m \geq 0$ and $n \geq 1$,*

$$F_{2n+1}^{2m+1} = 1 + \sum_{i=0}^{m} \left(\sum_{k=1}^{n} F_{2k}^{2i+1} \right) \sum_{j=i}^{m} \left[\binom{2m+1}{m-j} L_{2j+1} \cdot \frac{1}{5^{m-i}} \cdot \frac{2j+1}{j+i+1} \binom{j+i+1}{2i+1} \right].$$

Now we are going to give a rigorous proof of *some* of our observations in the previous section. For that, we need two lemmas. The first is

Lemma 6. *for real numbers x and positive integers m, $x-1$ divides $x^m + x^{m-1} + \cdots + x - m$.*

Proof. notice that

$$\begin{aligned} x^m + x^{m-1} + \cdots + x - m &= (x-1) \\ &\quad \times \left(x^{m-1} + 2x^{m-2} + \cdots + (m-1)x + m \right). \end{aligned}$$

□

The second is Lemma 1. We state it succinctly:

Lemma. *in Theorem 5, the coefficient for $\sum_{k=1}^{n} F_{2k}$ is $2m+1$.*

[1] This is the inverse of our Proposition 6.

Proof. see Section A. □

Let us start the proof of the main result:

Proof. we proceed by mathematical induction. Previously we showed that

$$4\sum_{k=1}^{n} F_{2k}^{3} = \left(F_{2n+1}-1\right)^{2} \cdot P_{2k,3},$$

$$44\sum_{k=1}^{n} F_{2k}^{5} = \left(F_{2n+1}-1\right)^{2} \cdot P_{2k,5},$$

where $P_{2k,3}, P_{2k,5}$ were polynomials in F_{2n+1}, with integer coefficients. Assume that $\left(F_{2n+1}-1\right)^2$ divides

$$L_1 L_3 \sum_{k=1}^{n} F_{2k}^{3},\ L_1 L_3 L_5 \sum_{k=1}^{n} F_{2k}^{5},\ L_1 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1}$$

for all $1, 3, \ldots, 2m+1$, and that the adjoining polynomials have integer coefficients.

Theorem 5 tells us

$$F_{2n+1}^{2m+3} = 1 + \sum_{i=0}^{m+1} \left(\sum_{k=1}^{n} F_{2k}^{2i+1}\right) \sum_{j=i}^{m+1} \left[\binom{2m+3}{(m+1)-j} L_{2j+1} \cdot \frac{1}{5^{(m+1)-i}} \cdot \frac{2j+1}{j+i+1} \binom{j+i+1}{2i+1}\right]. \tag{17}$$

Lemma 1 tells us the coefficient for $\sum_{k=1}^{n} F_{2k}$ is $2m+3$. We remove that term from the right side of (17), bring it and the 1 to the left side, and write

$$\begin{aligned} F_{2n+1}^{2m+3} - 1 - (2m+3)\sum_{k=1}^{n} F_{2k} &= \left(F_{2n+1}-1\right)\left(F_{2n+1}^{2m+2} + F_{2n+1}^{2m+1} + \cdots + F_{2n+1} + 1\right) \qquad (18) \\ &\quad - (2m+3)\left(F_{2n+1}-1\right) \\ &= \left(F_{2n+1}-1\right)\left(F_{2n+1}^{2m+2} + F_{2n+1}^{2m+1} + \cdots + F_{2n+1} - (2m+2)\right) \\ &= \left(F_{2n+1}-1\right)^{2} \\ &\quad \times \left(F_{2n+1}^{2m+1} + 2F_{2n+1}^{2m} + \cdots + (2m+1)F_{2n+1} + 2m+2\right), \end{aligned}$$

where we have used Lemma 6 to pull out the second factor of $F_{2n+1}-1$.

For the next step, in (17), the coefficient for $\sum_{k=1}^{n} F_{2k}^{2m+3}$ is L_{2m+3}. If we pull that sum out of the right-hand side of (17) as well, we get

$$\sum_{i=1}^{m} \left(\sum_{k=1}^{n} F_{2k}^{2i+1}\right) \sum_{j=i}^{m} \left[\binom{2m+3}{(m+1)-j} L_{2j+1} \cdot \frac{1}{5^{(m+1)-i}} \cdot \frac{2j+1}{j+i+1}\binom{j+i+1}{2i+1}\right] + L_{2m+3}\sum_{k=1}^{n} F_{2k}^{2m+3}.$$

Needless to say, the first sum is too difficult to work with. Let us rewrite the entire expression as

$$\sum_{i=1}^{m} \left[C_{2k,2i+1} \sum_{k=1}^{n} F_{2k}^{2i+1}\right] + L_{2m+3}\sum_{k=1}^{n} F_{2k}^{2m+3}, \tag{19}$$

where $C_{2k,2i+1}$ is some constant which depends on $\sum_{k=1}^{n} F_{2k}^{2i+1}$. (There is one catch: we are not sure if $C_{2k,2i+1}$ is an integer.) If we put together (18) and (19), we get

$$\left(F_{2n+1}-1\right)^2 \cdot P_{2n+1} = \sum_{i=1}^{m} \left[C_{2k,2i+1} \sum_{k=1}^{n} F_{2k}^{2i+1} \right] + L_{2m+3} \sum_{k=1}^{n} F_{2k}^{2m+3}, \qquad (20)$$

where P_{2n+1} is a polynomial in F_{2n+1}, with integer coefficients.

Now we are ready to make use of the inductive hypothesis. We multiply both sides of (20) by $L_1 L_3 \cdots L_{2m-1} L_{2m+1}$ and rewrite it as

$$\begin{aligned} L_1 \cdots L_{2m+1} \left(F_{2n+1}-1\right)^2 P_{2n+1} \cdot \quad = \quad & L_5 \cdots L_{2m+1} C_{2k,3} L_1 L_3 \sum_{k=1}^{n} F_{2k}^{3} \\ & + L_7 \cdots L_{2m+1} C_{2k,5} L_1 L_3 L_5 \sum_{k=1}^{n} F_{2k}^{5} \\ & + \cdots + C_{2k,2m+1} L_1 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1} \\ & + L_1 \cdots L_{2m+1} L_{2m+3} \sum_{k=1}^{n} F_{2k}^{2m+3} \\ & = L_5 \cdots L_{2m+1} C_{2k,3} \left(F_{2n+1}-1\right)^2 P_{2k,3} \\ & + L_7 \cdots L_{2m+1} C_{2k,5} \left(F_{2n+1}-1\right)^2 P_{2k,5} \\ & + \cdots + C_{2k,2m+1} \left(F_{2n+1}-1\right)^2 P_{2k,2m+1} \\ & + L_1 \cdots L_{2m+1} L_{2m+3} \sum_{k=1}^{n} F_{2k}^{2m+3}, \end{aligned}$$

where $P_{2k,3}, P_{2k,5}, \ldots, P_{2k,2m+1}$ are polynomials in F_{2n+1}, with integer coefficients. If we keep the term of $\sum_{k=1}^{n} F_{2k}^{2m+3}$ on the right side, move everything else to the left side, pull out the factor of $\left(F_{2n+1}-1\right)^2$, and collect the resulting polynomial in F_{2n+1}, we are finished. □

We state our main result:

Theorem 9. *(Fibonacci, incomplete) for $m \geq 0$ and $n \geq 1$,*

$$L_1 L_3 \cdots L_{2m-1} L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1} = \left(F_{2n+1}-1\right)^2 \cdot P_{2k,2m+1},$$

where $P_{2k,2m+1}$ is a polynomial in F_{2n+1}, of degree $2m-1$, with rational coefficients.

In the next section we will have more to say about the constants $C_{2k,2i+1}$.

5 Integer Sequences

When we say "integer sequence," what we have in mind is the prototypical expression concerning sums of powers, $\sum_{k=1}^{n} k^m$, where m is a positive integer (Edwards [1]). If we start with the telescoping sum

$$(n+1)^m - 1 = \sum_{k=1}^{n} \left[(k+1)^m - k^m\right], \tag{21}$$

we can rewrite it directly as

$$(n+1)^m \quad = \quad 1 + \sum_{j=0}^{m-1} \left[\binom{m}{j} \sum_{k=1}^{n} k^j \right]. \tag{22}$$

This expresses $\sum_{k=1}^{n} k^m$ in n. In matrix notation, the first several cases are

$$\begin{bmatrix} n+1 \\ (n+1)^2 \\ (n+1)^3 \\ (n+1)^4 \\ (n+1)^5 \\ (n+1)^6 \end{bmatrix} = \begin{bmatrix} 1 & & & & & 0 \\ 1 & 2 & & & & \\ 1 & 3 & 3 & & & \\ 1 & 4 & 6 & 4 & & \\ 1 & 5 & 10 & 10 & 5 & \\ 1 & 6 & 15 & 20 & 15 & 6 \end{bmatrix} \times \begin{bmatrix} n+1 \\ \sum k \\ \sum k^2 \\ \sum k^3 \\ \sum k^4 \\ \sum k^5 \end{bmatrix}. \tag{23}$$

(Again, we abbreviate $\sum_{k=1}^{n} k^m$ by $\sum k^m$.) The associated integer sequence is

$$1, 1, 2, 1, 3, 3, 1, 4, 6, 4, \ldots, \tag{24}$$

which is A074909 in the OEIS.

5.1 Lucas Numbers

We expand upon our work in Section 2 and derive Theorem 6. Doing so, we will discover a new integer sequence.

Theorem 6 of Swamy [10] is the following:[2]

Theorem 10. *for $j \geq 0$ and $n \geq 0$,*

$$L_{(2j+1)n} = \sum_{i=0}^{j} (-1)^{(j+i)(n+1)} \frac{2j+1}{j+i+1} \binom{j+i+1}{2i+1} L_n^{2i+1}.$$

If we replace n by $2k$ and sum from 1 to n, we get

$$\sum_{k=1}^{n} L_{2(2j+1)k} = \sum_{i=0}^{j} \left[(-1)^{(j+i)(2k+1)} \frac{2j+1}{j+i+1} \binom{j+i+1}{2i+1} \sum_{k=1}^{n} L_{2k}^{2i+1} \right].$$

[2] This is the inverse of our Proposition 3.

If we insert this expression into the result of Proposition 1, we get

$$\begin{aligned} L_{2n+1}^{2m+1} &= 1+\sum_{j=0}^{m}\left[(-1)^{m-j}\binom{2m+1}{m-j}L_{2j+1}\sum_{k=1}^{n}L_{2(2j+1)k}\right] \\ &= 1+\sum_{j=0}^{m}\left\{(-1)^{m-j}\binom{2m+1}{m-j}L_{2j+1}\left[\sum_{i=0}^{j}(-1)^{(j+i)(2k+1)}\frac{2j+1}{j+i+1}\binom{j+i+1}{2i+1}\sum_{k=1}^{n}L_{2k}^{2i+1}\right]\right\}. \end{aligned}$$

If we adjust the signs[3] and change the order of summation, we get Theorem 6:

Theorem. *for $m \geq 0$ and $n \geq 1$,*

$$L_{2n+1}^{2m+1} = 1+\sum_{i=0}^{m}\left(\sum_{k=1}^{n}L_{2k}^{2i+1}\right)\sum_{j=i}^{m}\left[(-1)^{m+i}\binom{2m+1}{m-j}L_{2j+1}\frac{2j+1}{j+i+1}\binom{j+i+1}{2i+1}\right].$$

One tiny detail remains before we can assert this sum produces only integers for coefficients: the possibly rational term of

$$\frac{2j+1}{j+i+1}\binom{j+i+1}{2i+1}. \tag{25}$$

We get around this in the following way.

Theorem 10, which is Theorem 6 of Swamy [10], actually is established in Filipponi [3]. Also, it is stated a bit differently. In our notation it is

Theorem 11. *for $j \geq 0$ and $n \geq 0$,*

$$L_{(2j+1)n} = \sum_{i=0}^{j}(-1)^{i(n+1)}\frac{2j+1}{2j+1-i}\binom{2j+1-i}{i}L_n^{2j+1-2i}.$$

The author points out that

$$\frac{2j+1}{2j+1-i}\binom{2j+1-i}{i} = \binom{2j+1-i}{i}+\binom{2j-i}{i-1}. \tag{26}$$

The expressions in Theorems 10 and 11 are the same thing: the first one counts up, the second one counts down. We can substitute one for the other. That means (25) always is an integer.

Now that we know the sum produces only integers for coefficients, what does it look like? Well, it is the same as the one for Fibonacci numbers, (16), but *with* changes of sign and *without* divisors of 5:

$$\begin{bmatrix} 1 \\ L_{2n+1} \\ L_{2n+1}^3 \\ L_{2n+1}^5 \\ L_{2n+1}^7 \\ L_{2n+1}^9 \end{bmatrix} = \begin{bmatrix} 1 & & & & & 0 \\ 1 & 1 & & & & \\ 1 & -15 & 4 & & & \\ 1 & 125 & -75 & 11 & & \\ 1 & -875 & 875 & -280 & 29 & \\ 1 & 5625 & -8250 & 4275 & -945 & 76 \end{bmatrix} \times \begin{bmatrix} 1 \\ \sum L_{2k} \\ \sum L_{2k}^3 \\ \sum L_{2k}^5 \\ \sum L_{2k}^7 \\ \sum L_{2k}^9 \end{bmatrix}. \tag{27}$$

[3] We replace $(-1)^{m-j+(j+i)2k+j+i}$ by $(-1)^{m+i}$.

The inverse is

$$\begin{bmatrix} 1 \\ \sum L_{2k} \\ \sum L_{2k}^3 \\ \sum L_{2k}^5 \\ \sum L_{2k}^7 \\ \sum L_{2k}^9 \end{bmatrix} = \begin{bmatrix} 1 & & & & & 0 \\ -1 & 1 & & & & \\ -4 & \frac{15}{4} & \frac{1}{4} & & & \\ -16 & \frac{625}{44} & \frac{75}{44} & \frac{1}{11} & & \\ -64 & \frac{69125}{1276} & \frac{11375}{1276} & \frac{280}{319} & \frac{1}{29} & \\ -256 & \frac{5034375}{24244} & \frac{510375}{12122} & \frac{140625}{24244} & \frac{945}{2204} & \frac{1}{76} \end{bmatrix} \times \begin{bmatrix} 1 \\ L_{2n+1} \\ L_{2n+1}^3 \\ L_{2n+1}^5 \\ L_{2n+1}^7 \\ L_{2n+1}^9 \end{bmatrix}, \quad (28)$$

which appears in Melham [5]. Also, it is a special case of Theorem 2. (Again, we abbreviate $\sum_{k=1}^{n} L_{2k}^{2m+1}$ by $\sum L_{2k}^{2m+1}$.) Our integer sequence is

$$1, 1, 1, 1, -15, 4, 1, 125, -75, 11, \ldots,$$

which we stated in (1). It seems to be new. The OEIS does not have an entry for it.

5.2 Fibonacci Numbers

When we last left the Fibonacci numbers in Section 4.2, we were talking about constants $C_{2k,2i+1}$. Let us refresh our memory.

The result of Theorem 5 is

$$F_{2n+1}^{2m+1} = 1 + \sum_{i=0}^{m} \left(\sum_{k=1}^{n} F_{2k}^{2i+1} \right) \sum_{j=i}^{m} \left[\binom{2m+1}{m-j} L_{2j+1} \cdot \frac{1}{5^{m-i}} \cdot \frac{2j+1}{j+i+1} \binom{j+i+1}{2i+1} \right].$$

For $i = m$, the sum

$$\sum_{j=m}^{m} \left[\binom{2m+1}{m-j} L_{2j+1} \cdot \frac{1}{5^{m-i}} \cdot \frac{2j+1}{j+i+1} \binom{j+i+1}{2i+1} \right]$$

is simply L_{2m+1}. For $i = 0$, by Lemma 1 the sum

$$\frac{1}{5^m} \sum_{j=0}^{m} \left[\binom{2m+1}{m-j} L_{2j+1} \left(2j+1\right) \right]$$

is equal to $2m + 1$. From the previous discussion of Lucas numbers, we know that

$$\sum_{j=i}^{m} \left[\binom{2m+1}{m-j} L_{2j+1} \frac{2j+1}{j+i+1} \binom{j+i+1}{2i+1} \right]$$

always produces an integer. For $1 \leq i \leq m-1$, does 5^{m-i} divide such a sum, thereby still producing an integer?

The experimental data in (16) says it does. Unfortunately, despite the proof of Lemma 1, we are unable to give a rigorous proof of the full result and must leave it as Conjecture 1:

Conjecture. *in Theorem 5, the coefficients for* $\sum_{k=1}^{n} F_{2k}^{2i+1}$ *are integers only. In other words, for* $0 \le i \le m$, 5^{m-i} *divides*

$$\sum_{j=i}^{m} \left[\binom{2m+1}{m-j} L_{2j+1} \frac{2j+1}{j+i+1} \binom{j+i+1}{2i+1} \right].$$

If we can establish this result then we will have a complete proof of Theorem 3, the original conjecture of Melham [5]. Also, as a byproduct we will have a second integer sequence,

$$1, 1, 1, 1, 3, 4, 1, 5, 15, 11, \ldots,$$

which we stated in (2). It also seems to be new.

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A Proof of Lemma 1

Now we will prove Lemma 1. Let us state it again. Theorem 5 tells us

$$F_{2n+1}^{2m+1} = 1+\sum_{i=0}^{m}\left(\sum_{k=1}^{n} F_{2k}^{2i+1}\right)\sum_{j=i}^{m}\left[\binom{2m+1}{m-j} L_{2j+1} \cdot \frac{1}{5^{m-i}} \cdot \frac{2j+1}{j+i+1}\binom{j+i+1}{2i+1}\right].$$

The coefficient for $\sum_{k=1}^{n} F_{2k}$ is

$$\frac{1}{5^m}\sum_{j=0}^{m}\left[\binom{2m+1}{m-j} L_{2j+1}\,(2j+1)\right]. \tag{29}$$

Our objective is to show this expression is equal to $2m+1$.

We start with Theorem 8 of Swamy [10]:

Theorem 12. *for $j \geq 0$ and $n \geq 0$,*

$$L_{(2m+1)n} = \sum_{j=0}^{m}\left[(-1)^{(m+j)n}\binom{m+j}{2j} 5^j L_n F_n^{2j}\right].$$

(We have changed the notation to match that of (29).) The first several cases are

$$\begin{aligned}
L_n &= L_n\\
L_{3n} &= (-1)^n L_n + 5L_nF_n^2\\
L_{5n} &= L_n + (-1)^{3n}\, 15L_nF_n^2 + 25L_nF_n^4\\
L_{7n} &= (-1)^{3n} L_n + 30L_nF_n^2 + (-1)^{5n}\, 125L_nF_n^4 + 125L_nF_n^6\\
L_{9n} &= L_n + (-1)^{5n}\, 50L_nF_n^2 + 375L_nF_n^4 + (-1)^{7n}\, 875L_nF_n^6 + 625L_nF_n^8.
\end{aligned} \tag{30}$$

The first several cases of the inverse relationship are

$$\begin{aligned}
L_n &= L_n\\
5L_nF_n^2 &= (-1)^{n+1} L_n + L_{3n}\\
5^2L_nF_n^4 &= 2L_n + (-1)^{n+1}\, 3L_{3n} + L_{5n}\\
5^3L_nF_n^6 &= (-1)^{3(n+1)}\, 5L_n + 9L_{3n} + (-1)^{n+1}\, 5L_{5n} + L_{7n}\\
5^4L_nF_n^8 &= 14L_n + (-1)^{3(n+1)}\, 28L_{3n} + 20L_{5n} + (-1)^{n+1}\, 7L_{7n} + L_{9n.}
\end{aligned} \tag{31}$$

It must be admitted the changes in sign make these examples difficult to work with. For the inverse, if we set $n = 1$ then we get

$$5^4 = 14L_1 + 28L_3 + 20L_5 + 7L_7 + L_9,$$

for example. In the proof of Lemma 1 this idea will come up again.

The general case of the inverse is

Proposition 7. *for $m \geq 0$ and $n \geq 1$,*

$$5^m L_n F_n^{2m} = \binom{2m}{0} L_{(2m+1)n} + \sum_{j=1}^{m} \left[(-1)^{j(n+1)} \left(\binom{2m}{j} - \binom{2m}{j-1} \right) L_{(2m+1-2j)n} \right].$$

We will prove it in two steps. The first step is to establish

Lemma 7. *for $m \geq 1$ and $n \geq 1$,*

$$5^m L_n F_n^{2m} = \sum_{j=0}^{m-1} \left[(-1)^{j(n+1)} \binom{2m}{j} L_{2(m-j)n} L_n \right] + (-1)^{m(n+1)} \binom{2m}{m} L_n.$$

The second step is to rewrite it using

Lemma 8. *for $m \geq n \geq 1$,*

$$L_m L_n = L_{m+n} + (-1)^n L_{m-n}.$$

Let us get started by proving Lemma 7:

Proof. we use the Binet forms:

$$\begin{aligned} F_n^{2m} &= \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)^{2m} = \frac{1}{\left(\sqrt{5}\right)^{2m}} \left(\alpha^n - \beta^n \right)^{2m} \\ &= \frac{1}{5^m} \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} \alpha^{(2m-j)n} \beta^{jn} \\ &= \frac{1}{5^m} \sum_{j=0}^{m-1} \left[(-1)^j \binom{2m}{j} \left(\alpha^{(2m-j)n} \beta^{jn} + \alpha^{jn} \beta^{(2m-j)n} \right) \right] + \frac{1}{5^m} (-1)^m \binom{2m}{m} (\alpha\beta)^{mn} \\ &= \frac{1}{5^m} \sum_{j=0}^{m-1} \left[(-1)^j \binom{2m}{j} (\alpha\beta)^{jn} \left(\alpha^{(2m-2j)n} + \beta^{(2m-2j)n} \right) \right] + \frac{1}{5^m} (-1)^{m+mn} \binom{2m}{m} \\ &= \frac{1}{5^m} \sum_{j=0}^{m-1} \left[(-1)^{j(n+1)} \binom{2m}{j} L_{2(m-j)n} \right] + \frac{1}{5^m} (-1)^{m(n+1)} \binom{2m}{m}. \end{aligned}$$

If we multiply both sides by $5^m L_n$, we get the desired result:

$$5^m L_n F_n^{2m} = \sum_{j=0}^{m-1} \left[(-1)^{j(n+1)} \binom{2m}{j} L_{2(m-j)n} L_n \right] + (-1)^{m(n+1)} \binom{2m}{m} L_n.$$

□

Let us prove Lemma 8:

Proof. we use the Binet forms as well:

$$\begin{aligned} L_m L_n &= (\alpha^m + \beta^m)(\alpha^n + \beta^n) \\ &= \alpha^{m+n} + \alpha^n \beta^m + \alpha^m \beta^n + \beta^{m+n} \\ &= \left(\alpha^{m+n} + \beta^{m+n}\right) + (\alpha\beta)^n \left(\alpha^{m-n} + \beta^{m-n}\right) \\ &= L_{m+n} + (-1)^n L_{m-n}. \end{aligned}$$

□

Now we rewrite Lemma 7:

$$\begin{aligned} 5^m L_n F_n^{2m} &= \sum_{j=0}^{m-1} \left[(-1)^{j(n+1)} \binom{2m}{j} L_{2(m-j)n} L_n\right] + (-1)^{m(n+1)} \binom{2m}{m} L_n \\ &= \sum_{j=0}^{m-1} \left[(-1)^{j(n+1)} \binom{2m}{j} \left(L_{(2m+1-2j)n} + (-1)^n L_{(2m-1-2j)n}\right)\right] \\ &+ (-1)^{m(n+1)} \binom{2m}{m} L_n \\ &= \binom{2m}{0} L_{(2m+1)n} + (-1)^n \binom{2m}{0} L_{(2m-1)n} \\ &+ (-1)^{n+1} \binom{2m}{1} L_{(2m-1)n} + (-1)^{2n+1} \binom{2m}{1} L_{(2m-3)n} \\ &+ \cdots + (-1)^{(m-1)(n+1)} \binom{2m}{m-1} L_{3n} + (-1)^{(m-1)(n+1)+n} \binom{2m}{m-1} L_n \\ &+ (-1)^{m(n+1)} \binom{2m}{m} L_n, \end{aligned}$$

which we group as

$$\begin{aligned} 5^m L_n F_n^{2m} &= \binom{2m}{0} L_{(2m+1)n} + \left[(-1)^n \binom{2m}{0} + (-1)^{n+1} \binom{2m}{1}\right] L_{(2m-1)n} \\ &+ \cdots + \left[(-1)^{mn+m-1} \binom{2m}{m-1} + (-1)^{mn+m} \binom{2m}{m}\right] L_n \\ &= \binom{2m}{0} L_{(2m+1)n} + (-1)^{n+1} \left[\binom{2m}{1} - \binom{2m}{0}\right] L_{(2m-1)n} \\ &+ \cdots + (-1)^{mn+m} \left[\binom{2m}{m} - \binom{2m}{m-1}\right] L_n, \end{aligned}$$

which gives us the final result of

$$5^m L_n F_n^{2m} = \binom{2m}{0} L_{(2m+1)n} + \sum_{j=1}^{m} \left[(-1)^{j(n+1)} \left(\binom{2m}{j} - \binom{2m}{j-1}\right) L_{(2m+1-2j)n}\right].$$

This establishes Proposition 7.

Now we are ready to prove Lemma 1. In the result of Proposition 7 we set $n = 1$ and multiply both sides by $2m + 1$. This gives us

$$\begin{aligned} 5^m (2m+1) &= \binom{2m}{0} L_{(2m+1)} (2m+1) \\ &+ \sum_{j=1}^{m} \left[\left(\binom{2m}{j} - \binom{2m}{j-1} \right) L_{(2m+1-2j)} (2m+1) \right]. \end{aligned} \tag{32}$$

For the expression in brackets, let us concentrate on the product

$$(2m+1) \left(\binom{2m}{j} - \binom{2m}{j-1} \right).$$

The basic identity

$$\binom{2m}{j-1} + \binom{2m}{j} = \binom{2m+1}{j}$$

allows us to rewrite it as

$$\begin{aligned} (2m+1) \left(\binom{2m+1}{j} - 2\binom{2m}{j-1} \right) &= (2m+1) \left[\frac{(2m+1)!}{j!\,(2m+1-j)!} - 2 \cdot \frac{(2m)!}{(j-1)!\,(2m+1-j)!} \right] \\ &= (2m+1) \left[\frac{(2m+1)!}{j!\,(2m+1-j)!} - \frac{2j\,(2m)!}{j!\,(2m+1-j)!} \right] \\ &= \frac{(2m+1)\,(2m+1)! - 2j\,(2m+1)!}{j!\,(2m+1-j)!} \\ &= (2m+1-2j) \binom{2m+1}{j}. \end{aligned}$$

(32) has become

$$\begin{aligned} 5^m (2m+1) &= \binom{2m}{0} L_{(2m+1)} (2m+1) \\ &+ \sum_{j=1}^{m} \left[\binom{2m+1}{j} L_{(2m+1-2j)} (2m+1-2j) \right]. \end{aligned}$$

If we change $\binom{2m}{0}$ to $\binom{2m+1}{0}$ then we get

$$2m+1 = \frac{1}{5^m} \sum_{j=0}^{m} \left[\binom{2m+1}{j} L_{(2m+1-2j)} (2m+1-2j) \right].$$

We have proved Lemma 1.